Lecture 6: Gauss Elimination

The objective is to solve for n unknowns given n linear equations. In terms of matrices, find the nx1 column vector x so that Ax = d where A is nxn and d is nx1. We will do this using the method called Gauss elimination, which requires about n^2 units of memory and about $n^3/3$ arithmetic operations. So, if n is not too large, this method is very useful; otherwise, one must go to other more specialized methods. We begin by considering several special matrices A.

A is a Diagonal Matrix.

$$Let\ n = 3.$$

$$\begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$Or,$$

$$a_{11}x_1 = d_1$$

$$a_{22}x_2 = d_2$$

$$a_{33}x_3 = d_3$$

$$So, if\ \ each\ a_{ii} \neq 0, then\ x_i = d_i / a_{ii}.$$

A is an Upper Triangular Matrix.

$$Let\ n = 3.$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$Or,$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= d_1 \\ a_{22}x_2 + a_{23}x_3 &= d_2 \\ a_{33}x_3 &= d_3 \end{aligned}$$

$So, if\ \ a_{33} \neq 0, then$

$$x_3 = d_3 / a_{33}.$$

$Solve\ the\ middle\ equation, if\ \ a_{22} \neq 0,$

$$a_{22}x_2 = d_2 - a_{23}x_3$$

$$x_2 = (d_2 - a_{23}x_3) / a_{22}.$$

$Solve\ the\ \ first\ equation, if\ \ a_{11} \neq 0,$

$$a_{11}x_1 = d_1 - a_{12}x_2 - a_{13}x_3$$

$$x_1 = (d_1 - a_{12}x_2 - a_{13}x_3) / a_{11}.$$

Proposition 3. Let A be a nxn upper or lower triangular matrix,
If each diagonal component is non-zero, then Ax = d has a solution.

Consider the following system with three unknowns and three equations, whose coefficient matrix is not upper or lower triangular.

$$2x_1 + 6x_2 + 8x_3 = 16$$

$$4x_1 + 15x_2 + 19x_3 = 38$$

$$2x_1 + 0x_2 + 3x_3 = 6$$

$Or,\ Ax = d$

$$\begin{bmatrix} 2 & 6 & 8 \\ 4 & 15 & 19 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 38 \\ 6 \end{bmatrix}$$

$Or,\ as\ a\ 3x4\ augmented\ \ matrix$

$$[A\, d] = \begin{bmatrix} 2 & 6 & 8 & 16 \\ 4 & 15 & 19 & 38 \\ 2 & 0 & 3 & 6 \end{bmatrix}.$$

We will try to transform this into an equivalent problem that has an upper triangular coefficient matrix, which may be solved by a backward substitution. Often this can be done by a combination of row (or equation) operations:

(i). interchange rows (or equations)

(ii). multiply rows (or equations) by a suitable constant and

(iii). add or subtract rows (or equations).

Row Operations on the Augmented Matrix:

We will try to transform the augmented matrix to an upper triangular matrix by using row operations to make the left most column 0 below row 1.

row 2 – 2(row 1) or multiply [A d] by $E_{21}(-2)$ and then

row 3 – 1(row 1) or multiply $E_{21}(-2)$ [A d] by $E_{31}(-1)$ to get

$$E_{31}(-1)E_{21}(-2)[A\,d] = \begin{bmatrix} 2 & 6 & 8 & 16 \\ 0 & 3 & 3 & 6 \\ 0 & -6 & -5 & -10 \end{bmatrix}.$$

Next, we must use row 2 to transform the –6 to 0.

row 3 + 2(row 2) or multiply $E_{31}(-1)$ $E_{21}(-2)$ [A d] by $E_{32}(2)$ to get

$$E_{32}(2)E_{31}(-1)E_{21}(-2)[A\,d] = \begin{bmatrix} 2 & 6 & 8 & 16 \\ 0 & 3 & 3 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Backward Substitution on the Transformed Augmented Matrix:

This is equivalent to the upper triangular system, which can be solved by backward substitution.

$$\begin{bmatrix} 2 & 6 & 8 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 6 \\ 2 \end{bmatrix}$$

$Solve\ \ for\ x_3, 1x_3 = 2\ so\ that\ x_3 = 2.$

$Solve\ \ for\ x_2, 3x_2 + 3x_3 = 6\ so\ that\ x_2 = 0.$

$Solve\ \ for\ x_1, 2x_1 + 6x_2 + 8x_3 = 16\ so\ that\ x_1 = 0.$

Gauss Elimination Method. Consider Ax = d.

Step 1:   Use row operations on the augmented matrix to transform Ax = d to an upper triangular problem.

Step 2:   Use backward substitution to solve the upper triangular problem.

See the Matlab demo gauss_el.m for additional examples. There are a number of very good computer implementations of the Gauss elimination method. In Matlab the simple command A\d can be used to compute the solution for most problems Ax = d.

Application to the Two-loop Circuit.

In lecture 4 we formulated the algebraic problem for the two-loop circuit problem:

$$\begin{bmatrix} 1 & -1 & 1 \\ R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ E_1 \\ E_2 \end{bmatrix}.$$

$Let\ R_1 = 1, R_2 = 2, R_3 = 3, E_1 = 10\ and\ E_2 = 20.$

$The\ augemented\ matrix\ is$

$$[A\,d] = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & -3 & 10 \\ 0 & -2 & -3 & 20 \end{bmatrix}.$$

Step 1: Use row operations.

row 2 – row 1 or multiply by $E_{12}(-1)$ to get

$$E_{21}(-1)[A\,d] = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -4 & 10 \\ 0 & -2 & -3 & 20 \end{bmatrix}.$$

row 3 + 2(row2) or multiply by $E_{32}(2)$ to get

$$E_{32}(2)E_{21}(-1)[A\,d] = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -11 & 40 \end{bmatrix}.$$

Step 2: Use backward substitution.

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & -11 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 40 \end{bmatrix}$$

$Solve\ \ for\ i_3, -11x_3 = 20\ so\ that\ i_3 = -40/11.$

$Solve\ \ for\ i_2, 1i_2 - 4i_3 = 10\ so\ that\ i_2 = -50/11.$

$Solve\ \ for\ i_1, 1i_1 - 1i_2 + 1i_3 = 0\ so\ that\ i_1 = -10/11.$

Homework.

1. Use Gauss elimination to solve Ax = d when

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 1 \\ 2 & 2 & 0 \end{bmatrix}\ and\ d = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

2. Prove Proposition 3 when A is lower triangular.