Lecture 6: Gauss Elimination

The objective is to solve for \( n \) unknowns given \( n \) linear equations. In terms of matrices, find the \( nx1 \) column vector \( x \) so that \( Ax = d \) where \( A \) is \( nxn \) and \( d \) is \( nx1 \). We will do this using the method called Gauss elimination, which requires about \( n^2 \) units of memory and about \( n^3/3 \) arithmetic operations. So, if \( n \) is not too large, this method is very useful; otherwise, one must go to other more specialized methods. We begin by considering several special matrices \( A \).

\( A \) is a Diagonal Matrix.

Let \( n = 3 \).

\[
\begin{bmatrix}
  a_{11} & 0 & 0 \\
  a_{22} & a_{23} & 0 \\
  a_{33} & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
  d_1 \\
  d_2 \\
  d_3 \\
\end{bmatrix}
\]

Or,

\( a_{11}x_1 = d_1 \)
\( a_{22}x_2 = d_2 \)
\( a_{33}x_3 = d_3 \)

So, if each \( a_{ii} \neq 0 \), then \( x_i = d_i / a_{ii} \).

\( A \) is an Upper Triangular Matrix.

Let \( n = 3 \).

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{22} & a_{23} & 0 \\
  a_{33} & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
  d_1 \\
  d_2 \\
  d_3 \\
\end{bmatrix}
\]

Or,

\( a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = d_1 \)
\( a_{22}x_2 + a_{23}x_3 = d_2 \)
\( a_{33}x_3 = d_3 \)
So, if \(a_{33} \neq 0\), then
\[x_3 = \frac{d_3}{a_{33}}.\]

Solve the middle equation, if \(a_{22} \neq 0\),
\[a_{22}x_2 = d_2 - a_{23}x_3,\]
\[x_2 = \frac{(d_2 - a_{23}x_3)}{a_{22}}.\]

Solve the first equation, if \(a_{11} \neq 0\),
\[a_{11}x_1 = d_1 - a_{12}x_2 - a_{13}x_3,\]
\[x_1 = \frac{(d_1 - a_{12}x_2 - a_{13}x_3)}{a_{11}}.\]

**Proposition 3.** Let \(A\) be a \(n \times n\) upper or lower triangular matrix,
If each diagonal component is non-zero, then \(Ax = d\) has a solution.

Consider the following system with three unknowns and three equations, whose coefficient matrix is not upper or lower triangular.
\[
\begin{align*}
2x_1 + 6x_2 + 8x_3 &= 16 \\
4x_1 + 15x_2 + 19x_3 &= 38 \\
2x_1 + 0x_2 + 3x_3 &= 6
\end{align*}
\]

Or, \(Ax = d\)
\[
\begin{bmatrix}
2 & 6 & 8 \\
4 & 15 & 19 \\
2 & 0 & 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
16 \\
38 \\
6
\end{bmatrix}
\]

Or, as a \(3 \times 4\) augmented matrix
\[
[A \ d] =
\begin{bmatrix}
2 & 6 & 8 & 16 \\
4 & 15 & 19 & 38 \\
2 & 0 & 3 & 6
\end{bmatrix}
\]

We will try to transform this into an equivalent problem that has an upper triangular coefficient matrix, which may be solved by a backward substitution. Often this can be done by a combination of row (or equation) operations:

(i). interchange rows (or equations)

(ii). multiply rows (or equations) by a suitable constant and

(iii). add or subtract rows (or equations).
Row Operations on the Augmented Matrix:

We will try to transform the augmented matrix to an upper triangular matrix by using row operations to make the left most column 0 below row 1.

row 2 – 2(row 1) or multiply \([A \ d]\) by \(E_{21}(-2)\) and then

\[
E_{31}(-1)E_{21}(-2)[A \ d] = \begin{bmatrix}
2 & 6 & 8 & 16 \\
0 & 3 & 3 & 6 \\
0 & -6 & -5 & -10
\end{bmatrix}.
\]

Next, we must use row 2 to transform the –6 to 0.

row 3 + 2(row 2) or multiply \(E_{31}(-1)\) \(E_{21}(-2)\) \([A \ d]\) by \(E_{32}(2)\) to get

\[
E_{32}(2)E_{31}(-1)E_{21}(-2)[A \ d] = \begin{bmatrix}
2 & 6 & 8 & 16 \\
0 & 3 & 3 & 6 \\
0 & 0 & 1 & 2
\end{bmatrix}.
\]

Backward Substitution on the Transformed Augmented Matrix:

This is equivalent to the upper triangular system, which can be solved by backward substitution.

\[
\begin{bmatrix}
2 & 6 & 8 & x_1 \\
0 & 3 & 3 & x_2 \\
0 & 0 & 1 & x_3
\end{bmatrix} = \begin{bmatrix}
16 \\
6 \\
2
\end{bmatrix}.
\]

*Solve for* \(x_1\), \(1x_3 = 2\) *so that* \(x_3 = 2\).

*Solve for* \(x_2\), \(3x_2 + 3x_3 = 6\) *so that* \(x_2 = 0\).

*Solve for* \(x_1\), \(2x_1 + 6x_2 + 8x_3 = 16\) *so that* \(x_1 = 0\).

**Gauss Elimination Method.** Consider \(Ax = d\).

Step 1: Use row operations on the augmented matrix to transform \(Ax = d\) to an upper triangular problem.

Step 2: Use backward substitution to solve the upper triangular problem.

See the Matlab demo gauss_el.m for additional examples. There are a number of very good computer implementations of the Gauss elimination method. In Matlab the simple command \(A\backslash d\) can be used to compute the solution for most problems \(Ax = d\).
Application to the Two-loop Circuit.

In lecture 4 we formulated the algebraic problem for the two-loop circuit problem:

\[
\begin{bmatrix}
1 & -1 & 1 \\
R_1 & 0 & -R_3 \\
0 & -R_2 & -R_3
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
E_1 \\
E_2
\end{bmatrix}.
\]

Let \( R_1 = 1, R_2 = 2, R_3 = 3, E_1 = 10 \) and \( E_2 = 20 \).

The augmented matrix is

\[
[A \ d] =
\begin{bmatrix}
1 & -1 & 1 & 0 \\
0 & 1 & -3 & 10 \\
0 & -2 & -3 & 20
\end{bmatrix}.
\]

Step 1: Use row operations.

row 2 – row 1 or multiply by \( E_{12}(-1) \) to get

\[
E_{21}(-1)[A \ d] =
\begin{bmatrix}
1 & -1 & 1 & 0 \\
0 & 1 & -4 & 10 \\
0 & -2 & -3 & 20
\end{bmatrix}.
\]

row 3 + 2(row2) or multiply by \( E_{32}(2) \) to get

\[
E_{32}(2)E_{21}(-1)[A \ d] =
\begin{bmatrix}
1 & -1 & 1 & 0 \\
0 & 1 & -4 & 10 \\
0 & 0 & -11 & 40
\end{bmatrix}.
\]

Step 2: Use backward substitution.

\[
\begin{bmatrix}
1 & -1 & 1 \\
0 & 1 & -4 \\
0 & 0 & -11
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
10 \\
40
\end{bmatrix}.
\]

Solve for \( i_3, -11x_3 = 20 \) so that \( i_3 = -40/11 \).

Solve for \( i_2, li_2 - 4i_3 = 10 \) so that \( i_2 = -50/11 \).

Solve for \( i_1, li_1 - li_2 + i_3 = 0 \) so that \( i_1 = -10/11 \).

Homework.

1. Use Gauss elimination to solve \( Ax = d \) when

\[
A =
\begin{bmatrix}
1 & 3 & 5 \\
0 & 2 & 1 \\
2 & 2 & 0
\end{bmatrix}
\text{ and } d =
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}.
\]

2. Prove Proposition 3 when \( A \) is lower triangular.