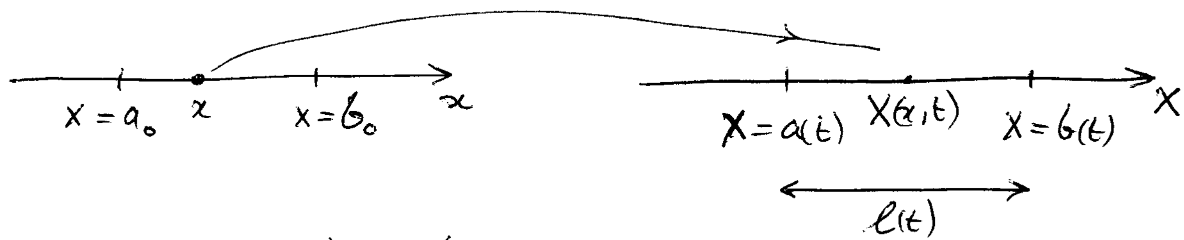
Compare pathlines of two points:

1-d. $\dfrac{dX}{dt} = u(X,t) = \dfrac{\partial}{\partial t} X(x,t)$



Let $l(t) = b(t) - a(t)$

$= \displaystyle\int_{a(t)}^{b(t)} 1 \, dX$ Subst. $X = X(x,t)$

$dX = \dfrac{\partial X}{\partial x} dx$

$= \displaystyle\int_{a_0}^{b_0} 1 \, \frac{\partial X}{\partial x} dx.$

$\therefore \dfrac{dl}{dt}$ = "strain rate" "stretch" $= \displaystyle\int_{a_0}^{b_0} \frac{\partial}{\partial t}\frac{\partial X}{\partial x} dx = \int_{a_0}^{b_0} \frac{\partial}{\partial x}\frac{\partial X}{\partial t} dx = \int_{a_0}^{b_0} \frac{\partial u}{\partial x} dx.$

$= \displaystyle\int_{a_0}^{b_0} \frac{\partial u}{\partial X}\frac{\partial X}{\partial x} dx. = \int_{a(t)}^{b(t)} \frac{\partial u}{\partial X}(X,t) \, dX$

$\dfrac{\partial u}{\partial X} = 0$: $l(t) =$ const. $\dfrac{\partial u}{\partial X} > 0 \Rightarrow l(t)$ increasing

$\displaystyle\lim_{l \to 0} \frac{1}{l}\frac{dl}{dt} = \frac{\partial u}{\partial X}(X,t).$

Question: what is result in 3-d?

3-d. Write $\underset{\sim}{X} = (\xi, \eta, \zeta)$ "xi, eta, zeta"

$\underset{\sim}{x} = (x, y, z)$. (see Chorin & Marsden)

strain tensor = "displacement gradient" = $\dfrac{\partial \underset{\sim}{X}}{\partial \underset{\sim}{x}}$

Jacobian $J = \det\left(\dfrac{\partial \underset{\sim}{X}}{\partial \underset{\sim}{x}}\right) := \begin{vmatrix} \xi_x & \eta_x & \zeta_x \\ \xi_y & \eta_y & \zeta_y \\ \xi_z & \eta_z & \zeta_z \end{vmatrix}$

$$= \xi_x \eta_y \zeta_z + \xi_y \eta_z \zeta_x + \xi_z \eta_x \zeta_y$$

$$- \xi_x \eta_z \zeta_y - \xi_y \eta_x \zeta_z - \xi_z \eta_y \zeta_x$$

multilinear in columns i.e. in ξ, η, ζ.

$$\boxed{\text{LEMMA} \quad \frac{\partial J}{\partial t}(x,t) = J(x,t) \, \mathrm{div}_{\underset{\sim}{x}} \, \underset{\sim}{u}(X(\underset{\sim}{x},t),t)}$$

Proof $\dfrac{\partial}{\partial t} J = \begin{vmatrix} \partial_t \xi_x & \eta_x & \zeta_x \\ \partial_t \xi_y & \eta_y & \zeta_y \\ \partial_t \xi_z & \eta_z & \zeta_z \end{vmatrix} + \begin{vmatrix} \xi_x & \partial_t \eta_x & \zeta_x \\ \xi_y & \partial_t \eta_y & \zeta_y \\ \xi_z & \partial_t \eta_z & \zeta_z \end{vmatrix} + \begin{vmatrix} \xi_x & \eta_x & \partial_t \zeta_x \\ \xi_y & \eta_y & \partial_t \zeta_y \\ \xi_z & \eta_z & \partial_t \zeta_z \end{vmatrix}$

Now $\partial_t \xi_x = \dfrac{\partial}{\partial x} \dfrac{\partial \xi}{\partial t} = \dfrac{\partial}{\partial x} u$ $\quad \{\underset{\sim}{u} = (u, v, w)\}$

etc. $\partial_t \eta_y = \dfrac{\partial}{\partial y} v \; \ldots$

Then $\dfrac{\partial J}{\partial t} = \begin{vmatrix} u_x & \eta_x & \zeta_x \\ u_y & \eta_y & \zeta_y \\ u_z & \eta_z & \zeta_z \end{vmatrix} + \begin{vmatrix} \xi_x & v_x & \zeta_x \\ \xi_y & v_y & \zeta_y \\ \xi_z & v_z & \zeta_z \end{vmatrix} + \begin{vmatrix} \xi_x & \eta_x & w_x \\ \xi_y & \eta_y & w_y \\ \xi_z & \eta_z & w_z \end{vmatrix}$

But $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial x} + \dfrac{\partial u}{\partial \zeta}\dfrac{\partial \zeta}{\partial x}$. (chain rule)

since $u = u(\underset{\sim}{X}, t)$ $\quad \underset{\sim}{X} = (\xi, \eta, \zeta) = \underset{\sim}{X}(\underset{\sim}{x}, t)$

1st det: use linearity.

$$\begin{vmatrix} u_x & \eta_x & \zeta_x \\ u_y & \eta_y & \zeta_y \\ u_z & \eta_z & \zeta_z \end{vmatrix} = \begin{vmatrix} u_\xi \xi_x & \eta_x & \zeta_x \\ u_\xi \xi_y & \eta_y & \zeta_y \\ u_\xi \xi_z & \eta_z & \zeta_z \end{vmatrix} + \underbrace{\begin{vmatrix} u_\eta \eta_x & \eta_x & \zeta_x \\ u_\eta \eta_y & \eta_y & \zeta_y \\ u_\eta \eta_z & \eta_z & \zeta_z \end{vmatrix}}_{0}$$

$$+ \underbrace{\begin{vmatrix} u_\zeta \zeta_x & \eta_x & \zeta_x \\ u_\zeta \zeta_y & \eta_y & \zeta_y \\ u_\zeta \zeta_z & \eta_z & \zeta_z \end{vmatrix}}_{0}$$

$$= u_\xi J$$

$\therefore$ sum: $\quad \underline{\dfrac{\partial J}{\partial t}} = u_\xi J + v_\eta J + w_\zeta J$

$$= \underline{\underline{J \operatorname{div} \underset{\sim}{u}}}. \qquad \#$$