• Steady State in 3D.

\[-\Delta u = f(x, y, z)\]

where \( u \) is defined on \( \Omega \subseteq \mathbb{R}^3 \) and \( \Delta u \equiv u_{xx} + u_{yy} + u_{zz} \)

with the boundary \( \partial \Omega \) given by \( u = g(x, y, z) \)

The unknown function \( u \) may represent temperature, concentration, potential, etc.

• Weak Equation.

Consider a test function \( \phi(x, y, z) \) with \( \phi|_{\partial \Omega} = 0 \). Multiply the differential equation by the function and integrate

\[-\iiint \Delta \phi = \iiint f \phi.\]

Now use the 3D divergence theorem whose conclusion is:

\[\iiint \nabla \cdot \mathbf{F} = \iint_{\partial \mathbb{O}} \mathbf{F} \cdot \mathbf{n} dS\]

Let \( \mathbf{F} = \nabla u \phi \) This equation can be expanded as

\[\nabla \cdot \mathbf{F} = \Delta u \phi + \nabla u \cdot \nabla \phi\]

\[\nabla u \cdot n \phi = \nabla u \cdot n \phi, \text{ where } |\mathbf{n}| = 1\]

\[= \frac{du}{dn} \phi\]
\begin{equation}
\int_{\partial \Omega} \mathbf{F} \cdot \mathbf{n} \, dS = \int_{\partial \Omega} \frac{d\mathbf{u}}{dn} \phi^0 \, dS = 0
\end{equation}

So, \( \int \int \int (\Delta \phi + \nabla \mathbf{u} \cdot \nabla \phi) = 0 \) and since \( \Delta \phi = -f \), the weak equation is

\begin{equation}
\int \int \int \nabla \cdot \nabla \phi = \int \int \int f \phi
\end{equation}

or

\begin{equation}
\int \int \int (u_x \phi_x + u_y \phi_y + u_z \phi_z) = \int \int \int f \phi
\end{equation}

Note that the left hand-side is \( a(u, \phi) \), and right hand-side is \( l(\phi) \)

- **Linear Shape Functions.**

We want to approximate \( u(x,y,z) \) by a linear functions

\[ u(x,y,z) \approx u^e(x,y,z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z \]

Determine the above four unknowns

\[ u^e = u_1^e N_1^e (x,y,z) + u_2^e N_2^e (x,y,z) + u_3^e N_3^e (x,y,z) + u_4^e N_4^e (x,y,z) \]

Where \( e \) is a tetrahedral element:

Note that the element nodes are points in space given as

\[(x_1,y_1,z_1), (x_2,y_2,z_2), (x_3,y_3,z_3) \text{ and } (x_4,y_4,z_4).\]

The equation for each \( N_i^e (x,y,z) \) has to be linear.
\[ N^e_i(x, y, z) = a_i + b_i x + c_i y + d_i z \]

\[
\begin{align*}
1 &= N^e_i(x_1, y_1, z_1) = a_i + b_i x_1 + c_i y_1 + d_i z_1 \\
0 &= N^e_i(x_2, y_2, z_2) = a_i + b_i x_2 + c_i y_2 + d_i z_2 \\
0 &= N^e_i(x_3, y_3, z_3) = a_i + b_i x_3 + c_i y_3 + d_i z_3 \\
0 &= N^e_i(x_4, y_4, z_4) = a_i + b_i x_4 + c_i y_4 + d_i z_4
\end{align*}
\]

Or, writing the above equations as a matrix form:

\[
A = \begin{pmatrix}
1 & x_1 & y_1 & z_1 \\
1 & x_2 & y_2 & z_2 \\
1 & x_3 & y_3 & z_3 \\
1 & x_4 & y_4 & z_4
\end{pmatrix}
\]

\[
\bar{e}_i = A \begin{pmatrix}
a_1 \\
b_1 \\
c_1 \\
d_1
\end{pmatrix}
\]

and a similar vector equation is for the other three unit vectors.

These four vector equations can be written as a matrix equation

\[
I = A \cdot \begin{pmatrix}
a_1 & a_2 & a_3 & a_4 \\
b_1 & b_2 & b_3 & b_4 \\
c_1 & c_2 & c_3 & c_4 \\
d_1 & d_2 & d_3 & d_4
\end{pmatrix}
\]

Then column \( j \) of \( A^{-1} \) is

\[
\begin{pmatrix}
a_j \\
b_j \\
c_j \\
d_j
\end{pmatrix}
\]

Also note that \( |\text{det}(A)| = 6 \cdot (\text{volume of the element}) = 6V \). This comes from the fact that

for the vectors A,B,C, \( Volume = |(A \times B) \cdot C| = |\text{det}\begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{pmatrix}| \)
• Element matrices.

\[ a(u^e, N_i^e) \approx l(N_i^e) \]

\[ a(u_1^e N_i^e + u_2^e N_i^e + u_3^e N_i^e + u_4^e N_i^e, N_i^e) \approx l(N_i^e) \]

\[ a(N_i^e, N_i^e) u_1^e + a(N_2^e, N_i^e) u_2^e + a(N_3^e, N_i^e) u_3^e + a(N_4^e, N_i^e) u_4^e \approx l(N_i^e) \]

\[ k_i^e = a(N_{i_1}^e, N_i^e) , \quad d_i^e = l(N_i^e) \]

\[ 4 \times 4 \quad \downarrow \quad 4 \times 1 \]

\[ k^e, d^e \]

\[ a(N_j^e, N_i^e) = \iiint (N_{ix}^e, N_{ix}^e) + (N_{iy}^e, N_{iy}^e) + (N_{iz}^e, N_{iz}^e) \]

where \( N_{ix}^e = b_i, N_{iy}^e = c_i, N_{iz}^e = d_i \)

So, \( a(N_j^e, N_i^e) = (b_i b_j + c_i c_j + d_i d_j) V \) where \( V = \frac{1}{6} |\text{det}(A)| \)

The right-hand term of the weak equation may be approximated by using a linear approximation of the given function \( f(x, y, z) \):

\[ l(N_i^e) = \iiint f(x, y, z) N_i^e(x, y, z) \approx \iiint f_1^e N_1^e + f_2^e N_2^e + f_3^e N_3^e + f_4^e N_4^e ) N_i^e \]

where \( (f_1^e N + f_2^e N_2^e + f_3^e N_3^e + f_4^e N_4^e ) = f_i \) is the linear approximations.
The following integral formulas are useful in the approximation of \( I(N_i^e) \):

\[
\iiint N_i^e = V / 4, \quad \iiint (N_i^e)^2 = V / 10 \quad \text{and} \quad \iiint N_i^e N_j^e = V / 20 \quad \text{for} \quad i \neq j
\]

- **Construction of all tetrahedral elements**

Approximate: \( \Omega \approx \) union of tetrahedra

\( \partial \Omega \approx \) union of some faces of tetrahedra

Now use 3D version of Delaunay “triangulation”

\[
ne \times 4
\]

\[
nod3 = delaunay3(xx, yy, zz)
\]

\( nn \) is the number of system nodes.

\( ne \) is the number of elements.

\( nod3(e, 1 \text{ or } 2 \text{ or } 3 \text{ or } 4) \) is the system node numbers.

\((xx, yy, zz)\) is a linear order of all nodes as arranged shown below:

![Diagram of tetrahedral elements](image)

The Matlab code gennodbox.m is an illustration of this.