

Differential Equation Model.

$$u_t = f(t,u) = c(usur(t) - u) \text{ with } u(0) \text{ given.}$$

In the heating of a body thermometer we wish to determine the time when $u(t)$ is very near $usur(t)$, the constant and unknown body temperature.

Method of Solution.

Use higher order Taylor polynomial approximations.

For example, the fourth order Taylor polynomial

$$u(t + h) = u(t) + u_t(t) h + u_{tt}(t) h^2/2 + u_{ttt}(t) h^3/6 + u_{tttt}(t) h^4/24 + u^{(5)}(C) h^5/120.$$

Compute the higher order derivatives by using the chain rule, for example,

$$\begin{aligned} u_{tt} &= f_t(t, u(t)) 1 + f_u(t, u(t)) u_t \\ &= f_t(t, u(t)) 1 + f_u(t, u(t)) f(t, u(t)). \end{aligned}$$

Repeating this will in general create a mess!

The advantage of this is we get a more accurate numerical solution, provided the solution has enough continuous derivatives. For example, consider the special case where u_{sur} is a constant

$$f(u) = c(u_{\text{sur}} - u).$$

Then the n^{th} derivative of u has the form $(-c)^{n-1}f(u)$.

This suggests the following algorithm

$$u^{k+1} = u^k + f(u^k) h - c f(u^k) h^2/2 + \\ c^2 f(u^k) h^3/6 - c^3 f(u^k) h^4/24.$$

This produces a fourth order error. In the following calculations note the Taylor error decreases by $1/16 = (1/2)^4$ when the time step decreases by $1/2$.

K (h = 10/K)	Euler Error	Taylor Error
5 (2.000)	1.3696	105.89 10 ⁻⁷
10 (1.000)	0.6693	6.4089 10 ⁻⁷
20 (0.500)	0.3309	0.3941 10 ⁻⁷
40 (0.250)	0.1645	0.0244 10 ⁻⁷

Matlab Implementation.

Use the function files ftay.m, ftayt.m, ftayu.m and the m-file tayde.m.

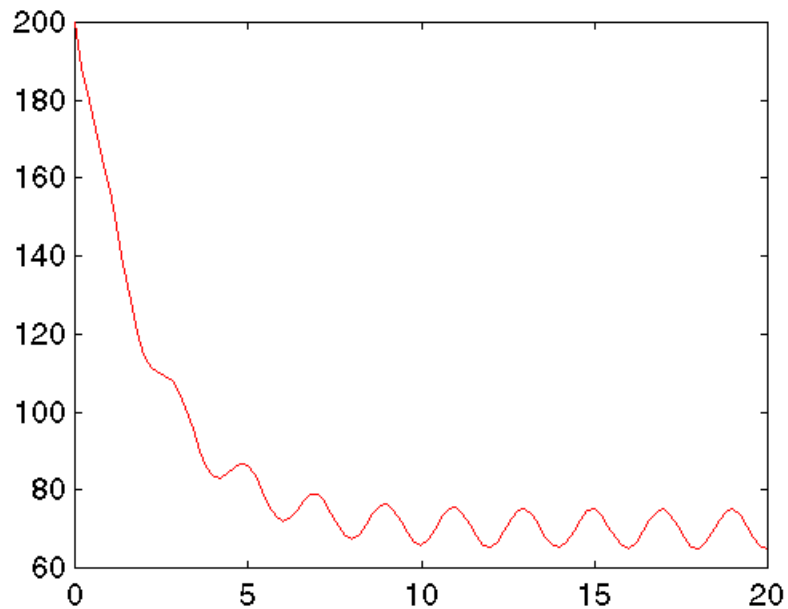
```
%your name, your student number,  
lesson number  
clear;  
y(1) = 70.;  
T = 20;  
KK = 100  
h = T/KK;  
t(1)= 0.;  
for k = 1:KK  
    k1 = ftay(t(k),y(k));  
    k2 = ftayt(t(k),y(k)) +  
        ftayu(t(k),y(k))*k1;  
    y(k+1) = y(k) +  
        h*(k1 + k2*h/2);  
    t(k+1) = t(k) + h;  
end  
plot(t,y)  
title('your name,your student  
number,lesson number')  
xlabel('time')  
ylabel('temperature')
```

```
function ftay = ftay(t,x)
    ftay = .5*(70.+
            30*sin(pi*t) - x);

function ftayt = ftayt(t,x)
    ftayt = .5*pi*30*cos(pi*t);

function ftayu = ftayu(t,x)
    ftayu = -.5;
```

The above is a numerical simulations of the temperature of a “cooling well stirred liquid” with oscillating surrounding temperature.



Consider the body temperature problem where

$$f(t,u) = c(102 - u) \text{ so that}$$

$$f_t = 0 \text{ and } f_u = -c.$$

We have assumed the body temperature is 102, the initial temperature is 70, and after 5 second the thermometer records 72.

The constant c in $f(t, u)$ can be found by using the improved Euler method:

$$(u(5) - u(0))/(5 - 0) = c (1/2) ((102 - u(0)) + (102 - u(5)))$$

$$(72 - 70)/(5 - 0) = c (1/2) ((102 - 70) + (102 - 72))$$

$$2/5 = c (1/2) (32 + 30)$$

$$c = 2/155.$$

your name, your student number, lesson number

