Lecture 12: Multilinear Approximation and Parameter Identification

In the previous lecture we approximated data by a function of a single variable, and this function was a polynomial. Now we will consider more two more complicated approximations to data. The multilinear approximation for three variable x, y and z is

$$r_i = d_i - (ax_i + by_i + cz_i)$$

where a, b and c are to found. The approximating function is assumed to be multilinear ax + by + cz. A more general single variable approximation has the form

$$r_i = d_i - u(x_i\ a,b,c)$$

where u could be a solution of a differential equation, which has unknown parameters a,b and c.

Application to Three-tank Mixing with Unknown Flow Rates. Consider the differential equations for the amounts in three tanks given by x(t), y(t) and z(t). The differential equation for tank x has the form

$$x' = ax + by + cz.$$

Usually, a, b, and c are given by the flow rates between the three tanks. If they are not known, then one must try to determine them from measurements made at various times. The following table contains some "pretend" measurements. The last column is an approximation of

$$x'(t_i) \cong d_i = (x_{i+1}-x_{i-1})/(t_{i+1}-t_{i-1}).$$

Since we can only compute d_i for i = 2,..,9, only eight times in the table will be used. Thus, in the matrix representation Ax = d A will be 8x3, $x = [\ a\ \ b\ \ c\ ]^T$ will be 3x1 unknown column vector and d will be the 8x1 computed column vector $d = [1.5\ \ 2.0\ \ 2.0\ \ 1.5\ \ 1.5\ \ 1.5\ \ 1.5\ \ 1.5]^T$.

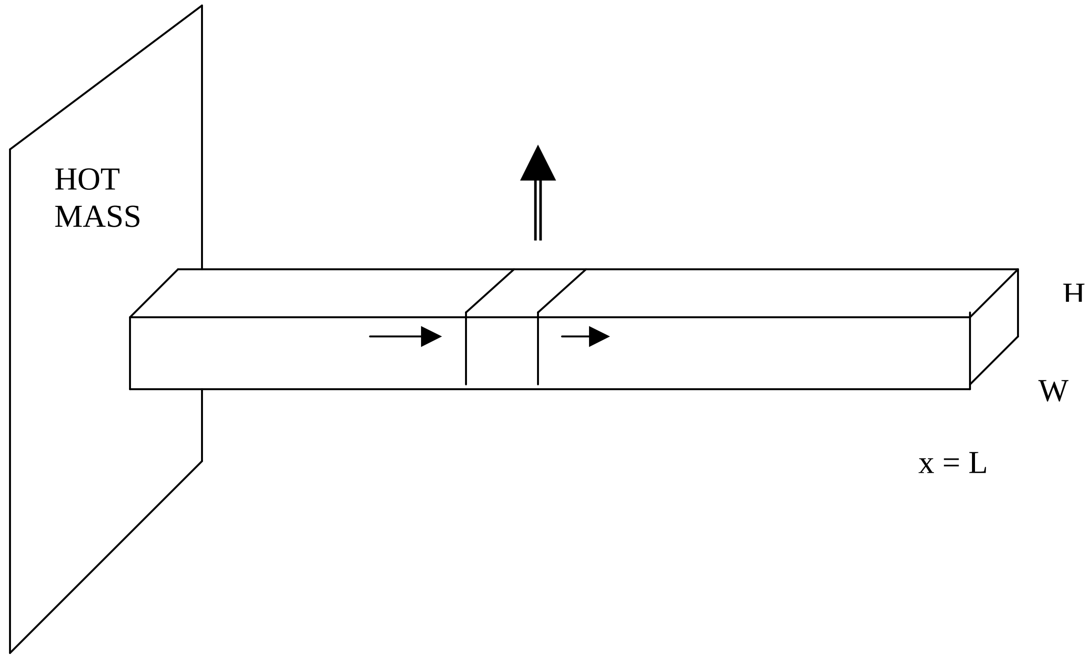
t_i	x_i	y_i	z_i	d_i
1	2	10	11	
2	3	9	12	1.5
3	5	8	13	2.0
4	7	7	16	2.0
5	9	5	18	1.5
6	11	4	21	1.5
7	12	3	23	1.5
8	14	2	26	1.5
9	15	1	29	1.5
10	17	1	33	

$$Ax = d$$

$$\begin{bmatrix} 3 & 9 & 12 \\ 5 & 8 & 13 \\ 7 & 7 & 16 \\ 9 & 5 & 18 \\ 11 & 4 & 21 \\ 12 & 3 & 23 \\ 14 & 2 & 26 \\ 15 & 1 & 29 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1.5 \\ 2.0 \\ 2.0 \\ 1.5 \\ 1.5 \\ 1.5 \\ 1.5 \\ 1.5 \end{bmatrix}.$$

The least squares solution is easily computed via Matlab, and this is done in the Matlab demo ls_multilin.m This gives the following solution a = 0.2849, b = 0.2361 and c = -0.1099, and the residual r = Ax – d gives $r^T r = 0.2313$.

Application to Parameter Identification for Steady State Heat Diffusion. Consider heat diffusing from a hot mass down a thin rod and into a surrounding cooler region. Suppose this rod has length equal to L in the x direction, and it is rectangular with other dimensions equal to H and W where both these are much smaller than L.



Change in the heat = 0 = (diffusion from the left)

-(diffusion to the right)

-(heat loss to the surrounding region)

Let u(x) be the steady state temperature in the long thin rode at position x.

The Fourier heat law gives

$0 = Ku_{xx} + c\,(2H + 2W)/(WH)\,(u_{sur} - u)$

$0 = u_{xx} + C(u_{sur} - u)$ where

$C = (c/K)\ (2H + 2W)/(WH)$

u_{sur} = given surrounding temperature.

K = thermal conductivity which may be known

c = unknown proportionality constant in heat loss to surrounding region.

So, C is not known. Two other unknowns are the heat diffusion rates at x = 0 and L:

$-u_x(0) = q_1/K = Q_1$ and $u_x(L) = q_2/K = Q_2.$

In summary, C, Q_1 and Q_2 are not known, $u = u(x, C, Q_1, Q_2)$ must satisfy:

$$0 = -u_{xx} + C(u_{sur} - u) \quad \text{for } x \text{ between } 0 \text{ and } L$$

$$-u_x(0) = q_1/K = Q_1 \text{ and } u_x(L) = q_2/K = Q_2.$$

One can show that $u(x, C, Q_1, Q_2)$ must have the form

$$u = C_1 e^{\sqrt{C}x} + C_2 e^{-\sqrt{C}x} + u_{sur} \text{ where}$$
$$Q_1 = -C_1\sqrt{C} + C_2\sqrt{C}$$
$$Q_2 = C_1\sqrt{C}e^{\sqrt{C}L} + C_2\sqrt{C}e^{-\sqrt{C}L}.$$

In order to find C, Q_1, Q_2 , we must make a number of observations for the steady state temperatures, d_i, as a function of space, x_i. Then we can form the residuals

$$r_i = d_i - u(x_i, C, Q_1, Q_2).$$

Choose C, Q_1, Q_2 so that $r^T r = \sum_i (d_i - u(x_i, C, Q_1, Q_2))^2$ is a minimum. Because of the complicated nature of $u(x_i, C, Q_1, Q_2)$, the normal equations are not applicable. Fortunately, Matlab has an implementation of a search algorithm called the Nelder-Mead simplex method, and this is in the Matlab command fminsearch.m A very sketchy outline of this method for three unknowns when $f = r^T r$ and the three variables are $x = (C, Q_1, Q_2)$ is as follows.

Step 1: Pick four points to form a tetrahedron.

Step 2: Order the points so that

$$f(x^0) \le f(x^1) \le f(x^2) \le f(x^3).$$

(The solution may be nearest to the plane formed by the first three points)

Step 3: Choose a new fourth point starting with some point on the line from the old fourth point and going through the center $x = (x^0 + x^1 + x^2)/3$ of the plane. Here there are a number of additional steps.

Step 4: After the new fourth point has been selected repeat steps 2-3.

Homework. Consider the multilinear approximation for three-tank mixing problem. Use the above data to approximate the coefficients e,f and g in the differential equation for $y' = ex + fy + gz$. The d_i will now be an approximation of y'. Repeat this for z'.