Lecture 24

Nonlinear Systems: Picard and Newton methods

(Lecture notes taken by Paul Thompson and Jason Andrus)

- Fixed point in 1D: \( x = g(x) \).

\[
x = g(x) \in \mathbb{R}^1
\]

Example: \( \sqrt{2} \) for \( x^2 - 2 = 0 \). Or,

\[
x^2 - 2 + x = x
\]

Picard Method:

\[
x^{m+1} = g(x^m)
\]

\[
x = g(x)
\]

\[
x^{m+1} - x = g(x^m) - g(x) = g'(c_m)(x^m - x)
\]

where the last part of the previous equation is accomplished by the mean value theorem.

Assume \( |g'| \leq r < 1 \) on some interval. Then by the mean value theorem \( g \) is contractive on \([a, b]\) (iff \( |g(x) - g(y)| \leq r|x - y|\))

\[
|x^{m+1} - x| \leq r|x^m - x| \leq r(r|x^{m-1} - x|) \leq \cdots \leq r^m |x^0 - x|
\]

If \( r < 1 \), then \( x^{m+1} \to x \)

Returning to Example:

\[
g(x) = x^2 - 2 + x
\]

\[
g'(x) = 2x - 0 + 1
\]

If \( g'(x) \) near \( \sqrt{2} \), then it is larger than 1!! Picard method would fails.
Example: ODE

\[ x' = f(x) \]
\[ x^{k+1} - x^k = \frac{f(x^{k+1})}{\Delta t} \]
\[ x^{k+1} = x^k + \Delta t \cdot f(x^{k+1}) = g(x^{k+1}) \]
\[ x = g(x) \]
\[ g(x) = x^k + \Delta t \cdot f(x) \]

\[ g'(x) = 0 + \Delta t \cdot f'(x) \]

If \(|f'(x)| \leq M\), then \(|g'(x)| \leq \Delta t \cdot M\) and we must choose \(\Delta t\) such that \(\Delta t \cdot M < 1\).

- **Picard method in nD.**

  \[ x = G(x) \in \mathbb{R}^n \]
  \[ x^{m+1} = G(x^m) \rightarrow x = g(x) \]

where we hope this converges to a fixed point

\[ x^{m+1} - x = G(x^m) - G(x) \]
\[ ||x^{m+1} - x|| \leq r ||x^m - x|| \]

A similar analysis as from above can be given and requires \(r < 1\).

Generally we change this into a one variable problem: \(G(x + tp) - G(x)\) where \(t\) is a scalar.

Example:

\[ -u_{xx} = f(u) \quad u(0) = u(1) = 0 \]

Use FDM:

\[ -u_{i-1} + 2u_i - u_{i+1} = h^2 f(u_i) \]

Or use: \(Au = h^2 f(u) \in \mathbb{R}^n\)

\[ A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, f(u) = \begin{bmatrix} f(u_1) \\ f(u_2) \\ \vdots \\ f(u_n) \end{bmatrix} \]

\[ u = A^{-1}(h^2 f(u)) = G(u) \]
\[ u^{m+1} = G(u^m) \text{ or } Au^{m+1} = h^2 f(u^m) \]
• **FEM example.**

\[
\begin{align*}
    u^e &= N_1 u_1^e + N_2 u_2^e \\
    f(u^e) &= f(N_1 u_1^e + N_2 u_2^e) \text{ or } N_1 f(u_1^e) + N_2 f(u_2^e) \\
    a(u^e, N_1^e) &= l(N_i) \\
    \int_0^L (N_{1x} u_1^e + N_{2x} u_2^e) N_{1x} &= \int_0^L (N_1 f(u_1^e) + N_2 f(u_2^e)) N_1 \\
    &\vdots \\
    Au &= h^2 B f(u)
\end{align*}
\]

Picard Method: \( Au^{m+1} = h^2 B f(u^m) \)

• **Root in 1D: \( f(x) = 0 \).**

**Newton's Method:**

\[
\begin{align*}
    f(x) &= \in \mathbb{R}^1 \\
    f(x) &= \in \mathbb{R}^1
\end{align*}
\]

The green curve represents the function whose zero one is trying to find \( f(x) \). The red line represents the tangent line to the curve at the initial guess and \( x^1 \) is located where the tangent line equals zero. The process is repeated until the iterative solution is sufficiently close to the exact solution. The formula used for this process can be seen below.
One can look at this method as a special Picard scheme. For convergence purposes we need 

\[ |g'(x)| < 1. \]

\[ g(x) = x - \frac{f(x)}{f'(x)} \]

\[ g'(x) = 1 - \frac{(f')^2 - ff''}{(f')^2} = \frac{f''(x)}{(f'(x))^2} f(x) \]

\[ g'(x) \leq \frac{|f''(x)|}{|(f'(x))^2|} |f(x)| \]

If \( |f''(x)| \leq M \) and \( |f'(x)| \geq m \) then:

\[ |g'(x)| \leq \frac{M}{m^2} |f(x)| < 1 \]

From the above statement it can be noted that for Newton’s Method the initial guess \((x^0)\) needs to be close to the exact solution.

Example Problem:

\[ x^2 - 2 = 0 = f(x) \]

\[ g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 2}{2x} \]

\[ x^{(0)} = 1 \]

\[ x^{(1)} = 1 - \frac{1 - 2}{2} = \frac{3}{2} \]

\[ x^{(2)} = \frac{9}{2} - \frac{4 - 2}{3} = \frac{17}{12} = 1.41667 \]

\[ \sqrt{2} = 1.4142 \]

Thus one notes that the series is converging to the exact solution.
Convergence Properties (Under Suitable Assumptions)

1) Local Convergence

$x^0$ should be "close" to $x$

2) Quadratic Convergence

$$|x^{m+1} - x| \leq C|x^m - x|^2$$

Matrix formation for Newton Method.

$$F(x) = 0 \in \mathbb{R}^n, \text{Need } F'$$

$n = 1$:

$$f' \approx \frac{\Delta f}{\Delta x} \Delta f \approx f' \Delta x$$

$n = 2$:

$$\begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = F(x)$$

$$\Delta f_1 = B + A = f_{1x_1} \Delta x_1 + f_{1x_2} \Delta x_2$$

$$\Delta f_2 = D + C = f_{2x_1} \Delta x_1 + f_{2x_2} \Delta x_2$$

$$\begin{bmatrix} \Delta f_1 \\ \Delta f_2 \end{bmatrix} \approx \begin{bmatrix} f_{1x_1} & f_{1x_2} \\ f_{2x_1} & f_{2x_2} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

$n > 2$:

$$[\Delta f_i] \approx [f_{ij}] [\Delta x_j]$$

$$\Delta f_i \approx \sum_{j=1}^{n} f_{ij} \Delta x_i$$

1D:

$$\Delta f \approx f' \Delta x$$

$$0 - f(x^m) = f'(x^m)(x^{m+1} - x^m)$$

nD:

$$\Delta F = F' \Delta x \in \mathbb{R}^n$$

$$0 - F(x^m) = F'(x^m)(x^{m+1} - x^m)$$

Note: Matrices $F(x^m)$ and $(x^{m+1} - x^m)$ will be dimension $n \times 1$, while matrix $F'(x^m)$ will be dimension $n \times n$. 
• Newton’s Method in nD

\[ F'(x^m)z = -F(x^m) \]
\[ x^{m+1} = z + x^m \]

Example Problem:

n = 2:

\[ x_1^2 + x_2^2 = 1, \quad f_1 = x_1^2 + x_2^2 - 1 \]
\[ x_1^2 - x_2^2 = d, \quad f_2 = x_1^2 - x_2^2 = d \]

We note at \( d > 1 \) the curves do not intersect, at \( d = 1 \) the curves are tangent at exactly two locations, and at \( d < 1 \) the curves will have four points of intersection.

\[ F' = \begin{bmatrix} 2x_1 & 2x_2 \\ 2x_1 & -2x_2 \end{bmatrix} \]

\[ \text{det}(F') = (-2x_1)(2x_2) - (2x_1)(2x_2) = -8x_1x_2 \]

Example Problem Using FDM

\[ -u_{xx} = f(u) \]
\[ -u_{i-1} + 2u_i - u_{i+1} = f(u_i) h^2 \]
\[ Au = h^2 f(u) \]
\[ f_i(u_1, u_2, \ldots, u_n) = 0 \]

Thus now let one build the derivative matrix \( F' \).

\[ f_{iu} = 0 \text{ if } j < i - 1 \]
\[ f_{iu} = -1 \text{ if } j = i - 1 \]
\[ f_{iu} = 2 - h^2 f(u_i) \text{ if } j = i \]
\[ f_{iu} = -1 \text{ if } j = i + 1 \]
\[ f_{iu} = 0 \text{ if } j > i + 1 \]

\[
F' = \begin{bmatrix}
2 - h^2 f'(u_1) & -1 & 0 & 0 \\
-1 & 2 - h^2 f'(u_2) & -1 & 0 \\
0 & -1 & \ddots & -1 \\
0 & 0 & -1 & 2 - h^2 f'(u_n)
\end{bmatrix}
\]

- **FEM example.**

\[ Au = h^2 Bf(u) \text{ or } F(u) = Au - h^2 Bf(u) = 0. \]

B is the mass matrix and both A and B are tri-diagonal as previously shown. Use the weak equation and following expressions for \( u \) and \( f \):

\[ u^e = N_1 u_1^e + N_2 u_2^e \]
\[ f(u) \approx N_1 f(u_1^e) + N_2 f(u_2^e) \]