Lecture 14: Computation of Eigenvalues and Eigenvectors

An eigenvector of a nxn A matrix is a nx1 non-zero column vector u such that Au = ru for some number r, called the eigenvalue of u. Three important properties are:

(i). ku is also and eigenvector for k a non-zero number.

(ii). r is a solution of the polynomial equation det(A – rI) = 0.

(iii). u is then chosen so that (A – rI)u = 0 , the zero vector.

For example, in the two-tank problem of the previous lecture we had

$$A = \begin{bmatrix} -1/3 & 1/12 \\ 1/3 & -1/3 \end{bmatrix}.$$

We used (ii) to find r = -1/2 and r = -1/6. By using r = -1/6 in (iii) we can find the eigenvector associated with this eigenvalue

$$(A - rI)u = 0$$

$$\begin{bmatrix} -1/3-(-1/6) & 1/12 \\ 1/3 & -1/3-(-1/6) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\textit{Or},$$

$$(-1/6)u_1 + (1/12)u_2 = 0$$

$$(1/3)u_1 + (-1/6)u_2 = 0.$$

$$\textit{Choose } u_2 = 1 \textit{ so that } u_1 = 1/2.$$

We will present a number of other examples. The first example shows that the eigenvalues and eigvectors may have complex numbers. The second example is an application to spring-mass with damping, and we show the eigenvalues are in fact the solution of the characteristic equation. The third example comes from an application to heat diffusion in a long thin rod. Other examples are given in the Matlab demo eigenval.m. Here the Matlab command [u d] = eig(A) generates two nxn matrices u and d. The columns of u are the eigenvectors of A, and d is a diagonal matrix with the corresponding eigenvalues.

Example 1. This matrix will give complex eigenvalues.

$$A = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}$$

First, find the eigenvalues:

$$\det(A - rI) = 0$$

$$\det\left(\begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} - r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} -1-r & 2 \\ -1 & -3-r \end{bmatrix}\right) = 0$$

$$(-1-r)\left(-3-r\right) - 2(-1) = 0$$

$$r^2 + 4r + 5 = 0$$

$$r = \frac{-4 \pm \sqrt{4^2 - 4\cdot 1\cdot 5}}{2} = -2 \pm i,\ i \equiv \sqrt{-1}.$$

Second, find the eigenvector for $r = -2 + i$:

$$(A - rI)u = 0$$

$$\begin{bmatrix} -1-(-2+i) & 2 \\ -1 & -3-(-2+i) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Or,

$$(1-i)u_1 + 2u_2 = 0$$

$$(-1)u_1 + (-1-i)u_2 = 0.$$

Let $u_1 = 1$ *so that* $u_2 = (i-1)/2$.

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ (i-1)/2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

Third, find the eigenvector for $r = -2 - i$:

Similarly, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} - i\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}.$

Example 2. This matrix evolves from the mass-spring with damping.

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \text{ where } m = \text{mass},\ k = \text{spring and } b = \text{damping}.$$

A comes from the equation of motion for the position y(t) of the mass

$$my'' + by' + ky = 0.$$

Let x_1 = y = position and x_2 = y' = velocity. The derivatives are

$x_1' = y' = x_2$ and

$x_2' = y'' = (-by' - ky)/m$

$= (-b/m)y' + (-k/m)y$

$= (-b/m)x_2 + (-k/m)x_1$.

The matrix form of this is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

First, find the eigenvalues:

$$\det(A - rI) = 0$$

$$\det(\begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} - r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 0$$

$$\det(\begin{bmatrix} -r & 1 \\ -k/m & -b/m - r \end{bmatrix}) = 0$$

$$(-r)\left(-b/m - r\right) - 1(-k/m) = 0$$

$$r^2 + (b/m)r + (k/m) = 0 \text{ , or } mr^2 + br + k = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = r_1 \text{ and } r_2.$$

Second, find the eigenvector for $r = r_1$ *:*

$$(A - rI)u = 0$$

$$\begin{bmatrix} 0 - r_1 & 1 \\ -k/m & -b/m - r_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Or,

$$-r_1 u_1 + 1u_2 = 0$$

$$-k/mu_1 + (-b/m - r)u_2 = 0.$$

Let $u_1 = 1$ *so that* $u_2 = r_1$.

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ r_1 \end{bmatrix}$$

Third, find the eigenvector for $r = r_2$ *:*

$$\text{Similarly, } u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ r_2 \end{bmatrix}.$$

Example 3. This example evolves from heat diffusion in a long thin rod. Let x be a 3x1 column vector whose components are approximations of the temperature of the left, center and right segments of the rod, and these components are functions of time. The Fourier heat law can be applied to give the following model:

$$x' = Ax + f$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = c\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

The c is a constant that depends on the thermal properties and size of the rod. The vector f represents any heat sinks or sources within the rod.

$$Let\ A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

$First,\ find\ the\ eigenvalues:$

$$\det(A - rI) = 0$$

$$\det(\begin{bmatrix} -2-r & 1 & 0 \\ 1 & -2-r & 1 \\ 0 & 1 & -2-r \end{bmatrix}) = 0$$

$$(-2-r)\big((-2-r)(-2-r)-1\big)-1(1(-2-r)-0) = 0$$

$$(-2-r)\big((-2-r)(-2-r)-2\big) = 0$$

$$(-2-r)\big(r^2+4r+2\big) = 0$$

$$So,\ r = -2, -2 \pm \sqrt{2}.$$

$Second,\ find\ the\ eigenvector\ \ for\ r = -2:$

$$(A - rI)u = 0$$

$$\begin{bmatrix} -2-(-2) & 1 & 0 \\ 1 & -2-(-2) & 1 \\ 0 & 1 & -2-(-2) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Or,$

$$0 \cdot u_1 + 1 \cdot u_2 + 0 \cdot u_3 = 0$$

$$1 \cdot u_1 + 0 \cdot u_2 + 1 \cdot u_3 = 0$$

$$0 \cdot u_1 + 1 \cdot u_2 + 0 \cdot u_3 = 0$$

$$So,\ u_2 = 0.\ Let\ u_1 = 1\ so\ that\ u_3 = -1.$$

$$Third,\ find\ the\ eigenvector\ \ for\ r = -2+\sqrt{2}:$$

$$(A - rI)u = 0$$

$$\begin{bmatrix} -2-(-2+\sqrt{2}) & 1 & 0 \\ 1 & -2-(-2+\sqrt{2}) & 1 \\ 0 & 1 & -2-(-2+\sqrt{2}) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Or,$

$$-\sqrt{2}\cdot u_1 + 1\cdot u_2 + 0\cdot u_3 = 0$$

$$1\cdot u_1 - \sqrt{2}\cdot u_2 + 1\cdot u_3 = 0$$

$$0\cdot u_1 + 1\cdot u_2 - \sqrt{2}\cdot u_3 = 0$$

$$Let\ u_2 = 1. Then\ u_1 = 1/\sqrt{2}\ and\ 1/\sqrt{2}\ .$$

In the next lecture we will present a number of applications of eigenvalues and eigenvectors to systems of differential equations. There are many other uses of eigenvalues and eigenvectors. Some of these are to modes of vibrations, probability models called Markov chains, and iterative methods for approximating the solution of very large algebraic systems.

Homework.

1. Find the second eigenvector in example 1.
2. In example 2 the eigenvalues may be complex. In particular, let m = k = b =1. Then find the eigenvectors in the form of u = v + iw where u and w are 2x1 real vectors.
3. Find the third eigenvector in example 3.
4. If r is an eigenvalue of A, show cr is an eigenvalue of cA where c is a real number. Apply this to example 3 where x' = (cA)x + f.
5. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.
6. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.