Lecture 1

FEM in 1D: classical, weak and energy formulations
(Notes taken by Brett Coonley and Cre’Shannon Thompson)

• Steady State String Problem.

String tied down at both ends with some pressure \( f \) applied to it and under tension \( T \).

\[
f = \text{pressure} \\
u(x) = \text{deformation (unknown)} \\
u(0) = u(L) = 0
\]

Assume this is a steady state, i.e. there is no time dependence in the function. There are three models that can be used to model this problem. And they are all equivalent, as we will show.

• Classical form - derivation from balance of forces.

The forces acting on the string should balance:

\[
0 \equiv fh + Tu_x \left( x + \frac{h}{2} \right) - Tu_x \left( x - \frac{h}{2} \right).
\]

Then

\[
Tu_x \left( x + \frac{h}{2} \right) - Tu_x \left( x - \frac{h}{2} \right) \approx fh
\]

Now by dividing both sides by \( h \) and taking the limit as \( h \to 0 \) we have

\[
-T \left( u_x \right)_x = f
\]

This is the classical boundary value model.

Note: For 2D case \(- \left( T \left( u_x \right)_x + T \left( u_y \right)_y \right) = f\).
• **Energy Form - derivation from minimum potential energy.**

\[
d\sigma = \sqrt{u_x^2 + 1}dx
\]

The potential energy of the segment is the difference between the arclength and \(dx\), and the sum of all of these segments is the integral.

Given \(\sqrt{u_x^2 + 1}\) let \(p = u_x^2\) then by approximating using Taylor series we obtain:

\[
\sqrt{p + 1} \approx \sqrt{0 + 1} + \frac{1}{2\sqrt{0 + 1}}(p - 0) = 1 + \frac{1}{2}p.
\]

Now by substituting \(p = u_x^2\) back into the equation we have

\[
\sqrt{u_x^2 + 1} - 1 \approx \frac{1}{2}u_x^2.
\]

The total potential energy, \(J(u)\), is then

\[
J(u) \equiv \frac{1}{2} \int_0^L Tu_x^2 - \int_0^L fu
\]

Note: The first integral is the potential energy from stretching (tension \(T\) in the string), and the second integral is the potential energy from the external force given by the pressure \(f\).

**Definition:** Let \(u,v\) be “suitable” functions (that is, so that \(J(u)\) exists) and we want to find the function \(u\) such that

\[
J(u) = \min_{u,v:S} J(v)
\]

In other words, the shape of the string should be such that the potential energy is at a minimum. The solution \(u\) thus obtained is called the **potential energy solution.**

• **Weak form.**

Assume \(u\) is classical

\[-T(u_x)_x = f.\]
Use a test function $\varphi$ satisfying $\varphi(0) = \varphi(L) = 0$ and multiply the differential equation by the test function and integrate by parts

$$-\int_{0}^{L} Tu_{xx} \varphi = \int_{0}^{L} f \varphi$$

$$-\varphi u_{x}\big|_{0}^{L} + \int_{0}^{L} T \varphi_{x} u_{x} dx = \int_{0}^{L} f \varphi dx$$

$$0 + \int_{0}^{L} T \varphi_{x} u_{x} dx = \int_{0}^{L} f \varphi dx.$$

**Definition:** $u$ is called a weak solution if and only if for all “suitable” $\varphi$ with zero at the boundaries

$$T \int_{0}^{L} u_{x} \varphi_{x} = \int_{0}^{L} f \varphi.$$

Note: Reason for weak solution is because in the equation only the first derivative appears.

- **Equivalence Theorem.**
  A. If $u$ is a classical solution, then $u$ is a weak solution
  B. If $u$ is a potential energy solution, then $u$ is a weak solution
  C. If $u$ is a weak solution, then $u$ is unique and so the classical solution equals the potential energy solution.

A. **Proof**

$$-\int_{0}^{L} Tu_{xx} \varphi = \int_{0}^{L} f \varphi \Rightarrow -\varphi u_{x}\big|_{0}^{L} + \int_{0}^{L} T u_{x} \varphi_{x} dx = \int_{0}^{L} f \varphi dx$$

$$\Rightarrow \int_{0}^{L} T u_{x} \varphi_{x} dx = \int_{0}^{L} f \varphi dx$$

B. **Proof**

Let $S = \{v : [0, L] \rightarrow \mathbb{R}, J(v) \text{ exists}, v(0), v(L) \text{ given}\}$.

$$J(u) = \min_{u, v \in S} J(v)$$
Since $\varphi(0) = \varphi(L) = 0$ such that $\varphi_x$ exists, then $\mu + \lambda \varphi \in S$. Because $u$ is a potential energy solution,

$$J(u) \leq J(u + \lambda \varphi)$$

We expect that $F'(\lambda = 0) = 0$ and $F''(\lambda = 0) > 0$. By the definition of $J$

$$F(\lambda) = J(u + \lambda \varphi) = \frac{1}{2} T \int_0^L (u + \lambda \varphi)_x^2 - \int_0^L f (u + \lambda \varphi).$$

Then the derivative of $F(\lambda)$ is as follows

$$F'(\lambda) = T \int_0^L (u_x + \lambda \varphi_x) \varphi_x - \int_0^L f \varphi$$

Let $\lambda = 0$ and we have

$$F'(0) = T \int_0^L u_x \varphi_x - \int_0^L f \varphi = 0$$

But this is required by the definition of a weak solution. This completes the proof.

Note: $F''(\lambda) = T \int_0^L \varphi_x^2 \, dx$ is positive, so that this is indeed minimum.

C. Proof

Let $u$, $\hat{u}$ be weak solutions. Then

$$T \int u_x \varphi_x = \int f \varphi \quad \text{and} \quad T \int \hat{u}_x \varphi_x = \int f \varphi.$$ 

Subtract these to get

$$T \int (u - \hat{u})_x \varphi_x = \int (f - f) \varphi = 0.$$

Let $\varphi = u - \hat{u}$ and note

$$T \int (u - \hat{u})_x^2 = 0 \Rightarrow (u - \hat{u})_x = 0 \Rightarrow (u - \hat{u}) = c$$

Since both $u$, $\hat{u}$ satisfy the same boundary conditions, then $c = 0$.

Thus if $u$ is a weak solution, then $u$ is unique.
• **Finite Difference Discrete Model.**

Let \( u(i\Delta x) \approx u_i \). Use the classical model with the first derivative approximated by finite differences:

\[
Tu_x(x + \frac{h}{2}) - Tu_x(x - \frac{h}{2}) = fh
\]

and

\[
-(T \frac{u_i+1-u_i}{\Delta x} - T \frac{u_i-u_i-1}{\Delta x}) f = f_i \Delta x
\]

\[
\begin{array}{c|c|c|c|c}
\Delta x = h & = L / n & n = 4 & u_1, u_2, u_3 = ?
\end{array}
\]

Or, the matrix form is \( Au = d \in \mathbb{R}^3 \)

\[
T \begin{bmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
= \begin{bmatrix}
f_1 \Delta x^2 + u_0 \\
f_2 \Delta x^2 \\
f_3 \Delta x^2 + u_4
\end{bmatrix}
\]

• **Finite Element Discrete Model: Galerkin**

Assume

\[
u(x) \approx \sum u_j \phi_j(x) \quad u_j \in \mathbb{R}.
\]

The \( \phi_j(x) \) are piece wise linear functions.
Let $\varphi(x) \equiv \varphi_i(x)$ and put it into weak equation. Then

$$T \int_0^L \left( \sum_j u_j \varphi_j(x) \right) \varphi_i(x) \, dx = \int_0^L f \varphi_i \Rightarrow \sum_j \left( T \int_0^L \varphi_j \varphi_i \, dx \right) u_j = \int_0^L f \varphi_i$$

Then we can rewrite it as $Au = d \in \mathbb{R}^3$ where $A$ is the same as in the finite difference model and the right side is a little different where for $i = 2$

$$d_i \approx \frac{1}{6} \Delta x^2 \left( f_{i-1} + 4f_i + f_{i+1} \right).$$

Note: Extensions to 2D space require linear shape function with three coefficients. In order to find these three points in the domain must be used. This is done by approximating the 2D space domain by a union of triangular elements. The weak equations can be derived from the partial differential equation as we did in the above 1D example.