Lecture 11

FEM in 1D and 2D: Quadratic Shape Functions

(Lecture notes taken by Paul Thompson and Jason Andrus)

- Steady state problem in 1D.

\[-u_{xx} + Cu = f\]
\[u(0), u(L) \text{ given.}\]

Find the weak equation by multiplying the differential equation by \(\varphi\) with \(\varphi(0) = \varphi(L) = 0\) and integrating by parts

\[a(u, \varphi) \equiv \int (u_x \varphi_x + Cu\varphi) = \int f \varphi \equiv l(\varphi).\]

- Quadratic shape functions in 1D.

In order to obtain more accurate approximations use quadratic and not linear shape functions.

Express the values as a function of unknown constants \(\alpha\)

\[u^e = \bar{\alpha}_1 + \bar{\alpha}_2 \xi + \bar{\alpha}_3 \xi^2\]
\[\xi = x - x_i\]

Then \(u\) can be written as a function of \(\xi\) and new values \(\alpha\)

\[u^e = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2\]

**Example Problem** – Use the below axis, assume \(u_1 = u(0), u_2 = u(h/2), u_3 = u(h)\)
\[ u_1^e = \alpha_1 + \alpha_2 0 + \alpha_3 0^2 \]
\[ u_2^e = \alpha_1 + \alpha_2 \frac{h}{2} + \alpha_3 (h/2)^2 \]
\[ u_3^e = \alpha_1 + \alpha_2 h + \alpha_3 h^2 \]

In matrix form
\[
\begin{pmatrix}
  u_1^e \\
  u_2^e \\
  u_3^e
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 & 0 \\
  1 & \frac{h}{2} & (h/2)^2 \\
  1 & h & h^2
\end{pmatrix}
\begin{pmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \alpha_3
\end{pmatrix}
\]

Let us define matrix \( A \) as
\[
A =
\begin{pmatrix}
  1 & 0 & 0 \\
  1 & \frac{h}{2} & (h/2)^2 \\
  1 & h & h^2
\end{pmatrix}
\]

This is written in vector form
\[
u^e = A\alpha \in \mathbb{R}^3
\]
\[A^{-1}u^e = \alpha
\]

The inverse of \( A \) is
\[
A^{-1} = B =
\begin{pmatrix}
  1 & 0 & 0 \\
  -3/h & 4/h & -1/h \\
  2/h^2 & -4/h^2 & 2/h^2
\end{pmatrix}
\]
Express \( u \) as a function of \( u_1, u_2, \) and \( u_3 \) through the use of shape function \( Q_i \), which will be defined later,

\[
\begin{align*}
  u^e &= Q_1(\xi)u_1^e + Q_2(\xi)u_2^e + Q_3(\xi)u_3^e \\

\end{align*}
\]

Let us analyze \( Q_i \)

\[
Q_1 = a_1 + b_1 \xi + c_1 \xi^2
\]

\[
Q_1(0) = 1, Q_1 \left( \frac{h}{2} \right) = 0, Q_1(h) = 0
\]

\[
\begin{pmatrix}
  1 \\
  0 \\
  0
\end{pmatrix} = \begin{pmatrix}
  1 & 0 & 0 \\
  1 & h/2 & (h/2)^2 \\
  1 & h & h^2
\end{pmatrix} \begin{pmatrix}
  a_1 \\
  b_1 \\
  c_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
  a_1 \\
  b_1 \\
  c_1
\end{pmatrix}
= A \begin{pmatrix}
  a_1 \\
  b_1 \\
  c_1
\end{pmatrix}, \begin{pmatrix}
  a_2 \\
  b_2 \\
  c_2
\end{pmatrix}
= A \begin{pmatrix}
  a_2 \\
  b_2 \\
  c_2
\end{pmatrix}, \begin{pmatrix}
  a_3 \\
  b_3 \\
  c_3
\end{pmatrix}
= A \begin{pmatrix}
  a_3 \\
  b_3 \\
  c_3
\end{pmatrix}
\]

This means the coefficients of the \( Q_i \) are in column \( i \) of the inverse of \( A \), which we wrote as \( B \).

\[
Q_1 = 1 - \frac{3}{h} \xi + \frac{2}{h^2} \xi^2
\]
\[ Q_2 = 0 + \left(\frac{4}{h}\right) \xi + (4/h^2)\xi^2 \]

\[ Q_3 = 0 - \left(\frac{1}{h}\right) \xi + (2/h^2)\xi^2 \]

- **Element matrices (3x3) in 1D.**

\[
a(u, \varphi) = l(\varphi) \\
a(u^e, Q_i) = l(Q_i) \\
a(u_1^e Q_1 + u_2^e Q_2 + u_3^e Q_3, Q_i) = l(Q_i) \\
a(Q_1, Q_i)u_1^e + a(Q_2, Q_i)u_2^e + a(Q_3, Q_i)u_3^e = l(Q_i)
\]
Thus the element matrix $k^e$ is dimension $3 \times 3$

$$k^e = [k^e_{ij}] \text{ and } k^e_{ij} = a(Q_j, Q_i).$$

The right hand side matrix must then be $3 \times 1$

$$d^e = [d^e_i] \text{ and } d^e_i = l(Q_i).$$

Use the following notation

$$\xi^m \text{ where } m_1 = 0, m_2 = 1 \text{ and } m_3 = 2.$$  

$$Q_i = \sum_{i=1}^3 B(i, i) \xi^{m_i} \text{ and } Q_j = \sum_{j=1}^3 B(j, i) \xi^{m_j}$$

From these definitions one can analyze $a(Q_j, Q_i)$

$$a(Q_j, Q_i) = \int (Q_j \xi + Q_i \xi) + C Q_j Q_i$$

$$a(Q_j, Q_i) = \sum_j B(j, j) m_j \xi^{m_j-1} \sum_i B(i, i) m_i \xi^{m_i-1} + C \sum_j B(j, j) \xi^{m_j} \sum_i B(i, i) \xi^{m_i}$$

$$g(i, j) = \int m_j \xi^{m_j-1} m_i \xi^{m_i-1} + C \xi^{m_j} \xi^{m_i}$$

$$a(Q_j, Q_i) = [B^T G B]_{ij}$$

Lastly the right hand side of the equation:

$$d^e_i = \int Q_i f = [B^T F]_i$$

$$F = \begin{pmatrix} 
\int f \xi^{m_1} \\
\int f \xi^{m_2} \\
\int f \xi^{m_3} 
\end{pmatrix}$$
• Error estimates for a test case: fem1d.m

Take an example Equation: \(- u_{xx} + u = 32\) where \(u(0) = 0\) and \(u(2) = 4\)

Define the error = \(u(ih) - u_i\) where we can choose to use either a linear or quadratic shape function for \(u_i\)

Table 1: Error Comparison for Weighting Functions

<table>
<thead>
<tr>
<th>n</th>
<th>error in linear fem</th>
<th>error in quadratic fem</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0.0978</td>
<td>2.75E-04</td>
</tr>
<tr>
<td>11</td>
<td>0.0248</td>
<td>2.12E-05</td>
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<tr>
<td>21</td>
<td>0.0062</td>
<td>1.49E-06</td>
</tr>
<tr>
<td>41</td>
<td>0.0015</td>
<td>9.80E-08</td>
</tr>
</tbody>
</table>

\[ O(h^2) \quad O(h^4) \]

• Extension to 2D.

\[ u^e(x, y) = \overline{a_1} + \overline{a_2}x + \overline{a_3}y + \overline{a_4}x^2 + \overline{a_5}xy + \overline{a_6}y^2 \]

Where the element is shown by:

Consider the change of coordinates shown as:
With the change of coordinates the element is now represented as:

\[ u^e(\xi, \eta) = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi^2 + \alpha_5 \xi \eta + \alpha_6 \eta^2 \]

The equations at the six nodes give the vector equation

\[
\begin{bmatrix}
  u_{1}^e \\
  u_{2}^e \\
  u_{3}^e \\
  u_{4}^e \\
  u_{5}^e \\
  u_{6}^e \\
\end{bmatrix} = 
\begin{bmatrix}
  1 & -b & 0 & (b)^2 & 0 & 0 \\
  1 & a-b & 0 & \left(\frac{a-b}{2}\right)^2 & 0 & 0 \\
  1 & 2 & 0 & a^2 & 0 & 0 \\
  1 & a & c & \left(\frac{a}{2}\right)^2 & ac & \left(\frac{c}{2}\right)^2 \\
  1 & 2 & 2 & \left(\frac{a}{2}\right)^2 & \frac{ac}{4} & \left(\frac{c}{2}\right)^2 \\
  1 & -b & c & \left(\frac{-b}{2}\right)^2 & -bc & \left(\frac{c}{2}\right)^2 \\
\end{bmatrix} \begin{bmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \alpha_3 \\
  \alpha_4 \\
  \alpha_5 \\
  \alpha_6 \\
\end{bmatrix}
\]

Which otherwise can be represented as:

\[ u^e = A \alpha \in R^6 \]

\[ A^{-1}u^e = \alpha \text{ or } Bu^e = \alpha \text{ where } B = A^{-1} \]

We can represent the element values of u as a combination of sources:

\[ u^e = u_1^e Q_1(\xi, \eta) + ... + u_6^e Q_6(\xi, \eta) \]

\[ Q_i(\xi, \eta) = a_i + b_i \xi + c_i \eta + d_i \xi^2 + e_i \xi \eta + f_i \eta^2 \]
Thus:

\[
\mathbf{\hat{\mathbf{e}}}_i = A \begin{bmatrix} a_i \\ b_i \\ c_i \\ d_i \\ e_i \\ f_i \end{bmatrix} \rightarrow B(:,i) = \begin{bmatrix} a_i \\ b_i \\ c_i \\ d_i \\ e_i \\ f_i \end{bmatrix} \quad \text{since} \ B = A^{-1}
\]

**Element matrices (6x6) \(6 \times 6\) in 2D.**

The element matrices \(k^e\) are \(6 \times 6\) matrices.

\[
k^e_{ij} = a(Q_j, Q_i) \text{ where } a \text{ comes from } -\Delta u = f
\]

\[
a(u, \varphi) = \iint u_x \varphi_x + u_y \varphi_y
\]

\[
a(Q_j, Q_i) = \iint_{e} Q_j \xi Q_i \xi + Q_j \eta Q_i \eta \; d\xi d\eta
\]

After a number of computations as in the 1D case this can be expressed as

\[
a(Q_j, Q_i) = [B^T GB]_{ij}
\]

\(G\) follows from the differential equation and the integration formulas:

\[
h(m, n) = \iint_{e} \xi^m \eta^n = \frac{c^{n+1}(a^{m+1} - (-b)^{m+1})m!n!}{(m+n+2)!}
\]

The values of \(a, b\) and \(c\) will change from one element to the next and can be computed from the coordinates of the element nodes:

\[
a = \frac{[(x_3 - x_5)(x_3 - x_i) - (y_5 - y_3)(y_3 - y_i)]}{r}
\]

\[
b = \frac{[(x_3 - x_5)(x_3 - x_i) + (y_5 - y_3)(y_3 - y_i)]}{r}
\]

\[
c = \frac{[(y_5 - y_3)(x_3 - x_i) + (x_3 - x_5)(y_3 - y_i)]}{r}
\]

\[
r = \left( (x_3 - x_i)^2 + (y_3 - y_i)^2 \right)^{1/2}
\]