Test Three in MA 341.002, Fall 2004

1. Use the *elimination method* to find the general solution for

   x' = 4x + y
   y' = -2x + y.

   (30 points)

2. Write the following second order ODE as a *first order 2d system* of ODEs

   2x'' + 6x' + 3x = sin(t) with x(0) = 1 and x'(0) = 4.

   (10 points)

3. Consider the 2x2 matrix

   $$A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}.$$

   (15 and 15 points)

   (a). Use *row operations and the Gauss-Jordan* method to find the inverse of A.
   (b). Find *both eigenvalues* and their corresponding *eigenvectors*.

4. Consider the 2d system given by the matrix in problem three

   $$\vec{x}' = A\vec{x} + \vec{f}$$
   $$where\ \vec{f} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

   (15 and 15 points)

   (a). Find a *particular* solution.
   (b). Find the solution *using eigenvalues and eigenvectors* that satisfies the initial condition

   $$\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

1. $x' = 4x + y$ ①
$y' = -2x + y$ ②

① $\longrightarrow$ $y = x' - 4x$ (+7)   OR ② $\rightarrow x = \frac{y'-y}{-2}$

② $\longrightarrow$ $(x' - 4x)' = -2x + (x' - 4x)$

$x'' - 4x' = -6x + x'$

$x'' - 5x' + 6x = 0$ (+8)

Let $x = e^{rt} \longrightarrow r^2 - 5r + 6 = 0$

$(r-2)(r-3) =$

$x(t) = c_1 e^{2t} + c_2 e^{3t}$ (+8)

$x' = c_1 e^{2t} 2 + c_2 e^{3t} 3$

$y = x' - 4x$

$= c_1 e^{2t} 2 + c_2 e^{3t} 3 - 4(c_1 e^{2t} + c_2 e^{3t})$

$= c_1(-2) e^{2t} + c_2(-1) e^{3t}$ (+7)

OR $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}$

2. $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 \equiv x$ and $x_2 \equiv x'$

$x_1' = x' = x_2$ (+2)

$x_2' = x'' = (\sin t - 3x - 6x')/2$

$= (\sin t - 3x_1 - 6x_2)/2$ (+2)

$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -3/2 & -6/2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ \frac{\sin t}{2} \end{bmatrix}$, $\vec{x}(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

(+2) (+2) (+2)

3 a. (a)

$$\left[\begin{array}{cc|cc} 4 & 1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{array}\right]$$

+2 $R2 + (R1)\frac{1}{2}$

$$\left[\begin{array}{cc|cc} 4 & 1 & 1 & 0 \\ 0 & 3/2 & 1/2 & 1 \end{array}\right]$$

+2 $(R1)\frac{1}{4}$:

+2 $(R2)\frac{2}{3}$:

$$\left[\begin{array}{cc|cc} 1 & 1/4 & 1/4 & 0 \\ 0 & 1 & 1/3 & 2/3 \end{array}\right]$$

+2 $(R1) - (R2)\frac{1}{4}$:

$$\left[\begin{array}{cc|cc} 1 & 0 & 2/12 & -2/12 \\ 0 & 1 & 1/3 & \frac{2}{3} \end{array}\right]$$

$$\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$ +3

(b) $\det(A - rI) = 0 \longrightarrow \det\begin{bmatrix} 4-r & 1 \\ -2 & 1-r \end{bmatrix} = 0$

$(4-r)(1-r) - 1(-2) =$

$r^2 - 5r + 6 = 0$

$r = 2, 3$ +5

$\underline{r=2}$ $\begin{bmatrix} 4-2 & 1 \\ -2 & 1-2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2u_1 + u_2 = 0 \longrightarrow u_1 = 1$ given $u_2 = -2$, $\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ +5

$-2u_1 - u_2 = 0$

$\underline{r=3}$ $\begin{bmatrix} 4-3 & 1 \\ -2 & 1-3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$u_1 + u_2 = 0 \longrightarrow u_1 = 1$ gives $u_2 = -1$, $\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ +5

$-2u_1 - 2u_2 = 0$

4 a. $\vec{x} = \Psi \vec{c} + \vec{x}_p$

$$= \left[\begin{bmatrix}1\\-2\end{bmatrix}e^{2t} \quad \begin{bmatrix}1\\-1\end{bmatrix}e^{3t}\right]\begin{bmatrix}c_1\\c_2\end{bmatrix} + \vec{a}$$

a) Choose $\vec{a}$ so that $\vec{a}' = A\vec{a} + \begin{bmatrix}1\\5\end{bmatrix}$ +5

$$\vec{0} = A\vec{a} + \begin{bmatrix}1\\5\end{bmatrix}$$

OR, use var. of para.

$$\vec{a} = -A^{-1}\begin{bmatrix}1\\5\end{bmatrix}$$

$$= -\begin{bmatrix}4 & 1\\-2 & 1\end{bmatrix}^{-1}\begin{bmatrix}1\\5\end{bmatrix}$$ +5

$$= -\frac{1}{6}\begin{bmatrix}1 & -1\\2 & 4\end{bmatrix}\begin{bmatrix}1\\5\end{bmatrix}$$

$$= -\frac{1}{6}\begin{bmatrix}-4\\22\end{bmatrix}$$

$$= \begin{bmatrix}2/3\\-11/3\end{bmatrix}$$ +5

b) $\vec{x}(0) = \left[\begin{bmatrix}1\\-2\end{bmatrix}e^{2\cdot 0} \quad \begin{bmatrix}1\\-1\end{bmatrix}e^{3\cdot 0}\right]\begin{bmatrix}c_1\\c_2\end{bmatrix} + \vec{a}$

$$\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}2/3\\-11/3\end{bmatrix} = \begin{bmatrix}1 & 1\\-2 & -1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$ +5

$$\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1 & 1\\-2 & -1\end{bmatrix}^{-1}\begin{bmatrix}1/3\\17/3\end{bmatrix}$$

$$= \begin{bmatrix}-1 & -1\\2 & 1\end{bmatrix}\begin{bmatrix}1/3\\17/3\end{bmatrix}$$

$$= \begin{bmatrix}-18/3\\19/3\end{bmatrix}$$ +5

$\vec{x}(t) = \Psi(t)\vec{c} + \vec{x}_p$ +