Test Two in MA 341.002, Fall 2004

1. Consider $y'' - y = -e^{2t}$.

(5, 15 and 5 points)

   (a) Find the homogeneous solution.
   (b) Find the particular solution (use either undetermined coefficients or variation of parameters).
   (c) Use these to solve the initial value problem (IVP) with $y(0) = 1$ and $y'(0) = -1$.

2. Consider a series LRC circuit with inductance equal to 4, resistance equal to 120, capacitance equal to 2200^{-1} and imposed voltage equal to $10 \cos(20t)$.

(5, 5, and 15 points)

   (a). State the differential equation for the charge.
   (b). State the differential equation for the current.
   (c). Find the particular solution for the current differential equation.

3. Use Laplace transforms to solve $y'' - 2y' + 5y = e^{2t}$ with $y(0) = 1$ and $y'(0) = 2$.

(25 points)

4. Use Laplace transforms to solve $y'' + y = \delta(t - 2) + 10u(t-5)$ with $y(0) = 0$ and $y'(0) = 0$.

(25 points)

Laplace Transform Rules 9-14:

9. $L(t^n f(t)) = (-1)^n d^n/ds^n F(s)$ where $F(s) = L(f(t)) \equiv \int_0^\infty e^{-st} f(t)dt.$
10. $L(e^{at} f(t)) = F(s-a)$.
11. $L(u(t-a)) = e^{-sa} 1/s$.
12. $L(u(t-a) f(t-a)) = e^{-sa} F(s)$.
13. $L(\delta(t-a)) = e^{-sa}$.
14. $L(f*g) = L(f) L(g)$ where $(f*g)(t) \equiv \int_0^t f(t-v)g(v)dv.$

$\boxed{OR}$

1. (b) var. of parameters:

$$y_p = \left(-\int \frac{y_2 f}{W}\right) y_1 + \left(\int \frac{y_1 f}{W}\right) y_2 \quad +5$$

$$y_1 = e^t, \quad y_2 = e^{-t}$$

$$W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix} = -2 \quad +1$$

$$f = -e^{2t}$$

$$y_p = \left(-\int \frac{e^{-t}(-e^{2t})}{-2}\, dt\right) e^t + \left(\int \frac{e^t(-e^{2t})}{-2}\, dt\right) e^{-t}$$

+3 +3

$$= \left(-\frac{1}{2} e^t\right) e^t + \left(\frac{1}{2}\,\frac{1}{3} e^{3t}\right) e^{-t}$$

$$= \left(-\frac{1}{2} + \frac{1}{6}\right) e^{2t}$$

$$= -\frac{1}{3} e^{2t} \quad +3$$

1. (a) $y = e^{rt} \rightarrow r^2 - 1 = 0 \rightarrow y_{homo} = c_1 e^t + c_2 e^{-t}$ +5

(b) $y_{part} = A e^{2t}$ +5 $\rightarrow y_{part}'' = A\, 2^2 e^{2t}$

$y_{part}'' - y_{part} = -e^{2t}$

$(4A - A)\, e^{2t}$ +5 $\rightarrow 3A = -1$ or $A = -1/3$

$y_{part} = -1/3\, e^{2t}$ +5

(c) $y = c_1 e^t + c_2 e^{-t} - 1/3\, e^{2t}$ +1

$y(0) = c_1 + c_2 - 1/3 = 1$ +1

$y' = c_1 e^t - c_2 e^{-t} - 1/3\, e^{2t} \cdot 2$

$y'(0) = c_1 - c_2 - 2/3 = -1$ +1 $\longrightarrow$ add eq.

$2c_1 + 0 - 1 = 0$

$y = \frac{1}{2} e^t + \frac{5}{6} e^{-t} - \frac{1}{3} e^{2t}$ $\qquad c_1 = 1/2,\ c_2 = 1 + 1/3 - \frac{1}{2}$ +2 $= 5/6$

2. (a) $LQ'' + RQ' + \frac{1}{C} Q = V$ +5

$4Q'' + 120Q' + \frac{1}{(2200)^{-1}} Q = 10 \cos(20t)$, $I \equiv Q'$

(b) $4I'' + 120 I' + 2200 I = 10(-20) \sin(20t)$ +5

(c) $I_p = A \cos 20t + B \sin(20t)$ +5

$I_p' = -20A \sin 20t + 20B \cos 20t$

$I_p'' = -400A \cos 20t - 400B \sin 20t$

$4(-400A \cos 20t - 400B \sin 20t)$

$+120(-20A \sin 20t + 20B \cos 20t)$ +5

$+2200(A \cos 20t + B \sin 20t) = -200 \sin 20t + 0 \cdot \cos 20t$

$\cos 20t\ (-1600A + 2400B + 2200A) =$

$\sin 20t\ (-1600B - 2400A + 2200B)$

So $-1600A + 2400B + 2200A = 0 \rightarrow 6A + 24B = 0$

$-1600B - 2400A + 2200B = -200 \rightarrow 6B - 24A = -2$

$A = -4B,\quad 6B - 24(-4B) = -2,\ B = \frac{-1}{51},\ A = \frac{+4}{51}$

$I_p = \frac{+4}{51} \cos 20t - \frac{1}{51} \sin 20t$ +5

3. $s^2\mathcal{L}(y) - s\cdot 1 - 2 - 2(s\mathcal{L}(y) - 1) + 5\mathcal{L}(y) = \mathcal{L}(e^{2t}) = \frac{1}{s-2}$ +2 +2 +2

$(s^2-2s+5)\mathcal{L}(y) = s + \frac{1}{s-2} = \frac{s(s-2)+1}{s-2} = \frac{s^2-2s+1}{s-2}$

$\mathcal{L}(y) = \frac{s^2-2s+1}{(\underbrace{s^2-2s+5}_{(s-1)^2+2^2})(s-2)} = \frac{A}{s-2} + \frac{Bs+C}{s^2-2s+5}$ +2 +2

$s^2-2s+1 = A(s^2-2s+5) + (Bs+C)(s-2)$
$= s^2(A+B)$
$+ s(-2A+C-2B)$
$+ 1\cdot(5A-2C)$

$1 = A+B \longrightarrow B = 1-A$

$-2 = -2A+C-2B \longrightarrow -2 = -2A+C-2(1-A)$

$C = 0$

$1 = 5A-2C \longrightarrow A = \frac{1}{5}$

$B = 4/5$

+5

$= \frac{1}{5}\frac{1}{s-2} + \frac{4}{5}\frac{s-1+1}{(s-1)^2+2^2}$

$= \frac{1}{5}\frac{1}{s-2} + \frac{4}{5}\frac{s-1}{(s-1)^2+2^2} + \frac{2}{5}\frac{2}{(s-1)^2+2^2}$ +1

$= \frac{1}{5}\mathcal{L}(e^{2t}) + \frac{4}{5}\mathcal{L}(e^{t}\cos(2t))$
$+ \frac{2}{5}\mathcal{L}(e^{t}\sin(2t))$

$y = \frac{1}{5}e^{2t} + \frac{4}{5}e^{t}\cos(2t) + \frac{2}{5}e^{t}\sin(2t)$ +3 +3 +3

4. $(s^2+1)\mathcal{L}(y) = e^{-s2} + 10\, e^{-s5}\frac{1}{s}$ +3 +3 +3

$\mathcal{L}(y) = e^{-s2}\frac{1}{s^2+1} + 10\, e^{-s5}\frac{1}{s(s^2+1)}$ +3

$= e^{-s2}\frac{1}{s^2+1} + 10\, e^{-s5}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$ +2

$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$ +5

$1 = A(s^2+1) + (Bs+C)s$
$= s^2(A+B)$
$+ s\,C$
$+ 1\cdot A$

$A = 1,\ C = 0,\ B = -1$

$= \mathcal{L}(u(t-2)\sin(t-2))$
$+ 10\,\mathcal{L}(u(t-5)(1-\cos(t-5)))$

$y = u(t-2)\sin(t-2)$ +3
$+ 10\, u(t-5)(1-\cos(t-5))$ +3