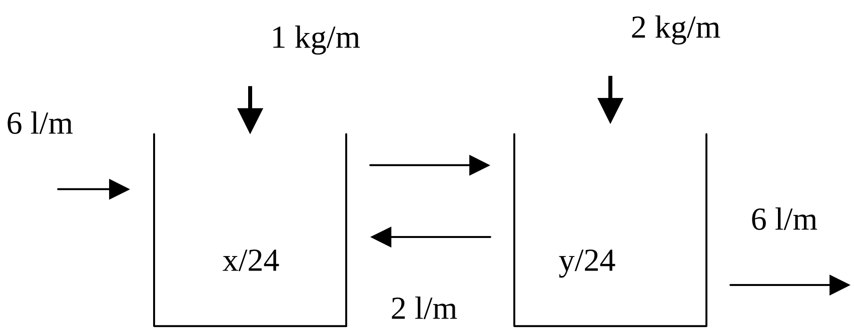
Lecture 13: Definition of Eigenvalues and Two-tank Mixing

In the next three lectures the time dependent solution of n linear differential equations with n unknown functions. Or, in terms of matrices find the unknown nx1 column vector $x^T = [x_1(t)\ \ x_2(t)\ \ \ldots\ \ x_n(t)]$ such that

$$x' = Ax + f$$

where A is nxn, f is nx1 and x(0) is nx1 and all are given. We have looked at the steady state solution where f is a constant vector and x' = 0 so that the steady state solution is $-A^{-1}f$.

One method for solving the time dependent problem is to use the eigenvalues and eigenvectors of A. As motivation for this approach, consider the two-tank mixing problem, which was introduced in lecture 4. Let x = x(t) and y = y(t) be the amounts of a chemical in the well-stirred left and right tanks.



$$x' = (-1/3)x + (1/12)y + 1$$
$$y' = (1/3)x + (-1/3)y + 2$$
Or,

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -1/3 & 1/12 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The solution of this system of differential equations can be obtained by either using elimination or the Matlab command desolve. However, both these methods become cumbersome for larger systems. The eigenvalue method does generalize nicely to larger systems. In order to motivate the definition of and eigenvalue, consider the solution of the two-tank problem

$$\boldsymbol{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} e^{(-1/2)t} + C_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} e^{(-1/6)t} + \begin{bmatrix} 6 \\ 12 \end{bmatrix}.$$

In order to verify that this is the solution of the above system, we must put the proposed solution into the left and right side of the system of differential equations.

$$\begin{aligned} Leftside = \boldsymbol{x}' &= \left(C_1 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} e^{(-1/2)t} + C_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} e^{(-1/6)t} + \begin{bmatrix} 6 \\ 12 \end{bmatrix} \right)' \\ &= C_1 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} (e^{(-1/2)t})' + C_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} (e^{(-1/6)t})' + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= C_1 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} e^{(-1/2)t} (-1/2) + C_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} e^{(-1/6)t} (1/2). \end{aligned}$$

$$\begin{aligned} Rightside &= A\boldsymbol{x} + \boldsymbol{f} \\ &= A(C_1 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} e^{(-1/2)t} + C_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} e^{(-1/6)t} + \begin{bmatrix} 6 \\ 12 \end{bmatrix}) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= C_1 e^{(-1/2)t} A \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + C_2 e^{(-1/6)t} A \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} e^{(-1/6)t} + A \begin{bmatrix} 6 \\ 12 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= C_1 e^{(-1/2)t} (-1/2) \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + C_2 e^{(-1/6)t} (-1/6) \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \; where \end{aligned}$$

$$A \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 & 1/12 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ -1/2 \end{bmatrix} = (-1/2) \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 & 1/12 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/12 \\ -1/6 \end{bmatrix} = (-1/6) \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} -1/3 & 1/12 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

The observation that allows this form of the solution to be generalized to n differential equations is that the column vectors in the solution are special examples of the following fact.

$$
\begin{aligned}
&\boldsymbol{x} = \boldsymbol{u}e^{rt} \textit{ is a solution of } \boldsymbol{x}' = A\boldsymbol{x} \textit{ provided}\\
&\quad \boldsymbol{u} \textit{ is non – zero column vector and}\\
&\quad r \textit{ is a number such that}\\
&\quad A\boldsymbol{u} = r\boldsymbol{u}.\\
&Leftside = \boldsymbol{x}' = (\boldsymbol{u}e^{rt})' = \boldsymbol{u}(e^{rt})' = \boldsymbol{u}\,e^{rt} r\\
&Rightside = A\boldsymbol{x} = A(\boldsymbol{u}e^{rt}) = e^{rt} A\boldsymbol{u}
\end{aligned}
$$

Definition. Let A be and nxn matrix. Let r be a single, possibly complex, number and u be a nx1 column vector with possibly complex numbers.

u is called an *eigenvector* of A if and only if

(i). u is a non-zero vector and

(ii). Au = ru.

r is called the *eigenvalue* associated with u.

Example. In the above two-tank problem $[-1/2\ 1]^T$ is an eigenvector whose eigenvalue is –1/2, and $[1/2\ 1]^T$ is an eigenvector whose eigenvalue is –1/6. Other examples are contained in the Matlab demo tank2_time.m, and in the next lecture.

Proposition 7. Let A be nxn.

1. If u is an eigenvector of A with eigenvalue r, then for any non-zero number, k, ku is also an eigenvector with eigenvalue r.
2. If u is an eigenvector of A with eigenvalue r, then det(A - rI) = 0.
3. If u is an eigenvector of A with eigenvalue r, then (A – rI)u = 0 (the zero vector).

Proof of 1. Au = ru

k(Au) = k(ru)

A(ku) = r(ku).

Proof of 2. Au = ru = r(Iu)

Au – r(Iu) = 0

(A – rI)u = 0

Since u is non-zero, A – rI must not have an inverse and so det(A – rI) = 0.

Property 2 can be used to find r, and then property 3 can be used to find u.

Consider the two-tank problem.

First, find the eigenvalues r via property 2.

$$A = \begin{bmatrix} -1/3 & 1/12 \\ 1/3 & -1/3 \end{bmatrix}$$

$$A - rI = \begin{bmatrix} -1/3 & 1/12 \\ 1/3 & -1/3 \end{bmatrix} - r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/3 - r & 1/12 \\ 1/3 & -1/3 - r \end{bmatrix}$$

$$\det(A - rI) = 0$$

$$\det(\begin{bmatrix} -1/3 - r & 1/12 \\ 1/3 & -1/3 - r \end{bmatrix}) = 0$$

$$(-1/3 - r)\left(-1/3 - r\right) - (1/12)(1/3) = 0$$

$$((-1/3 - r) - 1/6)((-1/3 - r) + 1/6) = 0$$

$$So,\ r = -1/2\ or\ -1/6.$$

Second, use property 3 to find the eigenvector u = [u_1 u_2]^T for the eigenvalue r = -1/2.

$$(A - rI)u = 0$$

$$\begin{bmatrix} -1/3 - (-1/2) & 1/12 \\ 1/3 & -1/3 - (-1/2) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$Or,$$

$$(1/6)u_1 + (1/12)u_2 = 0$$

$$(1/3)u_1 + (1/6)u_2 = 0.$$

$$Choose\ u_2 = 1\ so\ that\ u_1 = -1/2.$$

The freedom to choose u_2 to be any non-zero constant k is given by property 1.

The second eigenvector is determined in a similar way.

Homework.

1. Verify that u = [1 0 –1]^T and r = 2 are an eigenvector and eigenvalue for

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

2. Use property 3 to find the eigenvector associated with the other eigenvalue r = -1/6 of two-tank problem.