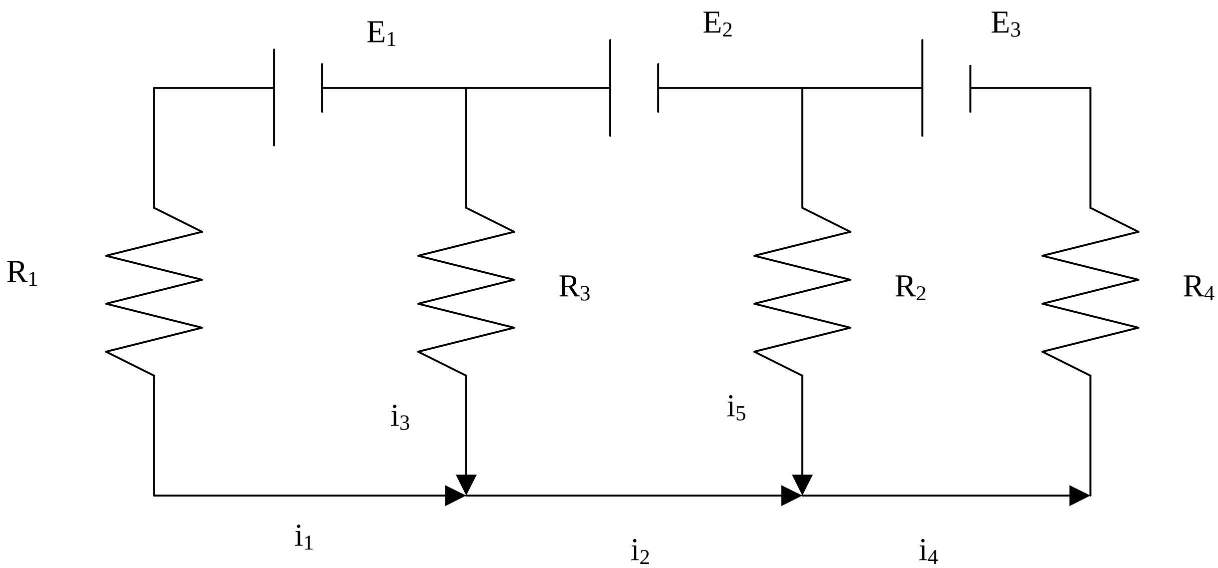
Lecture 9: Applications to Circuits, Structures and Mixing

In lecture 4 applications to steady state models for circuits, rigid structures and mixing tanks were introduced. Here more complicated versions of these applications will be studied. All three applications will illustrate how large systems of algebraic equations can be used to model important applications.

Application to Steady State Three-loop Circuit. Consider the following circuit with four resistors and three batteries. We wish to know the currents in the four resistors given the four resistances and three voltages of the batteries. By using Ohm's and Kirchhoff's laws we obtain and five equations for five currents.



$$i_1 + i_3 = i_2$$
$$i_2 + i_5 = i_4$$
$$E_1 = R_1 i_1 - R_3 i_3$$
$$E_2 = R_3 i_3 - R_2 i_5$$
$$E_3 = R_2 i_5 + R_4 i_4$$
$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ R_1 & 0 & -R_3 & 0 & 0 \\ 0 & 0 & R_3 & 0 & -R_2 \\ 0 & 0 & 0 & R_4 & R_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E_1 \\ E_2 \\ E_3 \end{bmatrix}.$$

So, the coefficient matrix, A, is 5x5, and the unknown vector is 5x1 vector of currents. In the Matlab demo circuit3.m the following values were used:

$R_1 = 1$, $R_2 = 2$, $R_3 = 3$, $R_4 = 4$, $R_5 = 5$ with

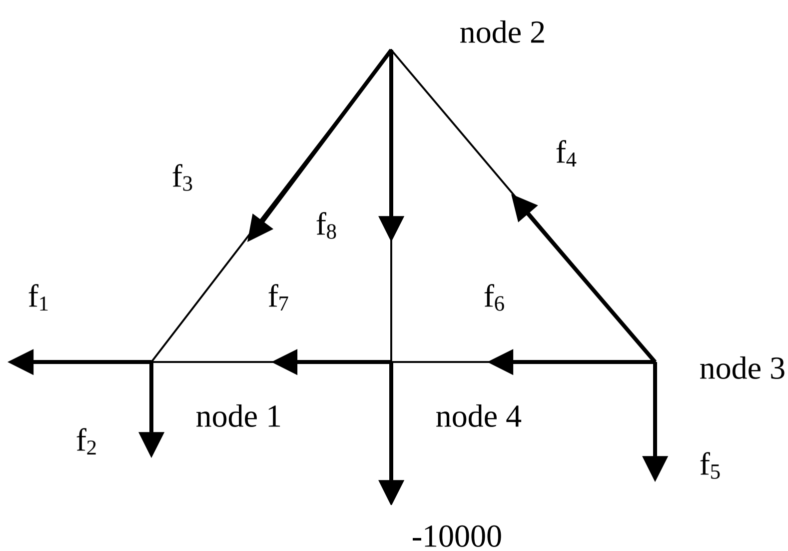
$E_1 = 10$, $E_2 = 20$ and $E_3 = 30$.

The solution that circuit3.m generated was

$i_1 = 16$, $i_2 = 18$, $i_3 = 2$, $i_4 = 11$ and $i_5 = -7$.

The fifth current is negative, which means the fifth current is going in the opposite direction as indicated in the above figure.

Application to Rigid Steady State Four-Node Structure. Consider the following structure with five beams and four nodes. You could think of this as a very simple model of a bridge with the left node fixed and the right node free in the horizontal direction.



Let the angles between the beams at node 1 and 3 both be equal to θ.

The balances of forces at the four nodes are a four 2x1 vector equations:

node 1: $[f_1 ,\ f_2] = [f_3\cos(\theta) + f_7 ,\ f_3\sin(\theta)]$

node 2: $[0 ,\ f_8] = [-f_3\cos(\theta) + f_4\cos(\theta) ,\ -f_3\sin(\theta) - f_4\sin(\theta)]$

node 3: $[0 ,\ f_5] = [-f_4\cos(\theta) - f_6 ,\ f_4\sin(\theta)]$

node 4: $[0 ,\ -10000] = [f_6 - f_7 ,\ -f_8]$

By setting the components equal we have

$-f_1 + \cos(\theta)f_3 + f_7 = 0,$

$-f_2 + \sin(\theta)f_3 = 0$

$-\cos(\theta)f_3 + \cos(\theta)f_4 = 0$

$-\sin(\theta)f_3 - \sin(\theta)f_4 - f_8 = 0$

$-\cos(\theta)f_4 - f_6 = 0$

$\sin(\theta)f_4 - f_5 = 0$

$f_6 - f_7 = 0$ and

$-f_8 = -10000.$

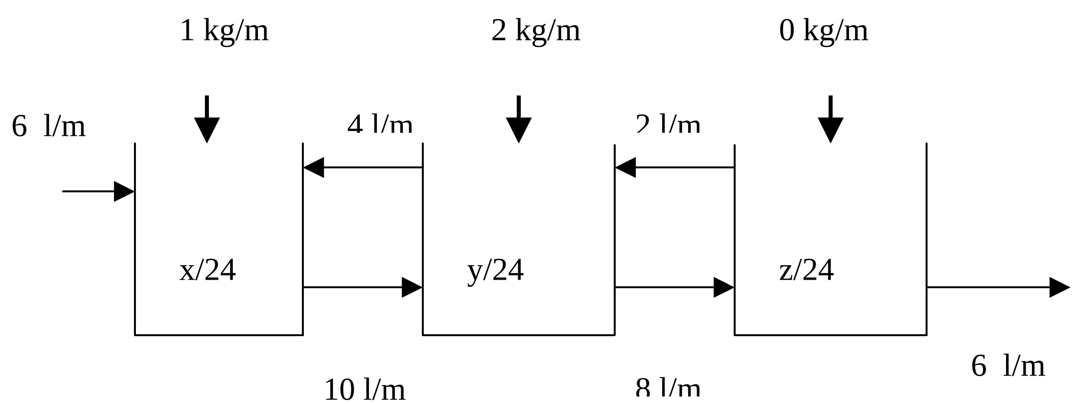
So, the coefficient matrix is 8x8, and the unknown vector is 8x1 for the magnitudes of the forces. Let c = cos(θ) and s = sin(θ).

$$\begin{bmatrix} -1 & 0 & -c & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c & c & 0 & 0 & 0 & 0 \\ 0 & 0 & -s & -s & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -c & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & s & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -10000 \end{bmatrix}.$$

One can experiment with different parameters. In the following table the solutions via the Matlab demo bridge1.m are for two different angles θ.

Forces	**θ = pi/12**	**θ = pi/3**
f_1	-0	0
f_2	-5,000	-5,000
f_3	-19,319	-5,774
f_4	-19,319	-5,774
f_5	-5,000	-5,000
f_6	8,660	2,887
f_7	8,660	2,887
f_8	10,000	10,000

Application to Steady State Three-tank Mixing. Consider three well-stirred 24 liter mixing tanks with amounts of a chemical equal to x(t), y(t) and z(t). All tanks are full, and the left tank pure water flowing into it from the left with a flow rate equal to 6 liters per minute (6 l/m). The right tank will have an outflow to the right of 6 liters per minute and a concentration equal to z/24 kilograms per liter (z/24 kg/l).



By applying rate of change of either tank is the rate in minus the rate out we get

$x' = (0 + 1 + 4\, y/24) - (10\, x/24)$ and

$y' = (2 + 10\, x/24 + 2\, z/24) - (4\, y/24 + 8\, y/24)$

$z' = (0 + 8\, y/24) - (2\, z/24 + 6/24\, z)$.

The steady state solution requires that all x', y' and z' be equal zero. This gives the algebraic equations

$$0 = 1 - (10/24)x + (4/24)x$$
$$0 = 2 + (10/24)x - (12/24)y + (2/24)z$$
$$0 = 0 + (8/24)y - (8/24)z$$

Or,

$$\begin{bmatrix} -10/24 & 4/24 & 0 \\ 10/24 & -12/24 & 2/24 \\ 0 & 8/24 & -8/24 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}.$$

The Matlab demo tank3.m solves this to give x = 7.2, y = 12. and z = 12.

Homework.

1. Use the Matlab demo circuit3.m to experiment with different value of the resistors and battery voltages.
2. Formulate a model for a four-loop circuit.
3. Use the Matlab demo bridge1.m to experiment with different values of the angle θ.
4. Formulate a model for a bridge where the angles at nodes 1 and 3 may be different.
5. Use the Matlab demo tank3.m to experiment with different flow rates. Be sure to that the tanks remain full.
6. Formulate a model for a tank system with four tanks.