Lecture 19

FEM and Navier-Stokes Equations: explicit methods

(Lecture notes taken by Athanas Mutiso and Billy Tallis)

- **Navier-Stokes Flow in 2D.**
  2D incompressible flow with rotation is allowed:
  Our unknowns are the x-velocity (u), the y-velocity (v), and the pressure (p). Use the chain rule to get
  \[
  \frac{d}{dt} u = u_x' + u_y y' + u_t
  \]
  x momentum:
  \[
  u_t + u u_x + v u_y + p_x - \mu \Delta u = f
  \]
  y momentum:
  \[
  v_t + v v_x + v v_y + p_y - \mu \Delta v = g
  \]
  Incompressibility:
  \[
  u_x + v_y = 0
  \]
  Note that \( \mu = \frac{1}{R_e} \), where \( R_e \) is the Reynolds number defined as:
  \[
  R_e = \frac{u_{\infty} L_{\infty}}{v}
  \]
  \( v = \frac{\text{viscosity}}{\rho} \)

  These equations plus boundary conditions and initial conditions define the problem. The conservative form is:
  \[
  (u^2)_x + (uv)_y = 2u u_x + u v_y + u v_y
  = u u_x + v u_y + u (u_x + v_y)
  = u u_x + v u_y + 0
  \]
  \[
  (uv)_x + (v^2)_y = u v_x + v v_y
  \]
  \( u_x \) can be approximated in different ways, including \( \frac{u_{i+1} - u_i}{\Delta x} \) and \( \frac{u_i - u_{i-1}}{\Delta x} \).

- **Explicit method.**

  The explicit method is a two-step process that gets rid of the non-linearity of the problem and moves the viscosity term to the right hand side.
1. Set \( p = 0 \) for now. (It will be reintroduced later to enforce the incompressibility condition.) We then have:

\[
\frac{u^{aux} - u^k}{\Delta t} = -\left(\left(u^k\right)_x^2 \right)_x - (u^k v^k)_y + \mu \Delta u^k + f^k \equiv F
\]

and a similar equation for \( v \) to define \( G \).

2. Choose \( p \) so that:

\[
\begin{align*}
    u^{k+1}_x &= u^{aux} - \Delta t p^k_x \\
    v^{k+1}_y &= v^{aux} - \Delta t p^k_y \\
    u^{k+1}_x + v^{k+1}_y &= 0
\end{align*}
\]

\[
\begin{align*}
    \left(u^{aux} - \Delta t p^{k+1}_x\right)_x + \left(v^{aux} - \Delta t p^{k+1}_y\right)_y &= 0 \\
    u^{aux}_x + v^{aux}_y - \Delta t \Delta p^{k+1} &= 0
\end{align*}
\]

- **Solution for the pressure.**

  In the 1D case, we have:

  \[
  -p_{xx} = f
  \]

  Integrating gives:

  \[
  -\int_0^L p_{xx} = \int_0^L f
  \]

  \[
  -p_x(L) + p_x(0) = \int_0^L f
  \]

  \[
  -g_2 + g_1 = \int_0^L f
  \]

  If \( p \) is a solution, then \( p + C \) is also a solution:

  \[
  \begin{align*}
    (p + C)_xx &= p_{xx} = f \\
    (p + C)_x(0) &= p_x(0) \\
    (p + C)_x(L) &= p_x(L)
  \end{align*}
  \]

  In the 2D case,

  \[
  \Delta t \Delta p = u^{aux}_x + v^{aux}_y
  \]

  Using Green’s theorem,

  \[
  \iint_{\Omega} u^{aux}_x + v^{aux}_y = \Delta t \int_{\partial\Omega} \frac{dp}{dn} d\sigma
  \]

  Also, we have the following stability condition:

  \[
  \frac{\Delta t}{\Delta x} |u| + \frac{\Delta t}{\Delta y} |v| + \Delta t \mu \left(\frac{2}{\Delta x^2} + \frac{2}{\Delta y^2}\right) < 1
  \]
• **Chorin’s semi-explicit method.**

This is a 3 step process that mixed implicit and explicit methods.

1. (explicit step)

\[
\begin{align*}
\frac{u^* - u^k}{\Delta t} &= -(u^k)_x - (u^k v^k)_y \\
\frac{v^* - u^k}{\Delta t} &= -(u^k v^k)_x - (v^k)_y
\end{align*}
\]

2. (implicit step)

\[
\begin{align*}
\frac{u^{**} - u^*}{\Delta t} &= \mu \Delta u^{**} \\
\frac{v^{**} - v^*}{\Delta t} &= \mu \Delta v
\end{align*}
\]

3. (pressure step)

\[
\begin{align*}
    u^{k+1} &= u^{**} - \Delta t p_x^{k+1} \\
    v^{k+1} &= v^{**} - \Delta t p_y^{k+1}
\end{align*}
\]

Choose \( p^{k+1} \) such that

\[
u_x^{k+1} + v_y^{k+1} = 0
\]

We have the stability condition

\[
\frac{\Delta t}{\Delta x} |u| + \frac{\Delta t}{\Delta y} |v| < 1
\]

• **FEM and weak equations for Navier-Stokes.**

\[
\begin{align*}
    u_t + (u^2)_x + (uv)_y - \mu \Delta u + p_x &= f \\
    \iint u_t \varphi + \iint (u^2)_x \varphi + \iint (uv)_y \varphi - \mu \iint \Delta u \varphi &= \iint (f - p_x) \varphi
\end{align*}
\]

Note: \((u, v) = (0,0)\) on \( \partial \Omega \). This is called the no-slip condition. We then let \( \varphi |_{\partial \Omega} = 0 \)

\[
\begin{align*}
    \iint u_t \varphi - \iint u^2 \varphi_x - \iint u v \varphi_x + \mu \iint u_x \varphi_x + u_y \varphi_y &= \iint (f - p_x) \varphi
\end{align*}
\]

Discretizing with:

\[
\begin{align*}
    u &\rightarrow \sum u_j \varphi_j \\
    v &\rightarrow \sum v_j \varphi_j \\
    \varphi &\rightarrow \varphi_i
\end{align*}
\]
\[ \iint u^2 \varphi_x \rightarrow \iint \left( \sum_{i \neq j} u_i \varphi_j \right)^2 \varphi_{ix} \]
\[ = \sum_{i} \sum_{j} \iint (\varphi_i \varphi_j u_i) \varphi_{ix} u_j \]
\[ = \sum_{i} \sum_{j} \iint (\varphi_i \varphi_j u_i) \varphi_{ix} u_j \]

The above defines \( C_x(u) \)

\[ \iint (uv) \varphi_y \rightarrow \iint \left( \sum_{i} u_i \varphi_j \right) \left( \sum_{i} v_i \varphi_j \right) \varphi_{iy} \]
\[ = \sum_{i} \sum_{j} \iint (\varphi_i \varphi_j u_i) \varphi_{iy} u_j \]

The above defines \( C_y(v) \)

Using the above

\[ B \frac{u_{aux} - u^k}{\Delta t} - C_x(u^k)u^k - C_y(v^k)u^k + Au^k = B \hat{f} \]

\( B \) is the mass matrix and \( A \) is the diffusion matrix.