Extra Credit 3 in MA 341.002

Due Date is Tuesday, November 16.
How much extra credit? Up to 10 percent on exam three.

Consider the initial value problem for a coupled mass-spring system with friction force on both masses equal to b times the velocity, see section 5.5 for the no friction case. Consult the Matlab/Maple links for lectures 34-37 in the old syllabus:

http://www4/eos/users/w/white/www/white/ma341/ma341hp.htm

(a). State the coupled system of two second-order differential equations.

(b). By hand use the elimination method to find the solution when
$m_1 = 2$, $m_2 = 1$, $k_1 = 4$, $k_2 = 2$, $b = 10$ and the initial conditions are
$x(0) = -1$, $x'(0) = 0$, $y(0) = 0$ and $y'(0) = 0$.
This will generate a fourth order polynomial whose roots must be found. You can use the Matlab command *roots* to find these. Use *dsolve* to confirm the by hand calculations.

(c). State the corresponding coupled system of four first-order differential equations. Indicate the initial conditions. Indicate the matrix version of these equations.

(d). Use eigenvectors and eigenvalues to find the solution of the initial value problem. The Matlab command *[u d] = eig(A)* can be used to find the four eigenvectors and eigenvalues. This solution should agree with the solution in part (b).

(a) $m_1 x'' = -k_1 x + k_2(y-x) - b x'$

$m_2 y'' = \quad - k_2(y-x) - b y'$

$x(0) = -1 \qquad y(0) = 0$

$x'(0) = 0 \qquad y'(0) = 0$

(b) $2x'' + 6x - 2y + 10x' = 0$ ①

$y'' + 2y - x + 10y' = 0$ ②

① $\rightarrow$ $y = \dfrac{2x'' + 6x + 10x'}{2} = x'' + 3x + 5x'$

$\downarrow$

② $\rightarrow$ $(x'' + 3 + 5x')'' + 2(x'' + 3x + 5x')^{*}$

$- x' + 10(x'' + 3x + 5x')' = 0$

③ $x'''' + x''' 15 + x'' 55 + x' 40 + 4 = 0$

$x(0) = -1$ ① $\Rightarrow$ $y(0) = x''(0) + 3x(0) + 5x'(0)$

$x'(0) = 0$ $0 = x''(0) + 3(-1) + 5 \cdot 0$

$x''(0) = 3$

① $\rightarrow$ $y'(0) = x'''(0) + 3x'(0) + 5x''(0)$

$0 = x'''(0) + 3 \cdot 0 + 5 \cdot 3$

$x'''(0) = -15$

Assume $x = e^{rt}$.

③ $\rightarrow$ $e^{rt}(r^4 + 15r^3 + 55r^2 + 40r + 4) = 0$

$e^{rt} \neq 0 \rightarrow r^4 + 15r^3 + 55r^2 + 40r + 4 = 0$

Let r_1, r_2, r_3, r_4 be solutions.

$$X = c_1 e^{r_1 t} + c_2 e^{r_2 t} + c_3 e^{r_3 t} + c_4 e^{r_4 t}$$

$$y = x'' + 3x + 5x'$$

$$= c_1 (r_1^2 + 3 + 5r_1) e^{r_1 t} + c_2 (r_2^2 + 3 + 5r_2) e^{r_2 t} + c_3 (r_3^2 + 3 + 5r_3) e^{r_3 t} + c_4 (r_4^2 + 3 + 5r_3) e^{r_3 t}$$

$$R_i \equiv r_i^2 + 3 + 5r_i$$

$$\vec{X} = \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \\ c_1 r_1 \\ c_1 R_1 \\ c_1 R_1 r_1 \end{bmatrix} e^{r_1 t} + \begin{bmatrix} c_2 \\ c_2 r_2 \\ c_2 R_2 \\ c_2 R_2 r_2 \end{bmatrix} e^{r_2 t} + \begin{bmatrix} c_3 \\ c_3 r_3 \\ c_3 R_3 \\ c_3 R_3 r_3 \end{bmatrix} e^{r_3 t} + \begin{bmatrix} c_4 \\ c_4 r_4 \\ c_4 R_4 \\ c_4 R_4 r_4 \end{bmatrix} e^{r_4 t}$$

$$= \left[\begin{bmatrix} 1 \\ r_1 \\ R_1 \\ R_1 r_1 \end{bmatrix} e^{r_1 t} \quad \begin{bmatrix} 1 \\ r_2 \\ R_2 \\ R_2 r_2 \end{bmatrix} e^{r_2 t} \quad \begin{bmatrix} 1 \\ r_3 \\ R_3 \\ R_3 r_3 \end{bmatrix} e^{r_3 t} \quad \begin{bmatrix} 1 \\ r_4 \\ R_4 \\ R_4 r_4 \end{bmatrix} e^{r_4 t} \right] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

$$= \underline{\hat{X}}(t) \, \hat{c}$$

(c) $\vec{x} = \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}$

$$\vec{x}' = \begin{bmatrix} x' \\ x'' \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-(k_1+k_2)}{m_1} & \frac{-b}{m_1} & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & \frac{-k_2}{m_2} & \frac{-b}{m_2} \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & -5 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & -10 \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}$$

(d) $\vec{x} = c_1 \vec{u}_1 e^{r_1 t} + c_2 \vec{u}_2 e^{r_2 t} + c_3 \vec{u}_3 e^{r_3 t} + c_4 \vec{u}_4 e^{r_4 t}$

where $i = 1, \cdots, 4$

$A\vec{u}_i = r_i \vec{u}_i \quad , \vec{u}_i \neq \vec{0}$

$[u \ d] = eig(A) \quad , \overset{4\times4}{A} , \overset{4\times4}{u} , \overset{4\times4}{d}$

$\vec{u}_i = u(:,i) \qquad r_i = d(i,i)$

<u>OR</u>

$\vec{x}(t) = \overset{4\times4}{X(t)} \overset{4\times1}{\vec{c}}$

$\vec{x}(0) = X(0) \vec{c}$

$\vec{c} = X(0)^{-1} \vec{x}(0)$

$X(0) = [\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3 \ \vec{u}_4]$

$\vec{x}(0) = \begin{bmatrix} -1 \\ 0 \\ +3 \\ -15 \end{bmatrix}$

PART (b).

```
EDU>> r = roots([1 15 55 40 4])

r =

   -9.8000
   -4.2779
   -0.8033
   -0.1188

EDU>> dsolve('D4x+15*D3x+55*D2x+40*Dx+4*x=0','D3x(0)=-
15','D2x(0)=3','Dx(0)=0','x(0)=-1','t')

Warning: Explicit solution could not be found.
```

$x(t) = C_1\, e^{r(1)\, t} + C_2\, e^{r(2)\, t} + C_3\, e^{r(3)\, t} + C_4\, e^{r(4)\, t}$

PART (c).

```
EDU>> A = [0 1 0 0;-3 -5 1 0;0 0 0 1;2 0 -2 -10]

A =

     0     1     0     0
    -3    -5     1     0
     0     0     0     1
     2     0    -2   -10

EDU>> [u d] = eig(A)

u =

    0.0020   -0.2267    0.7309    0.3792
   -0.0199    0.9699   -0.5871   -0.0450
    0.1015    0.0202   -0.2713    0.9178
   -0.9946   -0.0863    0.2179   -0.1090


d =

   -9.8000         0         0         0
         0   -4.2779         0         0
         0         0   -0.8033         0
         0         0         0   -0.1188
```

```
EDU>> x0 = [-1 0  3 -15]'

x0 =

    -1
     0
     3
   -15

EDU>> c = inv(u)*x0

c =

   14.5647
   -1.0168
   -2.2507
    1.0151


EDU>> syms x t

EDU>> x = c(1)*u(:,1)*exp(d(1,1)*t) +...
          c(2)*u(:,2)*exp(d(2,2)*t) +...
          c(3)*u(:,3)*exp(d(3,3)*t) +...
          c(4)*u(:,4)*exp(d(4,4)*t)
```

COMPARE eigenvectors:

```
EDU>> x1 = u(:,1)/u(1,1)

x1 =

    1.0000
   -9.8000
   50.0400
 -490.3914
```

```
EDU>> x2 = u(:,2)/u(1,2)

x2 =

    1.0000
   -4.2779
   -0.0890
    0.3806



EDU>> x3 = u(:,3)/u(1,3)

x3 =

    1.0000
   -0.8033
   -0.3712
    0.2982

EDU>> x4 = u(:,4)/u(1,4)

x4 =

    1.0000
   -0.1188
    2.4202
   -0.2875

EDU>> r

r =

   -9.8000
   -4.2779
   -0.8033
   -0.1188

EDU>> x1hat =[1  r(1)  r(1)^2+3+5*r(1)  (r(1)^2+3+5*r(1))*r(1)]'

x1hat =

    1.0000
   -9.8000
   50.0400
```

```
 -490.3914

EDU>> x1

x1 =

    1.0000
   -9.8000
   50.0400
 -490.3914


EDU>> x2hat =[1  r(2)  r(2)^2+3+5*r(2)  (r(2)^2+3+5*r(2))*r(2)]'

x2hat =

    1.0000
   -4.2779
   -0.0890
    0.3806

EDU>> x2

x2 =

    1.0000
   -4.2779
   -0.0890
    0.3806

EDU>> x3hat =[1  r(3)  r(3)^2+3+5*r(3)  (r(3)^2+3+5*r(3))*r(3)]'

x3hat =

    1.0000
   -0.8033
   -0.3712
    0.2982

EDU>> x3

x3 =

    1.0000
   -0.8033
   -0.3712
```

```
    0.2982


EDU>> x4hat =[1  r(4)  r(4)^2+3+5*r(4)  (r(4)^2+3+5*r(4))*r(4)]'

x4hat =

    1.0000
   -0.1188
    2.4202
   -0.2875


EDU>> x4

x4 =

    1.0000
   -0.1188
    2.4202
   -0.2875
```