Lecture 4: Applications and Matrix-Vector Products

In the next six lectures the solution of n linear algebraic equations with n unknowns will be studied. In many applications n may be as large as 100,000. Here we will focus on applications where n is relatively small, but n can be become very large for more realistic models of certain applications. Some models will include steady state circuits, ridged structures and mixing problems.

For ease of notation let n = 3. Then n linear equations can be either listed by 3 equations or by the use of 3x3 coefficient matrices.

Three Linear Equations:

$$\begin{aligned} &\textit{Let } x_j \textit{ be unknown, } a_{ij} \textit{ and } d_i \textit{ be given.} \\ &a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = d_1 \\ &a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = d_2 \\ &a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = d_3. \end{aligned}$$

Matrix Description of Three Linear Equations:

$$\textit{Let } A \textit{ be a } 3x3 \textit{ matrix and } d, x \textit{ be } 3x1 \textit{ row vectors.}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \textit{ and } d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

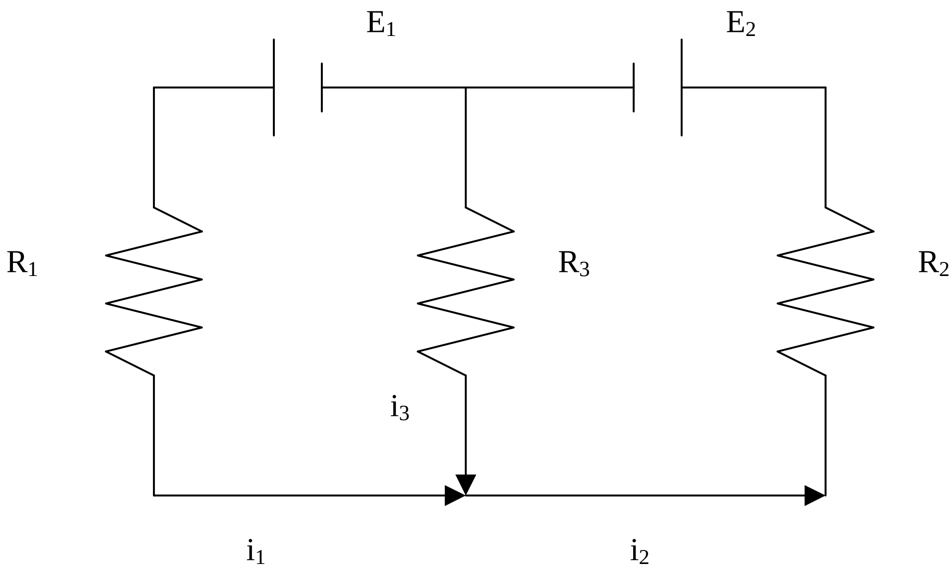
$$Ax = d$$

Generally, A can be mxn. In the lectures 3-9 m = n, in lectures 10-12 on least squares m > n and in lectures on eigenvalues 13-15 m < n. The following matrix notation is often used.

Matrix Notation. Let A = [a_{ij}] be mxn matrix with m rows and n columns, i = 1,…,m is row number and j = 1,…,n is a column number. For example,

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 7 & 8 & 6 \end{bmatrix}, m = 2, n = 3 \textit{ and } a_{23} = 6.$$

Application to Steady State Two-loop Circuit. Consider the following circuit with three resistors and two batteries. We wish to know the currents in the three resistors given the three resistances and two voltages of the batteries. By using Ohm's and Kirchhoff's laws we obtain and three equations for three currents.

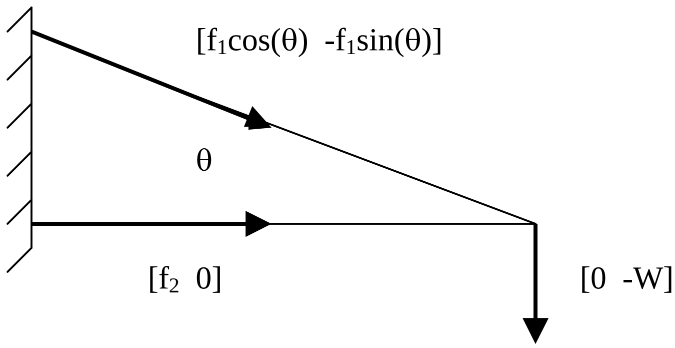


$$i_1 + i_3 = i_2$$
$$E_1 = R_1 i_1 - R_3 i_3$$
$$-E_2 = R_3 i_3 + R_2 i_2$$

$$\begin{bmatrix} 1 & -1 & 1 \\ R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ E_1 \\ E_2 \end{bmatrix}.$$

So, the coefficient matrix, A, is 3x3, and the unknown vector is 3x1 vector of currents. If there were more loops, then there would be more unknown currents. The resulting coefficient matrix would be nxn where n > 3, and this will be illustrated in lecture 9.

Application to Rigid Steady State One-Node Structure. Consider the following structure with two beams and one node with a weight at the node.



The balance of forces at the joining node of the two beams is

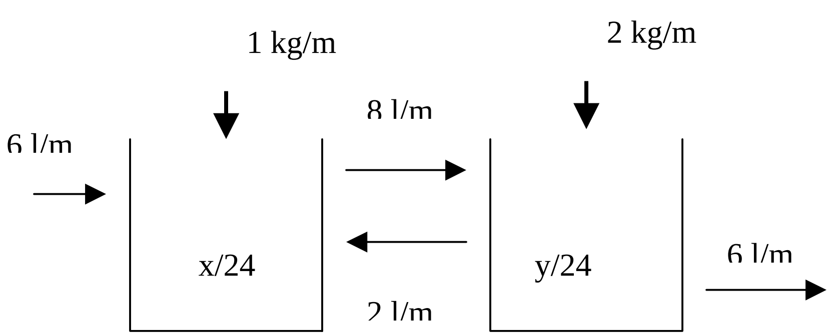
$$[0\ \ -W] = [f_1\cos(\theta)\ \ -f_1\sin(\theta)] + [f_2\ \ 0].$$

By setting the components equal we have

$$\cos(\theta)\, f_1 + 1 f_2 = 0$$
$$-\sin(\theta)\, f_1 + 0\, f_2 = -W$$
$$Or,$$
$$\begin{bmatrix} \cos(\theta) & 1 \\ -\sin(\theta) & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -W \end{bmatrix}.$$

So, the coefficient matrix is 2x2, and the unknown vector is 2x1 for the magnitudes of the forces in the two beams. More beams and nodes will generate more unknown forces, for example, see the bridge structure problem in lecture 9.

Application to Steady State Two-tank Mixing. Consider two well-stirred 24 liter mixing tanks with amounts of a chemical equal to x(t) and y(t).



Assume the left tank remains full, has concentration x(t)/24, has in flow at rate of 6 liter/min with zero concentration, has in flow from the right tank with flow at a rate of 2 liter/min with concentration y(t)/24, and has the chemical directly dumped in at a rate of 1 kg/min. Assume the right tank also remains full, has in flow from the left tank at a flow rate of 8 liter/min with concentration x(t)/24, and has the chemical directly dumped in at a rate of 2 kg/min. By applying rate of change of either tank is the rate in minus the rate out we get

x' = (0 + 1 + 2 y/24) – (8 x/24) and

y' = (2 + 8 x/24) –(2 y/24 + 6 y/24).

The steady state solution requires that both x' and y' equal zero. This gives the algebraic equations

$$0 = 1 - (1/3)x + (1/12)y$$
$$0 = 2 + (1/3)x - (1/3)y$$
$$Or,$$
$$\begin{bmatrix} -1/3 & 1/12 \\ 1/3 & -1/3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

Special Matrices.

Identity matrix for n = 3 $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Zero matrix for n = 3 $Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\textit{often just } 0 \textit{ is used})$

Diagonal matrix for n = 3 $D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

Elementary matrix for n = 3 $E_{21}(a) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Basic Matrix Operations. Let A, B and C be nxn matries, a,x be nx1 column vectors and c be a single number.

1. Scalar product cA = [c a_{ij}]; cA in a matrix whose ij components are ca_{ij}.
2. Addition of matrices A + B = [a_{ij} + b_{ij}]; A + B is a matrix whose ij components are a_{ij} + b_{ij}.
3. Row times a column $a^Tx = \Sigma a_j x_j$; a^Tx is a number equal to the sum of the products a_jx_j.
4. Martrix-vector product Ax = [$\Sigma a_{ij}x_j$]; Ax is a column vector whose i components are row j of A times x. This is the traditional dot product definition.

Examples. There are additional examples in the Matlab demo matvec.m

$$Let\ A = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\ and\ c = -2,$$

$$cA = \begin{bmatrix} -2 & -8 & -14 \\ 0 & 2 & -4 \\ -8 & -2 & -6 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 1x_1 + 4x_2 + 7x_3 \\ 0x_1 - 1x_2 + 2x_3 \\ 4x_1 + 1x_2 + 3x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} x_1 + \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix} x_3.$$

The above calculation of Ax is valid for all matrix-vector products. For n = 3

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{23}x_3 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} x_2 + \begin{bmatrix} a_{13} \\ a_{23} \\ a_{23} \end{bmatrix} x_3$$

$$= (col1\ of\ \ A)x_1 + (col2\ of\ \ A)x_2 + (col3\ of\ \ A)x_3$$

Other general properties are stated in Proposition 1.

Proposition 1. Let A, B, C be nxn matrics, x,y be nx1 column vectors and c a single number.

1. A + (B + C) = (A + B) + C
2. A + B = B + A
3. A + Z = A where Z is the zero matrix
4. $Ax = \Sigma_k$ (col k of $A)x_k$
5. (cA)x = A(cx)
6. A(x + y) = Ax + Ay.

Proof of 1. $A + (B + C) = [a_{ij}] + [(b_{ij} + c_{ij})]$

$= [a_{ij} + (b_{ij} + c_{ij})]$

$= [(a_{ij} + b_{ij}) + c_{ij}]$, by real products are distributative

$= [(a_{ij} + b_{ij})] + [c_{ij}]$

$= (A + B) + C$

Proof of 5. $(cA)x = \Sigma\, (c\, a_{ij})\, x_j$

$= \Sigma\, (a_{ij}\, c)\, x_j$, by real products commute

$= \Sigma\, a_{ij}\, (c\, x_j)$, by real products are associative

$= A(cx)$

Homework.

1. Verify the properties in Proposition 1 for the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 5 \\ 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 7 & 12 & 3 \\ -4 & 1 & 5 \\ 10 & 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 9 & 8 \\ 3 & 1 & 5 \\ 0 & -6 & 1 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, y = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \textit{and } c = 3.$$

2. Prove parts 2, 3, and 6 of Proposition 1.
3. Show for the 3x3 elementary matrix $E_{21}(a)\, E_{21}(-a) = I$.