Lecture 3: Surfaces and Optimization

A simple surface is a set of points of the form (x, y, f(x,y)) where z = f(x,y) is the height above or below the xy-plane. In order to graph surfaces, a finite collection of points in the xy-plane must be specified, and at each point f(x,y) must be evaluated. Then the points (x,y,f(x,y)) may be plotted and connected by straight lines. If there are enough points, then an accurate depiction of the surface will be given. We will illustrate this in an application to finding the shape of a 3d box, which will have specified volume and is to be a minimum cost. Here f(x,y) will be the cost as a function of the edges of the bottom of the box.

The points in the xy-plane can be specified in two ways: either list the points sequentially or list them in a 2d grid or mxn matrix where there are n grid points in the x direction and m grid points in the y direction. For each point in the xy-plane there must be a x coordinate, a y coordinate and a z coordinate equal to f(x,y). All this data can be stored in three mxn matrices, say, X for all the x coordinates, Y for all the y coordinates and F for all the z coordinates.

Example. Consider graphing the paraboloid $z = f(x,y) = 100 - x^2 - 4y^2$. Suppose we want 3 grid points in the x direction, say x = 0, 4, and 8, and 4 grid points in the y direction, say y = 0, 2, 4, 6. Then m = 4, n = 3 and we will need three 4x3 matrices to record all the data the 12 grid points.

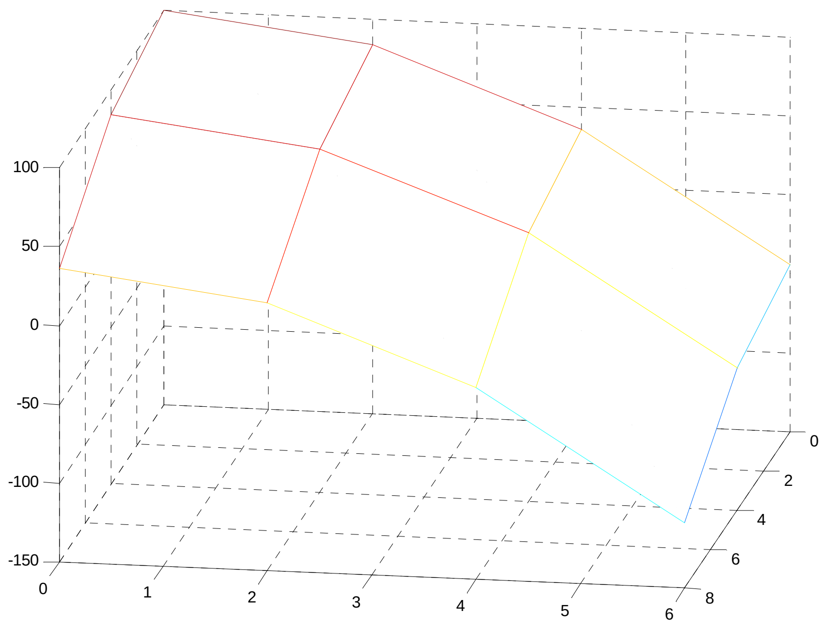
$$X = \begin{bmatrix} 0 & 4 & 8 \\ 0 & 4 & 8 \\ 0 & 4 & 8 \\ 0 & 4 & 8 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{bmatrix} \text{ and } F = \begin{bmatrix} 100 & 84 & 36 \\ 84 & 68 & 20 \\ 36 & 20 & -28 \\ -44 & -60 & -108 \end{bmatrix}$$

Now all 12 points should be plotted in space.

All this can be done by the following Matlab commands:

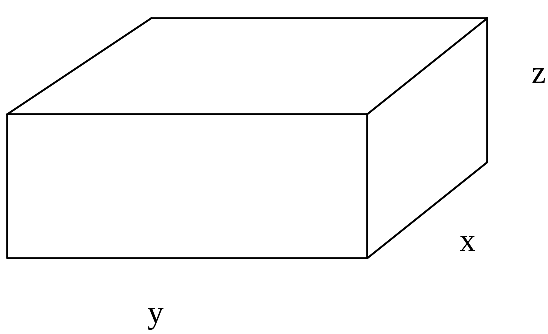
```
[X Y] = meshgrid(0:4:8,0:2:6);
F = 100 – x.^2 – 4*y.^2;
mesh(X, Y, F)
```

The first line generates the two matrices X and Y. The second line generates the third array that contains all the values of $100 - x^2 - 4y^2$; this is done by using the array operation .^2 The third line plots all 12 points and connects the adjacent points by straight lines to form a depiction of the surface by 6 rectangles.



Adjusting the input variable in meshgrid can generate more points, such as
[X   Y] = meshgrid(0:1:8,0:1:6) will generate 7x9 matrices.

Application to a Box With Minimum Cost. Consider a 3d box, which must have a volume equal to 1000 ft^3. The bottom will cost 3 $\$/ft^2$, the sides will cost $1\$/ft^2$ and there is to be no top. Let x and y be the length of the edges of the bottom and z be the height of the box.



volume = 1000 = xyz

z = 1000/(xy)

cost of bottom = 3 xy

cost of four sides = 1 (yz + yz + xz + xz)

total cost = cost of bottom + cost of sides

$C(x,y) = 3\,xy + 2yz + 2xz$

$= 3\,xy + 2y\,1000/(xy) + 2x\,1000/(xy)$

$= 3\,xy + 2000/x + 2000/y$

$= 3\,xy + 2000\,x^{-1} + 2000\,y^{-1}$

In order to find the x and y so that C(x,y) is a minimum, we can either graph C(x,y) and visually search for the minimum or we can use partial derivatives to locate where the tangent planes to C(x,y) are level. The Matlab demo box3d.m contains the graphical approach. The partial derivative approach is as follows:

Compute the partial derivatives of C(x,y), set both equal to zero and solve for x,y.

$C_x = (3\,xy + 2000\,x^{-1} + 2000\,y^{-1})_x$

$= 3\,y + 2000\,(-1)\,x^{-2} + 0$

$= 0$

$C_y = (3\,xy + 2000\,x^{-1} + 2000\,y^{-1})_y$

$= 3\,x + 0 + 2000\,(-1)\,y^{-2}$

$= 0$

The first equation gives $3y = 2000/x^2$. Put this into the second equation to get

$3x = 2000/(2000/3x^2)^2 = (9/2000)\,x^4$.

Therefore, both $x = y = (2000/3)^{1/3} = 8.7358$.

The $z = 1000/((2000/3)^{1/3})^2 = 13.1037$ and

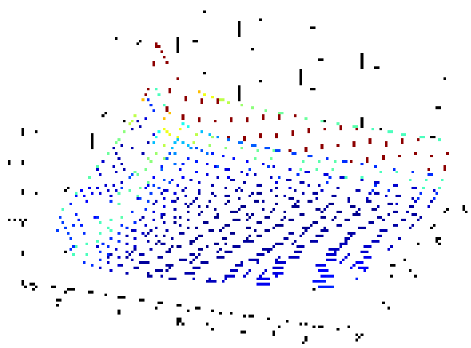
the minimum cost will be C(8.7358, 8.7358) = 689.8285.

The graphical solution in box3d.m clearly shows the above calculations generate the only level tangent plane and the tangent point on the surface is in fact the minimum cost. In box3d.m the Matlab command mesh was used to generate the surface.

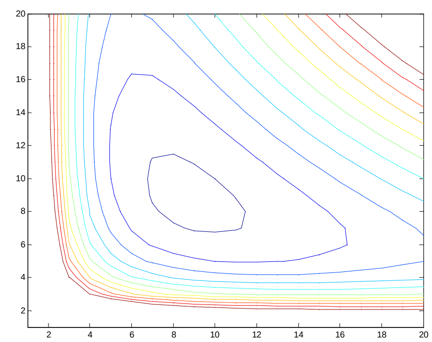
The following Matlab commands are in box3d.m:

```
[x y] = meshgrid(1:1:20,1:1:20);
c = 3*x.*y + 2000./y + 2000./x;
mesh(x,y,c)
```

The first line generates two 20x20 matrices that contain the x and y coordinates. The second line generates the 20x20 that contains the computed costs for all the points. Note how the third line uses the array operations .* and ./ to do all 400 calculations. The Matlab commands mesh and contour will generate surface/contour in 3d and contours in 2d. The curves generated by contour are the points where c = constants = 600:50:1200.



mesh(x,y,c)



contour(x,y,c,[600:50:1200])

Homework.

1. Using Matlab's command [X , Y] = meshgrid(0:1:8,0:1:6) graph the paraboloid $z = 100 - x^2 - 4y^2$. Print the arrays X, Y and F
2. If the sides of the box cost 2 $\$/ft^2$ and not 1 $\$/ft^2$, solve the box problem using partial derivatives.
3. If the sides of the box cost 2 $\$/ft^2$ and not 1 $\$/ft^2$, solve the box problem using a modified version of box3d.m.
4. Repeat problems 2 and 3 if the box must have at top which costs 1 $\$/ft^2$.