Sample Final Exam in MA 341.002, Fall 2004

1. Consider $y' + y = 70 + t$ and $y(0) = 200$.

(10) (a). Use the linear method to find $y(t)$.

(6) (b). List all possible methods that will solve or approximate the solution for this problem?

2. Consider the population model for $y(t)$ = the size of the population:

$$y' = (1/1000)(1500 - y)y - s$$

where $y(0) = 200$ is the initial population and s is the harvesting rate.

(5) (a). Find the steady state solutions when $s = 50$.

(13) (b). Use separation of variables to solve for $y(t)$.

3. Consider $y'' + y' + y = 2e^{-t}$, $y(0) = 0$ and $y'(0) = 0$.

(6) (a). Find the homogeneous solutions.

(6) (b). Find the particular solution via undetermined coefficients.

(6) (c). Find the solution that satisfies the initial conditions.

4. Consider a mass spring with the damping parameter equal zero.

(5) (a). If the mass is 2, the spring constant is 32, and external force equal to $10\cos(w t)$, state the differential equation which models this.

(5) (b). Find the particular solution when w is *not* equal to 4.

(5) (c). Describe what happens to the solution as w approaches 4.

5. Use Laplace transforms to solve $y'' + 9y = e^{3t}$, $y(0) = 1$ and $y'(0) = 0$.

(7) (a). Find $\mathbf{L}(y)$.

(8) (b). By the rules find $y(t)$.

6. Consider the system of ODEs

$$x' = 2x + y + 0 + 3t, \quad x(0) = 5 \text{ and}$$
$$y' = -3x - 2y - 4 + 0t, \quad y(0) = 10.$$

(2) (a). Find A and $\mathbf{f}$ so that $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ where $\mathbf{x} = [x \;\; y]'$.

(6) (b). By hand find the eigenvalues and eigenvectors of A.

(2) (c). Find the homogeneous solution.

(6) (d). Find the particular solution.

(2) (e). Use these to solve the initial value problem.

1. (a)

$y' + y = 70 + t$

$P = 1 \longrightarrow \mu = e^{\int P} = e^t$

$Q = 70 + t$

$y = \frac{1}{\mu}\left[\int \mu Q + C\right]$

$= e^{-t}\left[\int (70+t)e^t dt + C\right]$

$= e^{-t}\left[70e^t + te^t - e^t + C\right]$

$= 70 + t - 1 + Ce^{-t}$

$y(0) = 200 = 70 + 0 - 1 + C \cdot 1$

$C = 200 - 70 + 1 = 131$

$\int u\,dv = uv - \int v\,du$

$u = t \quad dv = e^t$

$\int te^t = te^t - e^t$

(b) Linear method, exact

Laplace transform

Euler, Improved Euler

(a) $y' = 0 = \frac{1}{1000}(1500 - y)y - 50$

$$0 = (1500 - y)y - 50{,}000$$

$$= -y^2 + 1500y - 50000$$

OR, $y^2 - 1500y + 50{,}000 = 0$

$$y = \frac{1500 \pm \sqrt{1500^2 - 4 \cdot 50{,}000}}{2}$$

$$= \frac{1500 \pm \sqrt{2{,}250{,}000 - 200{,}000}}{2}$$

$$= \frac{1500 \pm \sqrt{2{,}050{,}000}}{2} = \begin{cases} y_2 & (+) \\ y_1 & (-) \end{cases}$$

(b) $\frac{dy}{dt} = \frac{1}{1000}(1500 - y)y - 50$

$$= \frac{-1}{1000}\left[y^2 - 1500y + 50000\right] \qquad 0 < y_1 < y_2 < 15$$

$$= \frac{-1}{1000}(y - y_1)(y - y_2)$$

$$\frac{dy}{(y-y_1)(y-y_2)} = \frac{-1}{1000}\,dt$$

$$\int\left[\frac{1}{y_1-y_2}\frac{1}{y-y_1} + \frac{1}{y_2-y_1}\frac{1}{y-y_2}\right]dy = \int\frac{-1}{1000}\,dt + C$$

$$\frac{1}{y_1-y_2}\ln|y-y_1| + \frac{1}{y_2-y_1}\ln|y-y_2| = \left(\frac{-1}{1000}\right)t + C$$

$$\ln\frac{|y-y_2|}{|y-y_1|} = (y_2-y_1)\left(\frac{-1}{1000}t\right) + C$$

$$\ln\left|\frac{200-y_2}{200-y_1}\right| = (y_2-y_1)\cdot 0 + C$$

Solve for $y(t)$

here $y_1 < y(t) < y_2$

$$\frac{1}{(y-y_1)(y-y_2)} = \frac{A}{y-y_1} + \frac{B}{y-y_2}$$

$$1 = A(y-y_2) + B(y-y_1)$$

$$y = y_2 \rightarrow 1 = 0 + B(y_2 - y_1)$$

$$B = \frac{1}{y_2-y_1}$$

$$y = y_1 \Rightarrow 1 = A(y_1 - y_2) + 0$$

$$A = \frac{1}{y_1 - y_2}$$

$$\frac{dy}{dt}(0) = \frac{1}{1000}(1500 - 200)200 - 50$$

$$= \frac{260000}{1000} - 50$$

$$= 260 - 50 > 0$$

3. (a)

$$y'' + y' + y = 0$$

$$y = e^{rt} \longrightarrow r^2 + r + 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1}}{2}$$

$$= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$y_{homo} = c_1 e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

(b)

$$y_{part} = Ae^{-t}$$

$$y_{part}' = A(-1)e^{-t}$$

$$y_{part}'' = A(-1)^2 e^{-t}$$

$$Ae^{-t} + (-A)e^{-t} + Ae^{-t} = 2e^{-t}$$

$$(2A - A)\ e^{-t} =$$

$$\cancel{2}\ A = 2$$

$$y_{part} = 2e^{-t}$$

(c)

$$y = c_1 e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right) + 2e^{-t}$$

$$\longrightarrow y(0) = 0 = c_1 \cdot 1 + c_2 \cdot 0 + 2$$

$$y' = c_1\left[e^{-\frac{1}{2}t}\left(-\tfrac{1}{2}\right)\cos\left(\tfrac{\sqrt{3}}{2}t\right) + e^{-\frac{1}{2}t}\left(-\sin\left(\tfrac{\sqrt{3}}{2}t\right)\ \tfrac{\sqrt{3}}{2}\right)\right]$$

$$+ c_2\left[e^{-\frac{1}{2}t}\left(-\tfrac{1}{2}\right)\sin\left(\tfrac{\sqrt{3}}{2}t\right) + e^{-\frac{1}{2}t}\left(\cos\tfrac{\sqrt{3}}{2}t\ \ \tfrac{\sqrt{3}}{2}\right)\right]$$

$$+ 2e^{-t}(-1)$$

$$\longrightarrow y'(0) = 0 = c_1\left[-\tfrac{1}{2} + 0\right]$$

$$+ c_2\left[0 + \tfrac{\sqrt{3}}{2}\right]$$

$$\cancel{+}- 2$$

$$\rightarrow c_1 = -2 - c_2$$

$$\rightarrow 0 = \cancel{c_1}(-2 - c_2)\left(\tfrac{-1}{2}\right) + c_2\tfrac{\sqrt{3}}{2}\ \cancel{+}- 2$$

$$= 1 + \tfrac{1}{2}c_2 + c_2\tfrac{\sqrt{3}}{2}\ \cancel{+}- 2$$

$$c_2 = \frac{+1}{\frac{1}{2} + \frac{\sqrt{3}}{2}}$$

4. (a) $my'' + by' + ky = f(t)$

$2y'' + 0 \cdot y' + 32y = 10\cos(\omega t)$

$y'' + 16y = 5\cos(\omega t)$

(b) $y_{part} = A\cos\omega t + B\sin(\omega t)$

$y_{part}' = A\sin(\omega t)(-\omega) + B\cos(\omega t)\,\omega$

$y_{part}'' = A\cos(\omega t)(-\omega^2) + B\sin(\omega t)(-\omega^2)$

$y_{part}'' + 16 y_{part} = 5 \cdot \cos(\omega t) + 0 \cdot \sin\omega t$

$A(16-\omega^2)\cos(\omega t) + B(16-\omega^2)(\sin\omega t) =$

$A(16-\omega^2) = 5$ & $B(16-\omega^2) = 0$

$A = \frac{5}{16-\omega^2}$ $\quad B = 0$

$y_{part} = \frac{5}{16-\omega^2}\cos(\omega t)$, $\omega \neq 4$

(c) Pure resonance will occur

(i) use $\lim_{\omega \to 4}$ to find the part sol.

OR

(ii) use $y_{part} = A\,t\cos(4t) + B\,t\sin(4t)$

5. (a) $\mathcal{L}(y'' + 9y) = \mathcal{L}(e^{3t})$

$\mathcal{L}(y'') + 9\mathcal{L}(y) = \frac{1}{s-3}$

$s\mathcal{L}(y') - \underbrace{y'(0)}_{0} + 9\mathcal{L}(y) =$

$s[s\mathcal{L}(y) - y(0)] - 0 + 9\mathcal{L}(y) =$

$s^2\mathcal{L}(y) - s + 9\mathcal{L}(y) =$

$(s^2+9)\mathcal{L}(y) = \frac{1}{s-3} + s$

$\mathcal{L}(y) = \frac{1}{s^2+9}\,\frac{1}{s-3} + \frac{s}{s^2+9}$

(b) $\frac{1}{s^2+9}\,\frac{1}{s-3} = \frac{As+B}{s^2+9} + \frac{C}{s-3}$

$1 = (As+B)(s-3) + C(s^2+9)$

$0s^2+0s+1 = s^2(A+C) + s(+B-3A) + (-3B+9C)$

$0 = A+C$

$0 = +B - 3A \longrightarrow 0 = +B - 3(-C) = +B + 3C$

$1 = -3B + 9C \longrightarrow 1 = -3(-3C) + 9C = 18C$

$C = \frac{1}{18}, \quad A = -\frac{1}{18}, \quad B = -\frac{3}{18}$

$\mathcal{L}(y) = -\frac{1}{18}\,\frac{s}{s^2+9} - \frac{3}{18}\,\frac{1}{s^2+9} + \frac{1}{18}\,\frac{1}{s-3} + \frac{s}{s^2+}$

$= (1-\frac{1}{18})\underbrace{\frac{s}{s^2+9}}_{\mathcal{L}(\cos 3t)} - \frac{1}{18}\underbrace{\frac{3}{s^2+3^2}}_{\mathcal{L}(\sin 3t)} + \frac{1}{18}\underbrace{\frac{1}{s-3}}_{\mathcal{L}(e^{3t})}$

$y = \frac{17}{18}\cos(3t) - \frac{1}{18}\sin(3t) + \frac{1}{18}e^{3t}$

(a) $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 + 3t \\ -4 + 0\cdot t \end{bmatrix}, \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

(b) $\det \begin{bmatrix} 2-r & 1 \\ -3 & -2-r \end{bmatrix} = 0$

$(2-r)(-2-r) - (-3) =$

$r^2 - 4 + 3 =$

$(r-1)(r+1) = 0$

$\underline{r=1}$ $\begin{bmatrix} 2-1 & 1 \\ -3 & -2-1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$u_1 + u_2 = 0$

$-3u_1 - 3u_2 = 0$

$u_1 = 1 \longrightarrow u_2 = -1$

$\underline{r=-1}$ $\begin{bmatrix} 2+1 & 1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3u_1 + u_2 = 0$

$-3u_1 - u_2 = 0$

$u_1 = 1 \longrightarrow u_2 = -3$

(c) $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{1t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-1t}$

$= \underbrace{\begin{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t} & \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-t} \end{bmatrix}}_{X(t)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

(d) $\vec{f} = \begin{bmatrix} 0 \\ -4 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} t$

$\vec{X}_{part} = \vec{a} + \vec{b}\, t$

$\vec{X}_{part}' = A\, \vec{X}_{part} + \begin{bmatrix} 0 \\ -4 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} t$

$0 + \vec{b}\, 1 = A\vec{a} + A\vec{b}\, t + \begin{bmatrix} 0 \\ -4 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} t$

$\vec{0} = A\vec{b} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} \longrightarrow \vec{b} = -A^{-1}\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$= \_\_\_$

$\vec{b} = A\vec{a} + \begin{bmatrix} 0 \\ -4 \end{bmatrix}$

$A\vec{a} = \vec{b} - \begin{bmatrix} 0 \\ -4 \end{bmatrix} \longrightarrow \vec{a} = A^{-1}\left(\vec{b} - \begin{bmatrix} 0 \\ -4 \end{bmatrix}\right)$

$= \_\_\_$

(e) $\begin{bmatrix} x \\ y \end{bmatrix} = X(t)\, \vec{c} + \vec{a} + \vec{b}\, t$

$\begin{bmatrix} x(0) \\ y_0 \end{bmatrix} = X(0)\, \vec{c} + \vec{a} + \vec{b} \cdot 0$

$X(0)\, \vec{c} = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} - \vec{a} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} - \vec{a} \rightarrow \vec{c} = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}^{-1}\left(\begin{bmatrix} 5 \\ 10 \end{bmatrix} - \vec{a}\right)$