Lecture 15: Three-tank Mixing and Lead Poisoning

Eigenvalues and eigenvectors will be used to find the solution of a system for n unknown functions that satisfy n differential equations. The unknown functions will be written as an nx1 column vector $x = [x_1(t)\ \ x_2(t)\ \ \ldots\ \ x_n(t)]^T$, and the differential equations will be represented by an nxn matrix A

$$x' = Ax + f$$

where f and x(0) are given constant nx1 column vectors. Here we will assume that A has distinct real and non-zero eigenvalues. Less restrictive assumptions on A require more advanced course work. However, we will be able to give very interesting applications to time dependent mixing, lead poisoning and heat transfer (see homework problem 5).

The steady state solution requires x' = 0 so that 0 = Ax + f. We have assumed f is independent of time, and A has no zero eigenvalues so that A has an inverse matrix. Often the steady state solution is referred to as a particular solution, which we will denote by x_p

$$x_p = -A^{-1}f.$$

In order to form the homogeneous or time dependent part of the solution, we use the eigenvectors, u_j a non-zero nx1 column vector, and eigenvalues, r_j, so that

$$Au_j = r_j u_j.$$

The definition of eigenvalues, r, and eigenvectors, u, was motivated by the fact that ue^{rt} must satisfy x' = Ax. More generally any linear combination of the column vectors will also be a solution of x' = Ax. Other properties are summarized in Proposition 8. We will use the following notation:

$$\textit{Let } Au_j = r_j u_j \textit{ where } u_j \textit{ is non} - \textit{zero nx1 column vector.}$$

$$U(t) = \begin{bmatrix} u_1 e^{r_1 t} & u_2 e^{r_2 t} & \cdots & u_n e^{r_n t} \end{bmatrix} \textit{ is nxn matrix,}$$

$$U(t)c = \begin{bmatrix} u_1 e^{r_1 t} & u_2 e^{r_2 t} & \cdots & u_n e^{r_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$= u_1 e^{r_1 t} c_1 + u_2 e^{r_2 t} c_2 + \cdots + u_n e^{r_n t} c_n.$$

Proposition 8. Consider x' = Ax + f where f is a constant nx1 vector. Assume A has n distinct non-zero real eigenvectors. Define $x = U(t)c + x_p$ where c is a constant nx1 vector and $x_p = -Af$. Then the following are true:

1. $x = U(t)c + x_p$ satisfies x' = Ax + f.
2. A has an inverse matrix.
3. U(t) has an inverse matrix.
4. If $c = U(0)^{-1}(x(0) - x_p)$, then $x = U(t)c + x_p$ also satisfies x(0) equals a given nx1 column vector.
5. Moreover, if all the eigenvalues are negative, then as t goes to infinity x(t) converges to x_p.

Proof of 1.

$$
\begin{aligned}
Leftside = x' &= (U(t)c + x_p)' \\
&= (U(t)c)' + (x_p)' \\
&= (u_1 e^{r_1 t} c_1 + u_2 e^{r_2 t} c_2 + \cdots + u_n e^{r_n t} c_n)' + 0 \\
&= (u_1 e^{r_1 t} c_1)' + (u_2 e^{r_2 t} c_2)' + \cdots + (u_n e^{r_n t} c_n)' \\
&= u_1 c_1 (e^{r_1 t})' + u_2 c_2 (e^{r_2 t})' + \cdots + u_n c_n (e^{r_n t})' \\
&= u_1 c_1 e^{r_1 t} r_1 + u_2 c_2 e^{r_2 t} r_2 + \cdots + u_n c_n e^{r_n t} r_n .
\end{aligned}
$$

$$
\begin{aligned}
Rightside &= Ax + f \\
&= A(U(t)c + x_p) + f \\
&= A(U(t)c) + Ax_p + f \\
&= A(u_1 e^{r_1 t} c_1 + u_2 e^{r_2 \cdots t} c_2 + \cdots + u_n e^{r_n t} c_n) + A(-A^{-1} f) + f \\
&= (Au_1) e^{r_1 t} c_1 + (Au_2) e^{r_2 t} c_2 + \cdots + (Au_n) e^{r_n t} c_n - f + f \\
&= (r_1 u_1) e^{r_1 t} c_1 + (r_2 u_2) e^{r_2 t} c_2 + \cdots + (r_n u_n) e^{r_n t} c_n .
\end{aligned}
$$

Proof of 3. This will have to wait for a more advanced course.

Application to Three-tank Mixing. Consider the problem that was formulated in lecture 6 where n = 3 and x_1 = x the amount in the left tank, x_2 = y the amount in the center tank and x_3 = z the amount in the right tank.

$$x' = Ax + f$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} -10/24 & 4/24 & 0 \\ 10/24 & -12/24 & 2 \\ 0 & 8/24 & -8/24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$x_p = -A^{-1} f = \begin{bmatrix} 7.2 \\ 12 \\ 12 \end{bmatrix} \; and \;\; x(0) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

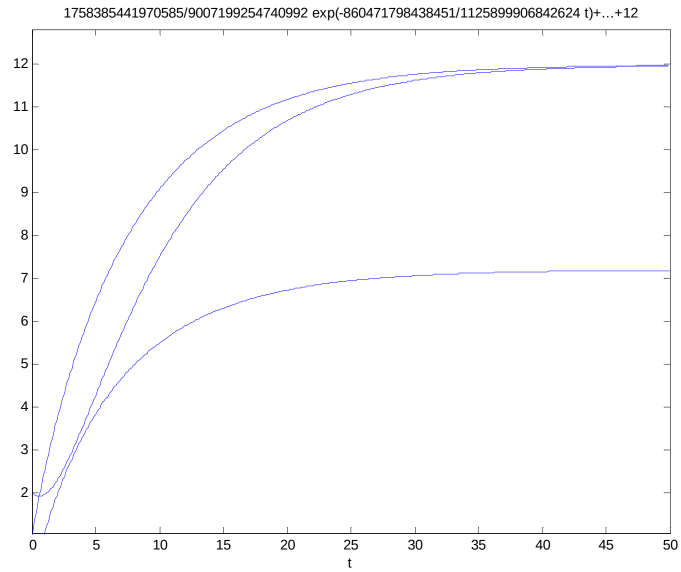
The Matlab command [u d] = eig(A) will produce two 3x3 matrices, and column j of u will be the eigenvector associated with the j diagonal component of d.

$$[u \quad d] = \begin{bmatrix} -.3546 & -.2201 & .2898 & -.7634 & & \\ .7396 & -.0758 & .5047 & & -.3593 & \\ -.5721 & .9725 & .8132 & & & -.1264 \end{bmatrix}.$$

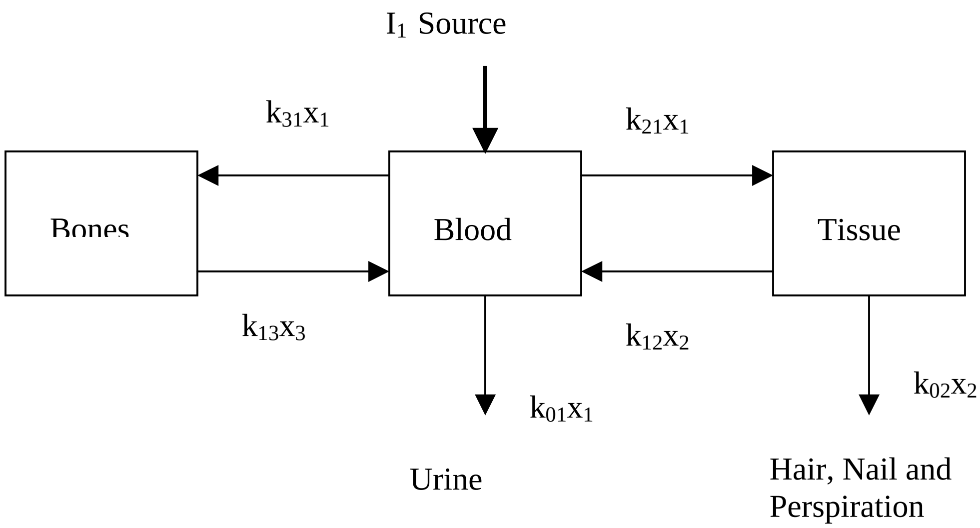
Here U(0) = u and c = $U(0)^{-1}(x(0) - x_p)$

$$c = U(0)^{-1}\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 7.2 \\ 12.0 \\ 12.0 \end{bmatrix} \right) = \begin{bmatrix} -00.3412 \\ 06.5060 \\ -20.3184 \end{bmatrix}.$$

The reader should consult the Matlab demo tank3_time.m for more details, and the creation of the following plot of the solutions.



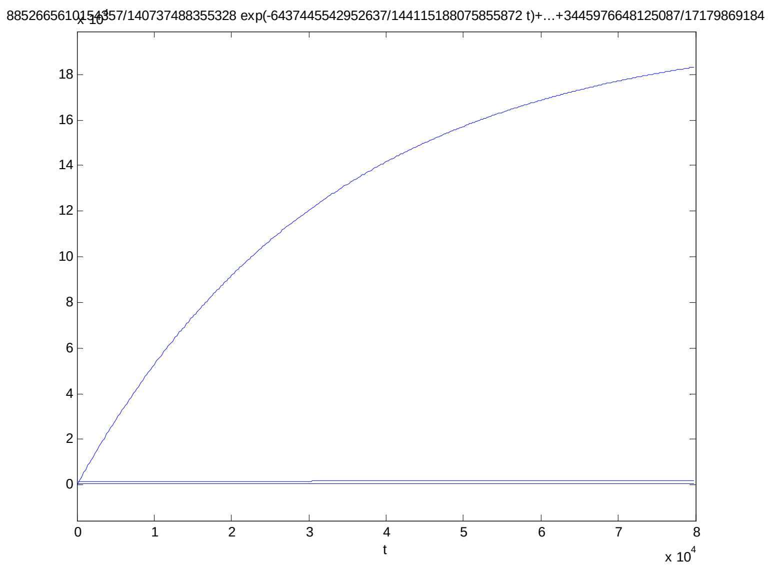
Application to Lead Poisoning. See chapter 7 in the ODE text by Borrelli and Coleman. A simplified model of lead poisoning in the human body is somewhat similar to the three-tank mixing problem. We will assume the lead is located in the blood ($x_1(t)$), tissue ($x_2(t)$) or bones ($x_3(t)$). The following graphic depicts the flow between these three regions.



The flow rates k_{ij} are assumed to be constants from group j to group i. These must be determined from measured data, and possibly by using a least squares curve fit to multilinear functions. By using the rate of change of group i is equal to the rate in minus the rate out, we can formulate the following system of differential equations.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} -(k_{01}+k_{21}+k_{31}) & k_{12} & k_{13} \\ k_{21} & -(k_{01}+k_{12}) & 0 \\ k_{31} & 0 & -k_{31} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} I_1 \\ 0 \\ 0 \end{bmatrix}.$$

Specific values of the constants are given in the Matlab demo lead3_time.m. The constant k_{01}can be controlled by medications, and the constant I_1 can be controlled by altering the environment. For the values use in lead3_time.m all the eigenvalues were negative, and therefore, the solution x(t) must converge to the steady state solution x_p = [1800 699 200583]. The following graph was generated for x(0) = $[0\ \ 0\ \ 0]^T$ where the vertical axis is scaled by 10000. The two curves for $x_1(t)$ and $x_2(t)$ are almost overlapping because of this scaling.



Homework.

1. Use tank3_time.m to experiment with different initial concentrations. Observe that the solutions always converge to the same steady state solution.
2. Use lead3_time.m to experiment with different I_1. How does this affect the steady state solution? Will it remain the limit of x(t)?
3. Use lead3_time.m to experiment with different medications by adjusting k_{01} away from .0211.
4. Prove parts 4 and 5 in Proposition 8.
5. Consider the heat diffusion in a rod in example 3 of lecture 14. The constant c in the matrix A has the form $(K/(\rho c_p))/(h^2)$ where K = thermal conductivity = .001,

   ρ = density = 1, c_p = specific heat =1, h = segment size = ¼ so that c = .016.

   Modify tank3_time.m so that this problem is solved with $f = [1\ \ 1\ \ 1]^T$ and $x(0) = [\ 0\ \ 0\ \ 0\ ]^T$. You may wish to compare your code with heat3_time.m. Experiment with different initial temperatures and observe that x(t) always converges to the steady state solution.