Lecture 8: Properties of Inverse Matrices

Not all matrices have inverse matrices! The following 2x2 matrix does not have an inverse because one cannot find the first column of the inverse $Ab_1 = e_1$.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \text{ and } e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\textit{Or},\ 1b_{11} + 1b_{21} = 1 \textit{ and}$$

$$2b_{11} + 2b_{21} = 0.$$

$$\textit{Then } b_{11} = -b_{21}, \textit{ and the first equation gives}$$

$$1(-b_{21}) + 1b_{21} = 1$$

$$0 = 1, \textit{ a contradiction}!$$

If A does have an inverse, then it can be used to solve $Ax = d$ where $x = A^{-1}d$. The proof of this is very simple and makes use of the associative property of a matrix product and the definition of an inverse matrix:

$A(A^{-1}d) = (A\, A^{-1})d = I\, d = d.$

Some other properties are given in Proposition 4.

Proposition 4. Let A, A_1 and A_2 be nxn matrices.

1. If $A^{-1} = B$, then A(col k of B) = e_k.
2. If A has an inverse matrix, then there is only one inverse matrix.
3. If A_1 and A_2 have inverses, then A_1A_2 has an inverse and $(A_1A_2)^{-1} = A_2^{-1}\, A_1^{-1}$.
4. If A has an inverse, then $x = A^{-1}d$ is the solution of $Ax = d$ and this is the only solution.
5. The following are equivalent:

    (i). A has an inverse.

    (ii). det(A) is not zero.

    (iii). $Ax = 0$ implies $x = 0$.

Proof of 2. Let B_1 and B_2 be inverses of A so that $AB_1 = I$ and $AB_2 = I$.

Subtract these two equations to get

$AB_1 – AB_2 = I – I = 0$

$A(B_1 – B_2) = 0$

$A^{-1}(A(B_1 – B_2)) = A^{-1}\, 0$

$(A^{-1}A)(B_1 – B_2) = 0$

$(I)(B_1 – B_2) = 0$

$(B_1 – B_2) = 0$

So, $B_1 = B_2$.

Proof of 3. Let A_1 and A_2 have inverses.

We must show $(A_1A_2)^{-1}(A_1A_2) = I$ and $(A_1A_2)(A_1A_2)^{-1} = I$.

$$\begin{aligned}(A_1A_2)^{-1}(A_1A_2) &= (A_2^{-1}A_1^{-1})(A_1A_2)\\ &= A_2^{-1}(A_1^{-1}(A_1A_2))\\ &= A_2^{-1}((A_1^{-1}A_1)A_2)\\ &= A_2^{-1}((I)A_2)\\ &= A_2^{-1}A_2\\ &= I\end{aligned}$$

Proof of 5. This will have to wait for a more detailed course on matrix algebra.

Example of Property 3 in Proposition 4. Find the inverse matrix of

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A_1 A_2$$

A_1 *is a diagonal matrix with an inverse*

$$A_1^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A_2 *is an elementary matrix with an inverse*

$$A_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$So, A^{-1} = A_2^{-1} A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 & 0 \\ 2/4 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Application of the Two-loop Circuit.

$$\begin{bmatrix} 1 & -1 & 1 \\ R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ E_1 \\ E_2 \end{bmatrix}.$$

Let $R_1 = 1, R_2 = 2, R_3 = 3, E_1 = 10$ *and* $E_2 = 20.$

$$Ax = d$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -3 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 20 \end{bmatrix}.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -3 \\ 0 & -2 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 10 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 6/11 & 5/11 & -3/11 \\ -3/11 & 3/11 & -4/11 \\ 2/11 & -2/11 & -1/11 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} -10/11 \\ -50/11 \\ -40/11 \end{bmatrix}.$$

Block Gauss Elimination and Inverse Matrices. Often an nxn matrix A is broken into blocks of matrices. This is done either to reflect substructures of a model or to accommodate memory hierarchy of a computer. Consider the following 2x2 block representation where A_{11} is $n_1 x n_1$, A_{22} is $n_2 x n_2$ and $n = n_1 + n_2$.

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \; and \; \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

If A_{11} has an inverse, then we can use a block elementary matrix to zero the 21-block.

$$\begin{bmatrix} I_1 & 0 \\ -A_{21}A_{11}^{-1} & I_2 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} I_1 & 0 \\ -A_{21}A_{11}^{-1} & I_2 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} d_1 \\ \hat{d}_2 \end{bmatrix} \quad where$$

$$\hat{A}_{22}=A_{22}-A_{21}A_{11}^{-1}A_{12}$$

$$\hat{d}_2=d_2-A_{21}A_{11}^{-1}d_1$$

If both A_{11} and $\hat{A}_{22}=A_{22}-A_{21}A_{11}^{-1}A_{12}$ have inverses,

the block backward substitution step can be executed

$$x_2=\hat{A}_{22}^{-1}\hat{d}_2$$

$$x_1=A_{11}^{-1}(d_1-A_{12}x_2).$$

Examples of this can be found in the Matlab demo blockg_el.m

Homework.

1. In the proof of part 3 in Proposition 4 prove the other equality.
2. In part 4 of Proposition 4 prove there is only one solution.
3. In part 5 in Proposition prove (i) implies (iii).
4. Use the inverse matrix to solve

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.$$

5. Use property 3 to find the inverse of

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 6 & 0 & 3 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$