Lesson 11: Vibrating String (discrete string) and ode45

11.1 Applied Problem.

Consider the string under tension, T, and assume the ends of the string are fixed. If the string is plucked, then the displacement of string will be function of space and time, u(x,t). The problem is to approximate the displacement given the initial displacement and velocity. Partitioning the string into a finite number of segments can form a system of ordinary differential equations, whose solution can be approximated using the Matlab command ode45.

11.2 Differential Equation Model.

Partition the string into four linear segments as illustrated below. View string as three masses lumped at the intersections of segments and with displacements u_i(t) where i = 1, 2 and 3. The mass of each segment is $\rho \Delta x$ so that mass times acceleration is $\rho \Delta x u_i''$. There are three forces: external from the pressure, $\Delta x f(t)$, from the tension on the right, $T(u_{i+1} - u_i)/\Delta x$ and from the left, $-T(u_i - u_{i-1})/\Delta x$. From Newton’s law of motion we have

$$\rho \Delta x u_i'' = \Delta x f(t) + T(u_{i+1} - u_i)/\Delta x - T(u_i - u_{i-1})/\Delta x.$$ 

Divide by $\rho \Delta x$ and define $\alpha = (T/\rho)/(\Delta x)^2$ to get the system of three second order differential equations.
ODE Model for Discrete String.

\[ \begin{align*}
    u_i'' &= \frac{1}{\rho} f + \alpha(u_{i+1} + u_{i-1}) - 2\alpha u_i \quad \text{where} \\
    i &= 1,\ldots,n-1, \\
    u_i(0) &= 0 \quad \text{and} \quad u_i'(0) = 0 \quad \text{for} \quad i = 1,\ldots,n-1 \quad \text{and} \\
    u_0(t) &= u_n(t) = 0 \quad \text{for} \quad t > 0.
\end{align*} \]

Equation (2) is the initial displacement and velocity equal to zero and (3) is the displacement at the left and right ends equal to zero.

Equation (1) may be put into the matrix version of a system of ODEs. For example, if the string is divided into four equal parts, then \( n = 4 \) and (1) may be written as six scalar equations

\[ \begin{align*}
    u_1' &= u_4 \\
    u_2' &= u_5 \\
    u_3' &= u_6 \\
    u_4' &= u_1'' = \frac{1}{\rho c} f + \alpha(u_2 + 0) - 2\alpha u_1 \\
    u_5' &= u_2'' = \frac{1}{\rho c} f + \alpha(u_1 + u_3) - 2\alpha u_2 \\
    u_6' &= u_3'' = \frac{1}{\rho c} f + \alpha(0 + u_2) - 2\alpha u_3.
\end{align*} \]

Or, using a vector equation and a 6x6 matrix in
\[ u' + A \ u = b \] where
\[
\begin{bmatrix}
  u_1' \\
  u_2' \\
  u_3' \\
  u_4' \\
  u_5' \\
  u_6'
\end{bmatrix},
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
  u_4 \\
  u_5 \\
  u_6
\end{bmatrix},
\begin{bmatrix}
  \sin(9.6812 \times t) \\
  \sqrt{2} \sin(9.6812 \times t) \\
  \sin(9.6812 \times t) \\
  0 \\
  0 \\
  0
\end{bmatrix}
\] and
\[ A = \alpha \begin{bmatrix}
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 \\
  2 & -1 & 0 & 0 & 0 & 0 \\
 -1 & 2 & -1 & 0 & 0 & 0 \\
  0 & -1 & 2 & 0 & 0 & 0
\end{bmatrix} \]
with \( \alpha = (T/\rho)/(\Delta x)^2 \).

Another model uses partial derivatives with respect to time and space. Let \( u_{xx} \) be approximated by \( [T(u_{i+1} - u_i)/\Delta x] - [T(u_i - u_{i-1})/\Delta x]/\Delta x \).

**PDE Model for Discrete String.**

\[ \rho u_{tt} = f + Tu_{xx}, \] (a partial differential equation)
\[ u(x,0), u_t(x,0) = \text{given and} \] (displacement and velocity)
\[ u(0,t), u(1,t) = \text{given} \] (boundary conditions).

### 11.3 Method of Solution.

In this lesson we will use the Matlab command ode45 to solve our system of six differential equations. This command is a robust implementation for systems of differential equations, which uses a variable step size method and the fourth and fifth order Runge-Kutta method.
11.4 Matlab Implementation.

Since there are three unknowns and three second order differential equations and we wish to use Matlab's ode45 scheme, the file, ypstringode.m, must contain the three right sides of the differential equations where

\[ \begin{align*}
    y(1) &= \text{displacement of the left segment,} \\
    y(2) &= \text{displacement of the center segment and} \\
    y(3) &= \text{displacement of the right segment.} \\
    y(4) &= \text{velocity of the left segment,} \\
    y(5) &= \text{velocity of the center segment and} \\
    y(6) &= \text{velocity of the right segment.}
\end{align*} \]

The stringode.m file contains the call to ode45, and the Matlab command plot generates the graphs of the three displacements. The initial displacements are 1.0, 2.0, 2.0 and initial velocities are 0.0, 0.0, 0.0. They are stored in the uo array in the stringode.m file.

**m-files ypstringode.m**

```matlab
function ypstringode = ypstringode(t,u)
    T = 10; rho = 1.;
    L = 1.; dx = L/4; cc = (T/rho)/(dx*dx);
    f = -10*sin(9.6812*t)*[1 1.414 1 0 0 0]/rho;
    ypstringode(1) = u(4);
    ypstringode(2) = u(5);
    ypstringode(3) = u(6);
    ypstringode(4) = -cc*(2*u(1) - u(2)) + f(1);
    ypstringode(5) = -cc*(-u(1) + 2*u(2) - u(3)) + f(2);
    ypstringode(6) = -cc*(2*u(3) - u(2)) + f(3);
    ypstringode = [ypstringode(1) ypstringode(2) ypstringode(3)...
                   ypstringode(4) ypstringode(5) ypstringode(6)]';
```

**m-files stringode.m.**

```matlab
clear; clf
T = 10; rho = 1.;
L = 1.; dx = L/4; cc = (T/rho)/(dx*dx);
uo = [1 2 2 0 0 0];
to = 0;
```
tf = 8;
[t u] = ode45('ypstringode', [to tf], uo);
maxk = size(t, 1);
x = 0:.25:1;
figure(1)
for k = 1:1:maxk
    plot(x, [0 u(k, 1:3) 0])
    axis([0 1 -6 6])
    title('string at various times')
    xlabel('position on string')
    % hold on
    pause
end
A = [0 0 0 1 0 0;
     0 0 0 1 0;
     0 0 0 1;
     2*cc -cc 0 0 0;
     -cc 2*cc -cc 0 0;
     0 -cc 2*cc 0 0];
eigenvalues = eig(A)
figure(2)
plot(t, u(:, 1), t, u(:, 2), t, u(:, 3))
xlabel('time')

11.5 Numerical Experiments.

The first graphical output has multiple curves each associated with a particular
time as given in the numerical output

times =
eigenvalues =

0  -23.3725
0.3098  -17.8885
1.0076  -9.6812
1.7680  23.3725
2.5689  17.8885
3.3798  9.6812
4.2189
5.0934
5.9623
6.8554
7.7659
The amplitudes increase as time increases. This is also depicted in the second graph, which has the displacements of the three segments versus time. The increasing amplitudes occur because the forcing term is $f = -10 \cdot \sin(9.6812 \cdot t) \cdot [1 \ 1.414 \ 1 \ 0 \ 0 0]/\rho$ and has the same frequency as associated with the smallest positive eigenvalue. The third figure uses 8.6912 in $f$ (not 9.6812) and indicates the discrete string is not resonating.
11.6 Additional Calculations.

Consider the discrete sting model with rho = 1.0, T = 5.0, L = 1.0 and variable w in the external force term \( f = -10\sin(w\cdot t)[1 1.414 1 0 0 0]/\rho; \)

(a). State the system of differential equations.

(b). Modify the ypstringode.m and stringode.m files.

(c). For \( w = 6.8456 \) execute the stringode.m file. Note the eigenvalues have changed because the tension is now smaller than \( T = 10 \).

(d). Repeat (c) using \( w = 5.8456 \) and \( w = 7.8456 \)

(e). Examine the three sets of three curves and numerical outputs.

What happens as \( w \) varies from 6.8456?