Lecture 5: Matrix-Matrix Products

In this lecture we will extend the matrix-vector product from nxn matrix times a nx1 column vector to a matrix-matrix product of a mxn matrix times a nxp matrix. Matrix-matrix product may not satisfy the commutative property AB = BA, and BA may not be defined even if AB is defined.

Definition. Let A be a mxn matrix and B be a nxp matrix.

$AB = [\ \Sigma_k a_{ik} b_{kj}\ ]$; AB is a nxp matrix whose ij components are

(row i of A) times (col j of B)

Examples.

1. Both A and B are 2x2, and AB is not equal to BA.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \; and \; B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1\cdot 0 + 2\cdot 1 & 1\cdot 1 + 2\cdot 0 \\ 3\cdot 0 + 4\cdot 1 & 3\cdot 1 + 4\cdot 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

2. A is 2x3 and B is 3x3, AB is 2x3 and BA is not defined.

$$A = \begin{bmatrix} 2 & 4 & 7 \\ 8 & 9 & 10 \end{bmatrix} \; and \; B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2\cdot 2 + 4\cdot -1 + 7\cdot 0 & 2\cdot -1 + 4\cdot 2 + 7\cdot -1 & 2\cdot 0 + 4\cdot -1 + 7\cdot -1 \\ 8\cdot 2 + 9\cdot -1 + 10\cdot 0 & 8\cdot -1 + 9\cdot 2 + 10\cdot -1 & 8\cdot 0 + 9\cdot -1 + 10\cdot -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -11 \\ 7 & 0 & -19 \end{bmatrix}$$

In Example two note that AB is also a matrix whose j column is A times column j of B. This is true in general, and some other properties of matrix-matrix products are listed in Proposition 2.

Proposition 2.

1. Let A be mxn, B be nxp and C be pxq. (AB)C = A(BC).
2. Let A and B be mxn, C be nxp. (A + B)C = AC + BC.
3. Let A be mxn, B be nxp and c be a single number. (cA)B = c(AB) = A(cB).
4. Let A be mxn, B be nxp. Column j of AB is A times column j of B, that is,

$$AB = A\begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}.$$

Proof of 1. Let A = $[a_{ij}]$, B = $[b_{jk}]$ and C = $[c_{kl}]$.

$$\begin{aligned}(AB)C &= [\sum_l (\sum_j a_{ij}b_{jk})c_{kl}] \\ &= [\sum_l (\sum_j (a_{ij}b_{jk})c_{kl})] \\ &= [\sum_l (\sum_j a_{ij}(b_{jk}c_{kl}))] \\ &= [\sum_j (\sum_l a_{ij}(b_{jk}c_{kl}))] \\ &= [\sum_j a_{ij}(\sum_l (b_{jk}c_{kl}))] \\ &= A(BC).\end{aligned}$$

Proof of 2. Let A = $[a_{ij}]$, B = $[b_{jk}]$ and C = $[c_{jk}]$.

$$\begin{aligned}A(B+C) &= [\sum_k a_{ik}(b_{jk}+c_{jk})] \\ &= [\sum_k (a_{ik}b_{jk}+a_{ik}c_{jk})] \\ &= [\sum_k a_{ik}b_{jk} + \sum_k a_{ik}c_{jk}] \\ &= [\sum_k a_{ik}b_{jk}]+[\sum_k a_{ik}c_{jk}] \\ &= AB+AC.\end{aligned}$$

Proof of 4. Let A = [a_{ij}]. B = [b_{jk}] and column k of B equals b_k.

$$AB = [\sum_j a_{ij} b_{jk}]$$
$$= \begin{bmatrix} \sum_j a_{ij} b_{j1} & \sum_j a_{ij} b_{j2} & \cdots & \sum_j a_{ij} b_{jp} \end{bmatrix}$$
$$= \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}.$$

More examples can be found in the Matlab demo matmat.m. Some useful matrix-matrix products are as follows:

1. AI = IA = A where A is any nxn matrix and I is the nxn identity matrix

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} \; where$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

2. Let A be a diagonal nxn matrix with non-zero diagonal components.

$$A = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} \; and \; B = \begin{bmatrix} 1/a_{11} & & & \\ & 1/a_{22} & & \\ & & \ddots & \\ & & & 1/a_{nn} \end{bmatrix}$$

$Then \; AB = I.$

3. Let A be an elementary matrix and B be the elementary matrix formed from A by changing the sign of the non-zero off diagonal component.

$$A = E_{32}(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{bmatrix} \; and \; B = E_{32}(-a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -a & 1 \end{bmatrix}$$

$Then \; AB = BA = I.$

4. Elementary matrices can be used to add and subtract multiples of various rows in a second matrix. For example, $E_{21}(-1)$ subtracts row 1 of A from row 2 of A:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } E_{21}(-1) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{21}(-1)A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$E_{31}(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E_{31}(-1)(E_{21}(-1)A) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$E_{31}(-1)E_{21}(-1) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$(E_{31}(-1)E_{21}(-1))A = E_{31}(-1)(E_{21}(-1)A).$$

Homework.

1. For the following matrices verify, if the appropriate matrix-matrix products are defined, the properties of Proposition 2:

$$A = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 & 2 \\ 5 & 6 & 8 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 7 \\ -1 & 2 & 1 \\ 7 & 2 & 3 \end{bmatrix}.$$

2. Use Matlab to do problem 1….see the Matlab demo matmat.m
3. Prove a variation of property 2, A(B + C) = AB + AC, in Proposition 2.
4. Prove property 3 in Proposition 2.