Test One in MA 341.002, Fall 2004

1. Consider $y' = t^3 (1-y)$ and $y(0) = 3$.
(20) Use separation of variables to find $y(t)$.

2. Consider $t^3 y' + 3t^2 y = t$ and $y(2) = 0$.
(20) Use the linear method to find $y(t)$.

3. Consider $y'' + 2y' + y = 0$ with $y(0) = 0$ and $y'(0) = 1$.
(20) Use the auxiliary equation to solve for $y(t)$.

4. Consider a 100 (gal.) mixing tank, which is initially full of a salt solution with concentration equal to 1 (lb./gal.). There is an inflow with rate equal to 2 (gal./min.) and concentration equals ½ (lb./gal.). Also the outflow has a rate equal to 3 (gal./min.) and concentration equals x(t)/volume where x(t) is the total amount of salt and volume is a function of time.

(a). Derive the differential equation for x(t) and state the initial condition.
(10)
(b). Find the general solution of this differential equation...do not find the
(10) constant C.

5. Consider a spring-mass system with nonzero damping. Suppose the mass is 1, the damping constant is 10, the spring constant is 30 and there is no external force. Let y(t) be the displacement from equilibrium with initial position equal to 1 and initial velocity equal to -2.

(a). State the differential equation for y(t) and the initial conditions.
(10)
(b). Find the general solution of this differential equation...do not find the
(10) constants C_1 and C_2.

$\boxed{\text{Test 1} \quad \text{F04}}$

1.

$\frac{dy}{dt} = t^3(1-y) \quad , \quad y(0) = 3$

(+4) $\int \frac{dy}{1-y} = \int t^3\,dt + C$

$-\ln|1-y| = \frac{t^4}{4} + c$ (+4)

(+4) $\ln(y-1) =$

$e^{\ln(y-1)} = e^{-t^4/4 + c} = \tilde{c}\,e^{-t^4/4}$

(+4) $y - 1 =$

$y(t) = 1 + \tilde{c}\,e^{-t^4/4}$

$y(0) = 3 = 1 + \tilde{c}\cdot 1 \longrightarrow \tilde{c} = 2$

$y(t) = 1 + 2e^{-t^4/4}$ (+4)

2.

$t^3\frac{dy}{dt} + 3t^2 y = t \quad , \quad y(2) = 0$

$\frac{dy}{dt} + \underbrace{3t^{-1}}_{P} y = \underbrace{t^{-2}}_{Q}$ (+4)

(+4) $\mu = e^{\int P} = e^{\int 3t^{-1}} = e^{3\ln t} = t^3$

(+4) $y = \frac{1}{\mu}\left[\int Q\mu + c\right]$

$= \frac{1}{t^3}\left[\int t^{-2}t^3 + c\right]$

$= \frac{1}{t^3}\left[\frac{t^2}{2} + c\right]$

(+4) $= \frac{1}{2}t^{-1} + c\,t^{-3}$

$y(t) = \frac{1}{2}t^{-1} + c\,t^{-3}$

$y(2) = 0 = \frac{1}{2}2^{-1} + c\,2^{-3}$

$= \frac{1}{4} + c\,\frac{1}{8}$ (+4)

$c = -2$

$y(t) = \frac{1}{2}t^{-1} - 2t^{-3}$

3.

$y'' + 2y' + y = 0 \quad , \quad y(0) = 0, \quad y'(0) = 1$

$y = e^{rt} \rightarrow 1\cdot r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

(+4) or $r = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot 1}}{2\cdot 1}$

$= -1$

(+4) $y = c_1 e^{-t} + c_2 t e^{-t}$

$y(0) = 0 = c_1\cdot 1 + c_2\cdot 0\cdot 1$

(+4) $0 = c_1$

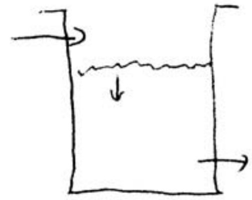
(+4) $y'(t) = c_1 e^{-t}(-1) + c_2\left(1\cdot e^{-t} + t e^{-t}(-1)\right)$

$y'(0) = 1 = c_1\cdot 1\cdot(-1) + c_2(1 + 0)$

$1 = 0 + c_2$

(+4) $y(t) = t e^{-t}$

4. (a) $x(t)$ = amount , vol. $= 100 - t$ (+2)



$\Delta x =$ in $-$ out $\quad \boxed{\frac{x(0)}{100-0} = 1, \quad x(0) = 100}$

(+2) $\cong 2\left(\frac{gal}{min}\right) \frac{1}{2}\left(\frac{lb}{gal}\right) \Delta t(min)$

(+2) $- 3\left(\frac{gal}{min}\right) \frac{x}{100-t}\left(\frac{lb}{gal}\right) \Delta t(min)$

$\frac{\Delta x}{\Delta t} \approx 1 - 3\frac{x}{100-t}$

$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} 1 - 3\frac{x}{100-t}$

(+2) $\frac{dx}{dt} = 1 - 3\frac{x}{100-t}$, $x(0) = 100$ (+2)

(b) $\frac{dx}{dt} + \underbrace{3\frac{1}{100-t}}_{P} x = \underbrace{1}_{Q}$ (+2)

(+2) $\mu = e^{\int P} = e^{-3\ln(100-t)} = (100-t)^{-3}$

(+2) $x = \frac{1}{(100-t)^{-3}}\left[\int 1 \cdot (100-t)^{-3} + C\right]$

(+2) $= (100-t)^{3}\left[-(100-t)^{-2}\frac{1}{-2} + C\right]$

(+2) $= \frac{1}{2}(100-t)^{1} + C(100-t)^{3}$

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5. (a) $my'' + by' + ky = 0$

$1 \cdot y'' + 10y' + 30y = 0$

$y(0) = \phi$, $y'(0) = -2$

(b) $y = e^{rt} \rightarrow r^2 + 10r + 30 = 0$

$r = \frac{-10 \pm \sqrt{10^2 - 4 \cdot 1 \cdot 30}}{2 \cdot 1}$ (+4)

$= -5 \pm i\sqrt{5}$

$y = c_1 e^{-5t}\cos\sqrt{5}t + c_2 e^{-5t}\sin\sqrt{5}t$ (+3) (+3)