Lecture 2

FEM in 1D: assembly of system matrix, cooling fin

(Lecture notes taken by Jaeseok Heo and Jason Andrus)

- Rayleigh-Ritz Discrete Model.

Using a generalized minimization we define:

$$\min \ G(u(x))$$

where $u(x) \in S$ and $G : S \rightarrow \mathbb{R}$

Substitute $u(x) \approx \sum_{j=1}^{N} u_j \varphi_j(x)$

where $\varphi_j(x)$ is a shape function that could be a trig function or polynomial, etc.

Thus the new problem is:

$$\min \ G\left(\sum_{j=1}^{n} u_j \varphi_j(x)\right)$$

where $u \in \mathbb{R}^n$ and $F : \mathbb{R}^n \rightarrow \mathbb{R}$

$$F(u) \equiv G\left(\sum_{j=1}^{n} u_j \varphi_j(x)\right).$$

Now Set:

$$\frac{\partial F(u)}{\partial u_j} = 0.$$  

This yields $n$ equations for $u_j \in \mathbb{R}$ thereby introducing many possible approximate solutions to the differential equation.

- Finite Element Discrete Model: Variational

Recall that we can represent the potential energy of the string by:

$$J(u) = \frac{T}{2} \left(\int_0^L u_x^2(x) - f u\right)$$

where the set of functions is defined as:
\[ S = \{ u: [0, L] \rightarrow \mathbb{R} \mid u(0) \text{ and } u(L) \text{ given and } \int_0^L u_i^2(x) < \infty \}. \]

Let us define \( u \) as:
\[ u \cong \sum_{j=0}^n u_j \phi_j(x) \]

Let us recall that \( \phi_j(x) \) is a shape function and can be represented as shown below:

For a domain with five nodes and four elements:

We can expect the space function to behave:

We can now take the partial derivative of the potential energy:
\[ \frac{\partial}{\partial u_i} J\left( \sum_{0}^{n} u_j \phi_j(x) \right) = \frac{\partial}{\partial u_i} \left( \frac{T}{2} \int_0^L \left( \sum_{0}^{n} u_j \phi_j(x) \right)^2_x - \int_0^L f \left( \sum_{0}^{n} u_j \phi_j(x) \right) \right) \]
Let the integral be broken into intervals by element:

\[
\int_0^{L} = \int_0^{h} + \int_h^{2h} + \ldots + \int_{L-h}^{L}
\]

Thus we get:

\[
\frac{\partial}{\partial u_i} J(\cdot) = \frac{\partial}{\partial u_i} \int_{x_i-h}^{x_i} + \frac{\partial}{\partial u_i} \int_{x_i}^{x_i+h}
\]

Introducing some notation let:

\[e_i = [x_i, x_i + h]\]

where \(u_i\) corresponds to system node \(i\).

If we restrict \(u^e(x) = u(x)|_{e_i}\) then we get the following behavior

where the circled numbers represent the corresponding element node numbers.

We can then represent the system by a combination of it’s nodes using the following expression:

\[
i = \text{nod}(e, \tilde{i}) \text{ where } \\
i = \text{system number} \\
e = \text{element number} \\
\tilde{i} = \text{element node number}
\]
Note that in a 1-D system the element node # will be 1 or 2, in a 2-D system the element node# will be 1, 2, or 3, and in a 3-D system the element node # will be 1, 2, 3, or 4

- **Linear shape function properties.**

Given a shape function we need to know: \( \int \phi^2 \), \( \int f \varphi \) where \( \varphi \) is a combination of \( N_1 \) and \( N_2 \).

\[
\int_0^L N_{1x}^e = \int_0^h \left( -\frac{x-h}{h} \right) dx = -\frac{1}{h}, h = -1
\]

\[
\int_0^L \left( N_{1x}^e \right)^2 = \left( \frac{1}{h} \right)^2 h = \frac{1}{h}
\]

**Also need to know:**

\[
\int_0^L \left( N_{1x}^e \right)^2 = \int_0^h \left( -\frac{x-h}{h} \right)^2 = \frac{2h}{6}
\]

In general this can be represented by:

\[
\int N_1^m N_2^n = \frac{m! \cdot n!}{(m+n+1)!} \cdot h
\]
- Assembly by nodes for string.

\[ T \int_0^L u \varphi_x = \int_0^L f \varphi \]

Let us define \( f_j \) as:

\[ f_j \equiv \sum f_j \varphi_j(x) \]

\[ T \int_0^L \left( \sum u_j \varphi_j(x) \right) \varphi_{i\alpha} = \int_0^L \left( \sum f_j \varphi_j(x) \right) \varphi_i \]

\[ \sum f_j \left( T \int_0^L \varphi_j(x) \varphi_{i\alpha} \right) u_j = \sum \left( \int_0^L \varphi_j(x) \varphi_i \right) f_j \]

These terms are nonzero if:

\[ j = i - 1 \]
\[ j = i \]
\[ j = i + 1 \]

Thus the nonzero terms are:

\[ -\frac{1}{h} u_{i-1} + \frac{2}{h} u_i - \frac{1}{h} u_{i+1} = \frac{h}{6} f_{i-1} + \frac{4h}{6} f_i + \frac{h}{6} f_{i+1} \]

We define \( u \) as:

\[ u = \sum_j u_j \varphi_j(x) \quad j: \text{all nodes} \]

The matrix \( A \) may be assembled by inspecting each row of the matrix, that is, fixing \( i \) and varying \( j = i-1, i \) and \( i+1 \). The 5x5 matrix below can be assembled either by rows or four 2x2 element matrices.
\[ A^{5 \times 5} = \begin{bmatrix} x & x \\ x & xv & v \\ v & vw & w \\ w & wz & z \\ z & z & 0 \end{bmatrix} \]

- Assembly by elements and element matrices for string.

\[ u = \sum_j u^e_j(x) \quad j : \text{all elements} \]

\[ J \left( \sum_j u^e_j(x) \right) = \sum_j \int_{e_j} ( ) \]

The integral can be broken into intervals:

\[ \frac{\partial}{\partial u_i} J \left( \sum_j u^e_j(x) \right) = \int_{e_{i-1}} + \int_{e_i} \]

\[ u_i = u_i^e \quad \text{or} \quad u_i = u_i^z \]

\[ \frac{\partial}{\partial u_i^e} J \left( u^e \right) = k_{11} u_i^e + k_{12} u_i^z - d_i^e \]

\[ \frac{\partial}{\partial u_i^z} J \left( u^e \right) = k_{21} u_i^e + k_{22} u_i^z - d_i^z \]

Element Matrices

\[ k^e = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = T \begin{bmatrix} 1 & -1 \\ \frac{1}{h} & -\frac{1}{h} \\ -\frac{1}{h} & 1 \\ -\frac{1}{h} & \frac{1}{h} \end{bmatrix} \]
• Heat in a cooling fin with derivative boundary condition.

\[
-Ku_{xx} + Cu = f
\]

\[u(0) \text{ given}
\]

\[Ku_x(L) = c(u_x - u(L))
\]

\[r = 0 : \text{insulated}
\]

\[c = \infty \quad u(L) = u_i : \text{given}
\]

\[0 < c < \infty : \text{partial insulation}
\]

\[TW(Ku_{xx}) = (2T + 2W)c(u_x - u)
\]

If \( c = \infty \), then the problem is

\[-Ku_{xx} + Cu = f
\]

\[u(0), u(L) \text{ given.}
\]

The weak equation of formed by

\[\int (-Ku_{xx} + Cu) \varphi = \int f \varphi, \quad \varphi(0) = \varphi(L) = 0
\]

\[\int (Ku_x \varphi_x + Cu \varphi) = \int f \varphi
\]

New element matrix is

\[
k^e = \begin{bmatrix}
\frac{K}{h} + C \frac{h}{3} & -\frac{K}{h} + C \frac{h}{6} \\
-\frac{K}{h} + C \frac{h}{6} & \frac{K}{h} + C \frac{h}{3}
\end{bmatrix}
\]