

TORI INVARIANT UNDER AN INVOLUTORIAL AUTOMORPHISM II

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Abstract. The geometry of the orbits of a minimal parabolic k -subgroup acting on a symmetric k -variety is essential in several area's, but its main importance is in the study of the representations associated with these symmetric k -varieties (see for example [6], [5], [20] and [31]). Up to an action of the restricted Weyl group of G , these orbits can be characterized by the H_k -conjugacy classes of maximal k -split tori, which are stable under the k -involution θ associated with the symmetric k -variety. Here H is a open k -subgroup of the fixed point group of θ .

This is the second in a series of papers in which we characterize and classify the H_k -conjugacy classes of maximal k -split tori. The first paper in this series dealt with the case of algebraically closed fields. In this paper we lay the foundation for a characterization and classification for the case of non algebraically closed fields. This includes a partial classification in the cases, where the base field is the real numbers, p -adic numbers, finite fields and number fields.

Introduction

Let G be a connected reductive linear algebraic group defined over a field k of characteristic not 2, θ an involution of G defined over k , $G_\theta = \{g \in G \mid \theta(g) = g\}$ the set of fixed points of θ and H a k -open subgroup of G_θ . Denote the set of k -rational points of G (resp. H) by G_k (resp. H_k). The variety G/H is called a symmetric variety and the variety G_k/H_k a symmetric k -variety.

Let P be a minimal parabolic k -subgroup of G and consider the double coset space $H_k \backslash G_k / P_k$. The geometry of these orbits is of fundamental importance in the representation theory of symmetric k -varieties. Examples include the study of Harish-Chandra modules with $k = \mathbb{C}$ (see for example [31]), the study of discrete series of affine symmetric spaces with $k = \mathbb{R}$ (see for example [20]) and the study of symmetric spaces over finite fields (see for example [17]). These

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orbits also play an equally important role in the relatively new area of representation theory of p -adic symmetric spaces (see [14]). Besides in representation theory, these orbits are also of importance in a number of other areas, including geometry (see [21] and [18]) and the cohomology of arithmetic subgroups (see [30]).

The double cosets $P_k \backslash G_k / H_k$ are characterized, up to the action of the restricted Weyl group of G , by the H_k -conjugacy classes of θ -stable maximal k -split tori. To be able to tackle the problems about the geometry of these double cosets, which arise in the representation theory of the corresponding symmetric k -varieties, a detailed characterization and possibly a classification of the H_k -conjugacy classes of θ -stable maximal k -split tori is needed.

This is the second in a series of papers, in which we characterize and classify the H_k -conjugacy classes of θ -stable maximal k -split tori for a number of different base fields, including algebraically closed fields, the real numbers, the p -adic numbers, finite fields and number fields. In the first paper in this series we dealt with the case of algebraically closed fields and we also proved most of the results about conjugacy classes of involutions in Weyl groups, which will be needed in the other papers of this series. In this second paper we will do a lot of the ground work for the remainder of this series. We will first show that this classification can be split into 2 problems, one of which is a classification of the H -conjugacy classes of the θ -stable maximal (quasi) k -split tori. We give a characterization of these conjugacy classes and also give a partial classification in the cases where the base field is the real numbers, p -adic numbers, finite fields and number fields.

In the next paper we will apply the results of this paper and give a classification of the H_k -conjugacy classes of θ -stable maximal k -split tori for $k = \mathbb{R}$. A classification for $k = \mathbb{Q}_p$ and k finite, will be dealt with in papers after that.

For $k = \mathbb{R}$ a characterization of the H_k -conjugacy classes of θ -stable maximal k -split tori was given by Matsuki in [19]. He characterizes the conjugacy classes in terms of systems of q -orthogonal sets of roots. This characterization only holds for real groups. Moreover it is quite technical and an actual classification, using this characterization, seems infeasible. In this paper we take a different approach to the classification problem, which leads to a more detailed characterization of the H_k -conjugacy classes, which actually holds for arbitrary base fields. Using this characterization and some additional results, specifically for the basefield in question, it is finally possible to classify the conjugacy classes in a number of cases, including $k = \mathbb{R}$. For more details, see [13]. The themes and results of this paper follow.

Let \mathcal{A}_k^θ be the set of θ -stable maximal k -split tori of G . The group H_k acts on \mathcal{A}_k^θ by conjugation and the orbit set \mathcal{A}_k^θ/H_k is the set of H_k -conjugacy classes θ -stable maximal k -split tori. If $A \in \mathcal{A}_k^\theta$ then we write $Z = Z_G(A)$ for the centralizer of A in G , $N = N_G(A)$ for the normalizer of A in G and $W = W(A) = N_G(A)/Z_G(A)$ the corresponding Weyl group. Let $\mathcal{V}_k = \{x \in G_k \mid x^{-1}\theta(x) \in N\}$ and $\mathcal{V} = \{x \in G \mid x^{-1}\theta(x) \in N\}$. Then $\mathcal{A}_k^\theta = \{xAx^{-1} \mid x \in \mathcal{V}_k\}$. Let V_k be the set of $H_k \times Z_k$ orbits on \mathcal{V}_k and V the set of $H \times Z$ orbits on \mathcal{V} . In [15, §6] it was shown that the set V_k characterizes the double cosets $H_k \backslash G_k / P_k$ and there is a bijection

$$(1) \quad \gamma_k : V_k/W \rightarrow \mathcal{A}_k^\theta/H_k.$$

Similarly the set V is related to the orbits $H \backslash G / P$ and there is a bijection $\gamma : V/W \rightarrow \mathcal{A}^\theta/H$. Here $\mathcal{A}^\theta = \{xAx^{-1} \mid x \in \mathcal{V}\}$, the set of θ -stable maximal quasi k -split tori of G . There is a natural map $\zeta : \mathcal{A}_k^\theta/H_k \rightarrow \mathcal{A}^\theta/H$, sending each H_k conjugacy class of a θ -stable maximal k -split torus onto its H conjugacy class. Its image \mathcal{A}_0^θ/H consists of the H -conjugacy classes θ -stable maximal k -split tori.

To classify \mathcal{A}_k^θ/H_k it suffices to classify the image and fibers of ζ . In some cases, including the standard pairs for $k = \mathbb{R}$, this map is actually one to one as we will prove in the following result.

Corollary 6.10. *Let $A_0 \in \mathcal{A}_0^\theta$ be a θ -stable maximal k -split torus of G with A_0^- a maximal (θ, k) -split torus of G and assume $W(A_0^-)$ has representatives in H_k . Then the map $\zeta : \mathcal{A}_k^\theta/H_k \rightarrow \mathcal{A}^\theta/H$ is one-to-one.*

To classify the image of ζ we will first need a classification of \mathcal{A}^θ/H . In this paper we will give characterizations of both \mathcal{A}^θ/H and \mathcal{A}_0^θ/H . The characterization of \mathcal{A}^θ/H for arbitrary base fields appears to be much more complicated than in the case of algebraically closed fields, i.e., the characterization of H -conjugacy classes of θ -stable maximal tori.

There is a natural conjugacy class of involutions associated with the H -conjugacy classes in \mathcal{A}^θ and \mathcal{A}_k^θ . Namely if $v \in V$, $x = x(v) \in \mathcal{V}$ a representative, $x^{-1}\theta(x) \in N$, then the corresponding Weyl group element $w_v \in W$ is a twisted involution, i.e. $\theta(w_v) = w_v^{-1}$. This gives a map $\varphi : V \rightarrow \mathcal{I}$, where $\mathcal{I} = \{w \in W \mid \theta(w) = w^{-1}\}$ is the set of twisted involutions in W . The Weyl group W acts on \mathcal{I} and every W -orbit contains an involution (see [15, §7]). In section 4 and 5 we show how one can choose a suitable representative $x(v) \in \mathcal{V}$, such that w_v becomes an involution. If $v \in V$ then we call these involutions θ -singular and if $v \in V_k$, then we will also call these involutions (θ, k) -singular.

The map $\varphi : V \rightarrow \mathfrak{L}$ is equivariant with respect to the actions of W on V , V_k and \mathfrak{L} . Let $\phi : V/W \rightarrow \mathfrak{L}/W$ and $\phi_k : V_k/W \rightarrow \mathfrak{L}/W$ be the corresponding orbit maps. In the case that k is an algebraically closed field the map ϕ is one-to-one. So together with the bijection γ_k in (1) we get a one-to-one correspondence between the H -conjugacy classes of θ -stable maximal tori and the W -conjugacy classes of θ -singular involutions (see [10]). Unfortunately, for arbitrary fields, the map ϕ is not always one-to-one. So the W -conjugacy classes of the θ -singular involutions only provide a first invariant to characterize the conjugacy classes in \mathcal{A}^θ (respectively \mathcal{A}_k^θ) and a characterization of the fibers of ϕ is also needed. These can also be described in terms of Weyl group elements of a maximal torus containing the maximal k -split torus. This characterization of the fibers is discussed in section 7. This characterization of the fibers is not yet very detailed. In practice it remains quite difficult to calculate the fibers in specific examples. A conjecture with a much more detailed description of these fibers is discussed in 8.34. Fortunately in many cases, including the standard pairs for $k = \mathbb{R}$, the map ϕ_k is actually one-to-one, so in these cases the fibers are trivial. This follows from the following result (see also 8.11):

Corollary 8.7. *Let $A_0 \in \mathcal{A}_0^\theta$ be a θ -stable maximal k -split torus of G with A_0^- a maximal (θ, k) -split torus of G and assume that $W(A_0^-)$ has representatives in H_k . Then there exists a one to one correspondence between the H_k -conjugacy classes of θ -stable maximal k -split tori and the $W(A_0)$ -conjugacy classes of (θ, k) -singular involutions of $W(A_0)$.*

A characterization of the image of ϕ in terms of θ -singular involutions is given in section 5 and a characterization of the image of ϕ_k in terms of (θ, k) -singular involutions is given in section 6. It then remains to classify the W -conjugacy classes of the θ -singular (respectively (θ, k) -singular) involutions of W . For an arbitrary θ -stable maximal k -split torus A_0 it is difficult to classify the θ -singular involutions, but in the case that A_0 contains a maximal (θ, k) -split torus of G we get a characterization similar to that in [10, 4.6]:

Theorem 8.16. *Let A_0 be a θ -stable maximal k -split torus of G with A_0^- a maximal (θ, k) -split torus of G and $w \in W(A_0)$, $w^2 = e$. Then the following are equivalent:*

- (i) w is θ -singular
- (ii) $(A_0)_w^- \subset A_0^-$

Finally, a classification of the W -conjugacy classes of the θ -singular involutions in the cases that k is algebraically closed, the real numbers, the p -adic

numbers and number fields is given in section 9. This includes a classification of the natural diagram of the conjugacy classes.

A brief summary of the contents is as follows. In section 1 we set the notation and discuss the relation between the orbits of a minimal parabolic k -subgroup acting on the symmetric k -variety and the H_k (resp. H -conjugacy) classes of θ -stable quasi k -split tori. After that in section 2 we give a number of results about the action of the Galois group and the involution on the root lattice. These results are needed in later sections. Section 3 discusses quasi k -split tori and their corresponding quasi parabolic k -subgroups. In section 4 we show that the classification of the conjugacy classes in \mathcal{A}_k^θ can be reduced to conjugacy classes of standard k -split tori. To these we can associate a natural involution in the Weyl group. In section 5 we prove a similar result for the θ -stable maximal quasi k -split tori and show that the corresponding standard involutions in the Weyl group are an invariant for the H -conjugacy classes in \mathcal{A}^θ . After that in section 6 we give a characterization of $\zeta(\mathcal{A}_k^\theta)/H$ and similarly in section 7, a characterization of \mathcal{A}^θ/H . In section 8 we give another description of the standard involutions, which occur in the characterizations of \mathcal{A}^θ/H and \mathcal{A}_k^θ/H . Finally, in section 9 we classify the W -conjugacy classes of the θ -singular involutions for k algebraically closed, the real numbers, the p -adic numbers and number fields.

Some of these results were announced in [11] and [12].

1. Preliminaries and Recollections

In this section we set the notations and discuss the relation between the orbits of minimal parabolic k -subgroup acting on a symmetric k -variety and the H - (resp. H_k)-conjugacy classes of θ -stable maximal (quasi) k -split tori. For this we will rephrase the characterization of these orbits in [15, 8] by giving another characterization of the orbits, which is geared more toward the conjugacy classes of θ -stable maximal (quasi) k -split tori. We will also prove a number of additional results. Our basic reference for reductive groups will be the papers of Borel and Tits [1, 2] and also the books of Humphreys [16] and Springer [26]. We shall follow their notations and terminology.

1.1. Notations. Given an algebraic group G , the identity component is denoted by G^0 . We use $L(G)$ (resp. \mathfrak{g} , the corresponding lower case German letter) for the Lie algebra of G . If H is a subset of G , then we write $N_G(H)$ (resp. $Z_G(H)$) for the normalizer (resp. centralizer) of H in G . We write $Z(G)$ for the center of G . The commutator subgroup of G is denoted by $D(G)$ or $[G, G]$.

An algebraic group defined over k shall also be called an algebraic k -group. For an extension K of k , the set of K -rational points of G is denoted by G_K or $G(K)$.

If G is a reductive k -group and A a torus of G then we denote by $X^*(A)$ (resp. $X_*(A)$) the group of characters of A (resp. one-parameter subgroups of A) and by $\Phi(A) = \Phi(G, A)$ the set of the roots of A in G . The group $X^*(A)$ can be put in duality with $X_*(A)$ by a pairing $\langle \cdot, \cdot \rangle$ defined as follows: if $\chi \in X^*(A)$, $\lambda \in X_*(A)$, then $\chi(\lambda(t)) = t^{\langle \chi, \lambda \rangle}$ for all $t \in k^*$. Let $W(A) = W(G, A) = N_G(A)/Z_G(A)$ denote the Weyl group of G relative to A . If $\alpha \in \Phi(G, A)$, then let U_α denote the unipotent subgroup of G corresponding to α . If A is a maximal torus, then U_α is one-dimensional. Given a quasi-closed subset ψ of $\Phi(G, A)$, the group G_ψ (resp. G_ψ^*) is defined in [1, 3.8]. If G_ψ^* is unipotent, ψ is said to be unipotent and often one writes U_ψ for G_ψ^* .

If A is a torus of G such that $\Phi(A)$ is a root system in the subspace of $X^*(A) \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by $\Phi(A)$ and if $W(A)$ is the corresponding Weyl group, then for each $\alpha \in \Phi(A)$ the subgroup $G_\alpha = Z_G((\ker \alpha)^0)$ is nonsolvable. If we choose now $n_\alpha \in N_{G_\alpha}(A) - Z_{G_\alpha}(A)$ and let s_α be the element of $W(A)$ defined by n_α , then there exists a unique one parameter subgroup $\alpha^\vee \in X_*(A)$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and $s_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha$ ($\chi \in X^*(A)$). We call α^\vee the *coroot* of α and denote the set of these α^\vee in $X_*(A)$ by $\Phi^\vee(A)$.

1.2. Throughout the paper G will denote a connected reductive algebraic k -group, θ an involution of G defined over k , $G_\theta = \{g \in G \mid \theta(g) = g\}$ the set of fixed points of θ and H a k -open subgroup of G_θ . The involution θ is also called a k -involution of G . The variety G/H is called a symmetric variety and the variety G_k/H_k is called a symmetric k -variety.

Given $g, x \in G$, the *twisted action* associated to θ is given by $(g, x) \mapsto g * x = gx\theta(g)^{-1}$. Let $Q = \{g^{-1}\theta(g) \mid g \in G\}$ and $Q' = \{g \in G \mid \theta(g) = g^{-1}\}$. The set Q is contained in Q' . Both Q and Q' are invariant under the twisted action associated to θ . There are only a finite number of twisted G -orbits in Q' and each such orbit is closed (see [22]). In particular, Q is a connected closed k -subvariety of G . Define a morphism $\tau : G \rightarrow G$ by

$$(1.1) \quad \tau(x) = x\theta(x^{-1}), \quad (x \in G).$$

The image $\tau(G) = Q$ is a closed k -subvariety of G and τ induces an isomorphism of the coset space G/G_θ onto $\tau(G)$. Note that $\tau(x) = \tau(y)$ if and only if $y^{-1}x \in G_\theta$ and $\theta(\tau(x)) = \tau(x)^{-1}$ for $x \in G$.

1.3. If $T \subset G$ is a torus and $\sigma \in \text{Aut}(G, T)$ an involution, then we write $T_\sigma^+ = (T \cap G_\sigma)^0$ and $T_\sigma^- = \{x \in T \mid \sigma(x) = x^{-1}\}^0$. It is easy to verify that the product

map

$$\mu : T_\sigma^+ \times T_\sigma^- \rightarrow T, \quad \mu(t_1, t_2) = t_1 t_2$$

is a separable isogeny. In particular $T = T_\sigma^+ T_\sigma^-$ and $T_\sigma^+ \cap T_\sigma^-$ is a finite group. (In fact it is an elementary abelian 2-group.) The automorphisms of $\Phi(G, T)$ and $W(G, T)$ induced by σ will also be denoted by σ . If $\sigma = \theta$ we reserve the notation T^+ and T^- for T_θ^+ and T_θ^- respectively. For other involutions of T , we shall keep the subscript.

Recall from [9] that a torus A is called θ -split if $\theta(a) = a^{-1}$ for every $a \in A$. If A is a maximal θ -split torus of G , then $\Phi(G, A)$ is a root system with Weyl group $W(A) = N_G(A)/Z_G(A)$ (see [22]). This is the root system associated with the symmetric variety G/H . To the symmetric k -variety G_k/H_k one can also associate a natural root system. To see this we consider the following tori:

Definition 1.4. A k -torus A of G is called (θ, k) -split if it is both θ -split and k -split.

Consider a maximal (θ, k) -split torus A in G . In [15, 5.9] it was shown that $\Phi(G, A)$ is a root system and $N_{G_k}(A)/Z_{G_k}(A)$ is the Weyl group of this root system. We can also obtain this root system by restricting the root system of G_k . Namely let $A^0 \supset A$ be a θ -stable maximal k -split torus of G . Then $A = (A^0)_\theta^-$ and $\Phi(G, A)$ can be identified with $\overline{\Phi}_\theta = \{\alpha | A \neq 0 \mid \alpha \in \Phi(G, A^0)\}$.

1.5. P_k -orbits on G_k/H_k . Let P be a minimal parabolic k -subgroup of G . There are several way's in which one can characterize the double cosets $P_k \backslash G_k / H_k$. One can characterize them as the P_k -orbits on the symmetric k -variety G_k/H_k (using the θ -twisted action), one can take the H_k -orbits on the flag variety G_k/P_k or one can consider the $P_k \times H_k$ -orbits on G_k . All these characterizations are essentially the same. For more details see [15]. We will use the $P_k \times H_k$ -orbits on G_k to characterize $P_k \backslash G_k / H_k$. In the following we will first review briefly this characterization.

Let A be a θ -stable maximal k -split torus of P , $N = N_G(A)$, $Z = Z_G(A)$ and $W = W(A) = N_G(A)/Z_G(A)$ the corresponding Weyl group. As in [15, 6.7] set $\mathcal{V}_k = \{x \in G_k \mid \tau(x) \in N_k\}$. The group $Z_k \times H_k$ acts on \mathcal{V}_k by $(x, z) \cdot y = xyz^{-1}$, $(x, z) \in Z_k \times H_k$, $y \in \mathcal{V}_k$. Let V_k be the set of $(Z_k \times H_k)$ -orbits on \mathcal{V}_k . Similarly let $\mathcal{V} = \{x \in G \mid \tau(x) \in N\}$. Then $Z \times H$ acts on \mathcal{V} by $(x, z) \cdot y = xyz^{-1}$, $(x, z) \in Z \times H$, $y \in \mathcal{V}$. Denote the set of $(Z \times H)$ -orbits on \mathcal{V} by V . If $v \in V_k$ (resp. V), we let $x(v) \in \mathcal{V}_k$ (resp. V) be a representative of the orbit v in \mathcal{V}_k (resp. V). These sets V_k (resp. V) are essential in the study of orbits of minimal parabolic subgroups on the symmetric k -variety G_k/H_k .

The inclusion map $\mathcal{V}_k \rightarrow G_k$ induces a bijection of the set V_k of $(Z_k \times H_k)$ -orbits on \mathcal{V}_k onto the set of $(P_k \times H_k)$ -orbits on G_k (see [15]). Similarly the set V is associated with the set of $H \times P$ orbits on G (see [13]). The set V is finite, but the set V_k is in general infinite. In a number of cases one can show that there are only finitely many $(P_k \times H_k)$ -orbits on G_k . If k is algebraically closed, the finiteness of V_k was proved by Springer [27]. The finiteness of the orbit decomposition for $k = \mathbb{R}$ was discussed by Wolf [33], Rossmann [24] and Matsuki [19]. For general local fields this result can be found in Helminck-Wang [15]. An example that in most cases the set V_k is infinite can be found in [15, 6.12].

1.6. W-action on V and V_k . The Weyl group W acts on both V and V_k . The action of W on V (resp. V_k) is defined as follows. Let $v \in V$ and let $x = x(v)$. If $n \in N$, then $nx \in \mathcal{V}$ and its image in V depends only on the image of n in W . We thus obtain a (left) action of W on V , denoted by $(w, v) \rightarrow w \cdot v$ ($w \in W, v \in V$).

To get another description of \mathcal{V}_k and \mathcal{V} we define first the following.

Definition 1.7. A torus A of G is called a *quasi k -split torus* if A is conjugate under G with a k -split torus of G .

Since all maximal k -split tori of G are conjugate, also all maximal quasi k -split tori of G are conjugate. If A is a maximal quasi k -split torus of G , then $\Phi(G, A)$ is a root system in $(X^*(A), X_0)$ in the sense of [1, §2.1] where X_0 is the set of characters of A which are trivial on $(A \cap [G, G])^0$; moreover the Weyl group of $\Phi(G, A)$ is $N_G(A)/Z_G(A)$.

1.8. Let \mathcal{A} be the variety of maximal quasi k -split tori of G . This is an affine variety, isomorphic to $G/N_G(A)$, on which θ acts. Let \mathcal{A}^θ be the fixed point set of θ , i.e. the set of θ -stable maximal quasi k -split tori. It is an affine variety on which H acts by conjugation.

Similarly let \mathcal{A}_k denote the set of maximal k -split tori of G and let \mathcal{A}_k^θ be the fixed point set of θ i.e., the set of θ -stable maximal k -split tori. The group H_k acts on \mathcal{A}_k^θ by conjugation.

If $x \in \mathcal{V}_k$, then $x^{-1}Ax$ is again a maximal k -split torus and conversely any θ -stable maximal k -split torus in \mathcal{A}_k^θ can be written as $x^{-1}Ax$ for some $x \in \mathcal{V}_k$. Similarly any θ -stable maximal quasi k -split torus is of the form $x^{-1}Ax$ for some $x \in \mathcal{V}$.

If $v \in V_k$ (resp. $v \in V$), then $x(v)^{-1}Ax(v) \in \mathcal{A}_k^\theta$ (resp. \mathcal{A}^θ). This determines maps of V_k , resp. V to the orbit sets \mathcal{A}_k^θ/H_k resp. \mathcal{A}^θ/H . It is easy to check that

these maps are independent of the choice of the representative $x(v)$ for v and are constant on W -orbits. So we also get maps of orbit sets: $\gamma_k : V_k/W \rightarrow \mathcal{A}_k^\theta/H_k$ and $\gamma : V/W \rightarrow \mathcal{A}^\theta/H$. In fact we have a bijection:

Proposition 1.9. *Let G , \mathcal{A}^θ , \mathcal{A}_k^θ , γ and γ_k be as above. Then we have the following.*

- (i) $\gamma_k : V_k/W \rightarrow \mathcal{A}_k^\theta/H_k$ is bijective.
- (ii) $\gamma : V/W \rightarrow \mathcal{A}^\theta/H$ is bijective.

Proof. (i). Since any θ -stable maximal k -split torus is of the form $x^{-1}Ax$ for some $x \in \mathcal{V}_k$, the map γ_k is surjective.

Let $v_1, v_2 \in V_k$ with $\gamma_k(v_1) = \gamma_k(v_2)$. Let $x_1 = x(v_1), x_2 = x(v_2) \in \mathcal{V}_k$ be representatives of v_1, v_2 and let $A_1 = x_1^{-1}Ax_1, A_2 = x_2^{-1}Ax_2$. Since $\gamma_k(v_1) = \gamma_k(v_2)$ there exists $h \in H_k$ such that $hA_2h^{-1} = hx_2^{-1}Ax_2h^{-1} = x_1^{-1}Ax_1 = A_1$. So we may assume that $A_1 = A_2$. But then by [15, 6.10] there exists $n \in N_G(A_1)$ such that $P_k n x_1 H_k = P_k x_2 H_k$. If $w \in W(A_1)$ is the image of n in $W(A_1)$, then $w \cdot v_1 = v_2$, what proves (i).

The proof of (ii) follows using a similar argument as in (i). \square

1.10. Let \mathcal{A}_0^θ denote the set of θ -stable quasi k -split tori of G , which are H -conjugate to a θ -stable maximal k -split torus. Then $\mathcal{A}_0^\theta/H \subset \mathcal{A}^\theta/H$ can be identified with the set of H -conjugacy classes of θ -stable maximal k -split tori of G . There is a natural map

$$(1.2) \quad \zeta : \mathcal{A}_k^\theta/H_k \rightarrow \mathcal{A}^\theta/H,$$

sending the H_k -conjugacy class of a θ -stable maximal k -split torus onto its H -conjugacy class. Then \mathcal{A}_0^θ/H is precisely the image of ζ . On the other hand the inclusion map $\mathcal{V}_k \rightarrow \mathcal{V}$ induces a map $\eta : V_k \rightarrow V$, where η maps the orbit $Z_k g H_k$ onto ZgH . This map is W -equivariant. Denote the corresponding orbit map by $\delta : V_k/W \rightarrow V/W$ and write V_0 for the image of η in V . Denote the restriction of γ to V_0/W by γ_0 . Then γ_0 maps V_0/W onto \mathcal{A}_0^θ/H . This all leads to the following diagram:

$$\begin{array}{ccc} V_k/W & \xrightarrow{\gamma_k} & \mathcal{A}_k^\theta/H_k \\ \downarrow \delta & & \downarrow \zeta \\ V_0/W & \xrightarrow{\gamma_0} & \mathcal{A}_0^\theta/H \end{array}$$

Since γ_0 and γ_k are bijections, it follows that there is a bijection between the fibers of δ and ζ . For a θ -stable maximal k -split torus A , the fiber $\zeta^{-1}(A)$ consists of all θ -stable maximal k -split tori, which are H -conjugate to A , but not

H_k -conjugate. That these fibers can be infinite can be seen from example [15, 6.12].

Another way to characterize the orbits in \mathcal{A}_0^θ/H and \mathcal{A}_k^θ/H_k is by using the natural map from V into W , induced by the map $\tau|\mathcal{V} : \mathcal{V} \rightarrow N_G(T)$. In the next subsection we analyze this map in a bit more detail.

1.11. Twisted involutions. Recall that an element $a \in W(A)$ is a *twisted involution* if $\theta(a) = a^{-1}$ (see [27, §3] or [15, §7]). Let

$$(1.3) \quad \mathfrak{I} = \mathfrak{I}_\theta = \mathfrak{I}(W(A), \theta) = \{w \in W \mid \theta(w) = w^{-1}\}$$

be the set of twisted involutions in $W(A)$. If $v \in V$, then $\varphi(v) = \tau(x(v))Z_G(A) \in W(A)$ is a twisted involution. The element $\varphi(v) \in \mathfrak{I}$ is independent of the choice of representative $x(v) \in \mathcal{V}$ for v . So this defines a map $\varphi : V \rightarrow \mathfrak{I}$.

We can define a map $\varphi_k : V_k \rightarrow \mathfrak{I}$ in a similar manner. Namely if $v \in V_k$, then let $\varphi_k(v) = \tau(x(v))Z_G(A) \in W(A)$. Again this is a twisted involution. From the above observations we get the following relation between φ_k and φ :

$$(1.4) \quad \varphi_k = \varphi \circ \eta$$

The maps φ (resp. φ_k) play an important role in the study of the Bruhat order on V (resp. V_k). For more details, see [23] and [8].

1.12. The Weyl group W acts also on \mathfrak{I} . This action comes from the *twisted action* of W on (the set) W , which is defined as follows: if $w, w_1 \in W$, then $w * w_1 = ww_1\theta(w)^{-1}$. If $w_1 \in W$, then $W * w_1 = \{w * w_1 \mid w \in W\}$ is the *twisted W -orbit* of w_1 . Now \mathfrak{I} is stable under the twisted action, so that we get a twisted action of W on \mathfrak{I} . The images of φ and φ_k in \mathfrak{I} are unions of twisted W -orbits, as follows from the following result:

Lemma 1.13. *Let $w \in W$ and $v \in V$ (resp. V_k). Then $\varphi(w \cdot v) = w * \varphi(v)$ (resp. $\varphi_k(w \cdot v) = w * \varphi_k(v)$).*

Proof. The proof is immediate from the above observations. \square

1.14. From this result it follows now that the maps $\varphi : V \rightarrow \mathfrak{I}$ (resp. $\varphi_k : V_k \rightarrow \mathfrak{I}$) are equivariant with respect to the action of W on V (resp. V_k) and the twisted action of W on \mathfrak{I} . So there are natural orbit maps $\phi : V/W \rightarrow \mathfrak{I}/W$ and $\phi_k : V_k/W \rightarrow \mathfrak{I}/W$. From (1.4) and 1.10 we get the following relation between ϕ and ϕ_k .

$$\phi_k = \phi \circ \delta$$

Since γ and γ_k are one-to-one, we also get embeddings of \mathcal{A}_0^θ/H and \mathcal{A}_k^θ/H_k into \mathfrak{I}/W . This indicates that the W -orbits of twisted involutions can be

used as an invariant to characterize the conjugacy classes in \mathcal{A}_0^θ/H and \mathcal{A}_k^θ/H_k . In fact in Proposition 8.10 we will show that we can use conjugacy classes of involutions in the Weyl group W , instead of W -orbits of twisted involutions.

Remarks 1.15. (1) If k is algebraically closed, then it follows from [23, 2.5] that the map ϕ_k is one-to-one. So in this case the classes in $\varphi_k(V_k)/W$ completely characterize the H_k -conjugacy classes of θ -stable maximal k -split tori. The map ϕ_k is also one-to-one in a number of other cases, like the standard pairs (G_k, H_k) for $k = \mathbb{R}$. For a further discussion of this see 6.11 and [13].

(2) An example that the map ϕ_k is not always one-to-one can be found in 3.10.

1.16. In this paper we will need several properties of the twisted involutions, therefore we collect some of their properties in the remainder of this section. A first description of these twisted involutions, in the case that $\theta(\Phi^+) = \Phi^+$, was given by Springer in [27]. For $k = \bar{k}$ there exists a θ -stable Borel subgroup (see [28]), so then this condition is satisfied. However if $k \neq \bar{k}$ then G does not necessarily have θ -stable minimal parabolic k -subgroups. One can easily generalize the description in [27] and give a similar description of the twisted involutions, when $\theta(\Phi^+) \neq \Phi^+$. This follows essentially from the results in [15], although the results are not explicitly stated there. In the following we will review this characterization and prove a few additional results. First we need a few facts about real, complex and imaginary roots.

1.17. In the remainder of this section, let A be a θ -stable maximal quasi k -split torus of G , $\Phi = \Phi(A)$ the root system of A with respect to G , Φ^+ a set of positive roots of Φ , Δ the corresponding basis, $W = W(A)$ the Weyl group of A and $\Sigma = \{s_\alpha \mid \alpha \in \Delta\}$. The Weyl group W is generated by Σ . Let $E = X^*(A) \otimes_{\mathbb{Z}} \mathbb{R}$. If $\sigma \in \text{Aut}(\Phi)$, then we denote the eigenspace of σ for the eigenvalue ξ , by $E(\sigma, \xi)$. For a subset Π of Δ denote the subset of Φ consisting of integral combinations of Π by Φ_Π . Then Φ_Π is a subsystem of Φ with Weyl group W_Π . Let w_Π^0 denote the longest element of W_Π with respect to Π .

The involution θ of G induces an automorphism of W , also denoted by θ , given by

$$\theta(w) = \theta \circ w \circ \theta, \quad w \in W.$$

If s_α is the reflection defined by α , then $\theta(s_\alpha) = s_{\theta(\alpha)}$, $\alpha \in \Phi$.

1.18. The roots of Φ can be divided in three subsets, related to the action of θ , as follows.

- (a) $\theta(\alpha) \neq \pm\alpha$. Then α is called *complex* (relative to θ).
- (b) $\theta(\alpha) = -\alpha$. Then α is called *real* (relative to θ).

(c) $\theta(\alpha) = \alpha$. Then α is called *imaginary* (relative to θ).

For $\alpha \in \Phi$ let $A_\alpha = \{a \in A \mid s_\alpha(a) = a\}^0$, $G_\alpha = Z_G(A_\alpha)$ and $\overline{G}_\alpha = [G_\alpha, G_\alpha]$. If α is either real or imaginary, then G_α is θ -stable.

These definitions carry over to the Weyl group W and the set V . We define first real, complex and imaginary elements for the Weyl group, which then will imply similar definitions for V .

Definition 1.19. Given $w \in \mathcal{I}_\theta$, an element $\alpha \in \Phi$ is called *complex* (resp. *real*, *imaginary*) relative to w if $w\theta\alpha \neq \pm\alpha$ (resp. $w\theta\alpha = -\alpha$, $w\theta\alpha = \alpha$). We use the following notation,

$$\begin{aligned} C'(w, \theta) &= \{\alpha \in \Phi^+ \mid -\alpha \neq w\theta\alpha < 0\}, & R(w, \theta) &= \{\alpha \in \Phi^+ \mid -\alpha = w\theta\alpha\}, \\ C''(w, \theta) &= \{\alpha \in \Phi^+ \mid \alpha \neq w\theta\alpha > 0\}, & I(w, \theta) &= \{\alpha \in \Phi^+ \mid \alpha = w\theta\alpha\}. \end{aligned}$$

We will omit θ from this notation if there is no ambiguity as to which involution we consider.

1.20. These definitions carry now over to V as follows. Fix $v \in V$. Write $x = x(v)$ and let $A_1 = x^{-1}Ax$. Then A_1 is a θ -stable maximal torus. The inner automorphism $\text{Int}(x^{-1})$ defines an isomorphism ω_v of $\Phi(A)$ onto the root system $\Phi_1 = \Phi(A_1)$. We say that α is real, complex, or imaginary for v if $\alpha_1 = \omega_v(\alpha)$ is real, complex, or imaginary in the sense of 1.18 for A_1 . If α is real, complex, or imaginary for v , then we also say that the reflection $s_\alpha \in W$ is real, complex, or imaginary for v .

The notion of real, complex, and imaginary roots for v and $w = \varphi(v)$ is the same:

Lemma 1.21. *Let $v \in V$, $w = \varphi(v)$ and $\alpha \in \Phi$. Then α is real, complex, or imaginary for v if and only if α is real, complex, or imaginary for w .*

Proof. Let $x = x(v)$, $A_1 = x^{-1}Ax$ and ω_v be as above. Then $\tau(x) = x\theta(x)^{-1}$ is a representative for $w = \varphi(v)$ in $N_G(A)$. Let $\alpha \in \Phi(A)$ and $\alpha_1 = \omega_v(\alpha) \in \Phi(A_1)$. Then $\omega_v^{-1}\theta(\alpha_1) = \omega_v^{-1}\theta(\omega_v(\alpha)) = \omega_v^{-1}\theta(\omega_v)(\theta(\alpha)) = \text{Int}(x\theta(x)^{-1})(\theta(\alpha)) = w\theta(\alpha)$. The result follows. \square

We will use the same notation for v as for $w = \varphi(v)$. In particular we will write $C'(v)$, $C''(v)$, $R(v)$, $I(v)$ for $C'(w)$, $C''(w)$, $R(w)$, $I(w)$ respectively. It follows from Lemma 1.21 that this notation does not depend on $v \in V$.

1.22. To get a similar description of the twisted involutions as in [27], we pass to another involution, which leaves Φ^+ invariant. Let $w_0 \in W$ such that

$$(1.5) \quad \theta(\Phi^+) = w_0(\Phi^+).$$

and let $\theta' = \theta w_0$. Then w_0 and θ' satisfy the following conditions:

Proposition 1.23. *Let Φ , Φ^+ , θ , w_0 and θ' be as above. Then we have the following properties:*

- (i) $w_0 \in \mathfrak{I}_\theta$.
- (ii) $\theta'(\Phi^+) = \Phi^+$.
- (iii) θ' is an involution of Φ .
- (iv) $\mathfrak{I}_{\theta'} = \mathfrak{I}_\theta \cdot w_0$.

Proof. (i). Since $\theta(\Phi^+) = w_0(\Phi^+)$ it follows that $w_0^{-1}\theta(\Phi^+) = \Phi^+$ and hence $\theta(w_0^{-1})(\Phi^+) = \theta w_0^{-1}\theta(\Phi^+) = \theta(\Phi^+) = w_0(\Phi^+)$. So $\theta(w_0^{-1}) = w_0$, what proves (i).

(ii) follows from the fact that θ is an involution and $\theta(\Phi^+) = w_0(\Phi^+)$. Namely then $\theta'(\Phi^+) = \theta w_0(\Phi^+) = \Phi^+$. So θ' equals the id or a diagram automorphism. All diagram automorphisms are involutorial, except when Φ is of type D_4 . That θ' is in fact involutorial follows as follows:

$$(\theta')^2 = \theta w_0 \theta w_0 = \theta(w_0) w_0 = w_0^{-1} w_0 = \text{id}.$$

This proves (iii).

(iv). Let $w \in \mathfrak{I}_{\theta'}$ and consider $w w_0^{-1}$. Note that from (i) it follows that $\theta w_0 = w_0^{-1} \theta$. But then

$$\theta(w w_0^{-1}) = \theta w w_0^{-1} \theta = \theta w \theta w_0 = w_0 w_0^{-1} \theta w \theta w_0 = w_0 \theta' w \theta' = w_0 w^{-1}.$$

So $\mathfrak{I}_{\theta'} \subset \mathfrak{I}_\theta \cdot w_0$. A similar argument shows the opposite inclusion, what proves the result. \square

Remark 1.24. By choosing a suitable minimal parabolic k -subgroup, w_0 becomes an involution (see also Lemma 2.15).

For the real, complex and imaginary roots we can show the following:

Lemma 1.25. *If $w \in \mathfrak{I}_\theta$ and $w' = w w_0$, then $w' \theta' = w \theta$. In particular we have*

$$\begin{aligned} I(w, \theta) &= I(w', \theta'), & R(w, \theta) &= R(w', \theta') \\ C'(w, \theta) &= C'(w', \theta'), & C''(w, \theta) &= C''(w', \theta') \end{aligned}$$

Proof. The first statement follows from Proposition 1.23. Namely,

$$w' \theta' = w w_0 \theta w_0 = w w_0 \theta w_0 \theta \theta = w w_0 \theta (w_0) \theta = w w_0 w_0^{-1} \theta = w \theta.$$

The remaining statements follow from this, since if $\alpha \in \Phi^+$, then $w' \theta'(\alpha) = w \theta(\alpha)$. \square

Remark 1.26. Instead of working with θ we can now work with θ' and w' . Combining this result with the description of twisted involutions in [27] and [15], we get the following characterization of twisted involutions.

Proposition 1.27. [15, 7.9] *If $w \in \mathcal{L}_\theta$ and $w' = ww_0 \in \mathcal{L}_{\theta'}$, then there exist $s_1, \dots, s_h \in \Sigma$ and a θ' -stable subset Π of Δ satisfying the following conditions:*

- (i) $w' = s_1 \dots s_h w_\Pi^0 \theta'(s_h) \dots \theta'(s_1)$ and $l(w') = 2h + l(w_\Pi^0)$.
- (ii) $w_\Pi^0 \theta' \alpha = -\alpha$, $\alpha \in \Phi_\Pi$ (i.e. $\Phi_\Pi^+ \subset R(w_\Pi^0, \theta')$).

Moreover if $w' = t_1 \dots t_m w_\Lambda^0 \theta'(t_m) \dots \theta'(t_1)$, where $t_1, \dots, t_m \in \Sigma$ and Λ a θ' -stable subset of Δ satisfying conditions (i) and (ii), then $m = h$, $s_1 \dots s_h \Pi = t_1 \dots t_h \Lambda$ and

$$s_1 \dots s_h \theta'(s_h) \dots \theta'(s_1) = t_1 \dots t_h \theta'(t_h) \dots \theta'(t_1).$$

1.28. **W-action on $\mathcal{L}_{\theta'}$.** Similar as in 1.12 we have an action of the Weyl group W on $\mathcal{L}_{\theta'}$. Namely if $w \in W$ and $a' \in \mathcal{L}_{\theta'}$, then define an action $w *' a = wa' \theta'(w)^{-1}$. Since $\mathcal{L}_{\theta'} = \theta w_0$ and $a' = aw_0$ for some $a \in \mathcal{L}_\theta$, we get

$$w *' a = waw_0 \theta' w^{-1} \theta' = waw_0 w_0^{-1} \theta w^{-1} \theta w_0 = wa \theta w^{-1} \theta w_0 = (w * a) w_0.$$

This means that the right translation by w_0 gives an isomorphism $\iota_{w_0} : \mathcal{L}_\theta \rightarrow \mathcal{L}_{\theta'}$, which is equivariant with respect to the action of W on \mathcal{L}_θ and $\mathcal{L}_{\theta'}$. So we also get a bijection

$$(1.6) \quad \iota : \mathcal{L}_\theta / W \rightarrow \mathcal{L}_{\theta'} / W.$$

The following result is useful in the study of twisted involutions and will be used in the sequel.

Lemma 1.29. *Let $a \in \mathcal{L}_\theta$, $a' = aw_0 \in \mathcal{L}_{\theta'}$ and $w \in W$. Then $E(a' \theta', -1) = E(a \theta, -1)$ and $w(E(a \theta, -1)) = E((w * a) \theta, -1)$.*

Proof. The first statement follows from Lemma 1.25 and the second statement is immediate from the definition of the twisted W -action on \mathcal{L}_θ . \square

2. Some results about group actions on root systems

Most questions about H -conjugacy classes of θ -stable maximal quasi k -split tori can be reduced to questions about the underlying root systems. For arbitrary base fields we have both the action of the Galois group and the action of the involution on the root system. Both these actions can be described by choosing suitable orders on the root system. In this section we collect a number of the results about these actions, which are needed in the sequel.

2.1. (Γ, θ) -indices. We first discuss a number of properties of the fine structure associated with a symmetric k -variety. These results will be needed in the classification of the conjugacy classes in section 9 and also in a number of proofs. Some of these results come from [7], [29] and [9] and are modified to fit in the setting of symmetric k -varieties.

Let G be a reductive k -group and $\theta \in \text{Aut}(G)$ a k -involution. Let A be a θ -stable maximal k -split torus, $T \supset A$ a θ -stable maximal k -torus of G , $X = X^*(T)$ and $\Phi = \Phi(T)$. There is a finite Galois extension K/k such that T splits over K . Denote the Galois group $\text{Gal}(K/k)$ of K/k by Γ and let $\Gamma_0 \subset \text{Aut}(X, \Phi)$ be the subgroup corresponding to the action of Γ on (X, Φ) . Similarly let $\Theta = \{1, -\theta\} \subset \text{Aut}(X, \Phi)$ be the subgroup spanned by $-\theta|_T$ and let $\Gamma_\theta = \Gamma_0 \cdot \Theta$ the subgroup of $\text{Aut}(X, \Phi)$ generated by Γ_0 and Θ . In the following $\mathcal{E} \subset \text{Aut}(X, \Phi)$ will be one of Γ_0 , Θ or Γ_θ .

2.2. For $\sigma \in \mathcal{E}$ and $\chi \in X$ we will also write χ^σ or $\sigma(\chi)$ for the element $\sigma \cdot \chi \in X$. Write $W = W(\Phi)$ for the Weyl group of Φ . For any closed subsystem Φ_1 of Φ let $W(\Phi_1)$ denote the finite group generated by the s_α for $\alpha \in \Phi_1$. Now define the following:

$$(2.1) \quad X_0 = X_0(\mathcal{E}) = \left\{ \chi \in X \mid \sum_{\sigma \in \mathcal{E}} \chi^\sigma = 0 \right\}$$

This is a co-torsion free submodule of X , invariant under the action of \mathcal{E} . Let $\Phi_0 = \Phi \cap X_0$. This is a closed subsystem of Φ invariant under the action of \mathcal{E} . Denote the Weyl group of Φ_0 by W_0 and identify it with $W(\Phi_0)$. Put $W^\mathcal{E} = \{w \in W \mid w(X_0) = X_0\}$, $\bar{X}_\mathcal{E} = X/X_0(\mathcal{E})$ and let π be the natural projection from X to $\bar{X}_\mathcal{E}$. If we take $S = \{t \in T \mid \chi(t) = e \text{ for all } \chi \in X_0\}$ and $Y = X^*(S)$, then Y may be identified with $\bar{X}_\mathcal{E} = X/X_0$. Let $\bar{\Phi}_\mathcal{E} = \pi(\Phi - \Phi_0(\mathcal{E}))$ denote the set of *restricted roots of Φ relative to \mathcal{E}* .

Remark 2.3. In the case that $\mathcal{E} = \Gamma$, then X_0 is the annihilator of a maximal k -split torus S of T . Similarly in the case that $\mathcal{E} = \Theta$, then X_0 is the annihilator of a θ -split torus S of T . In both these cases $\bar{\Phi}_\mathcal{E}$ is the root system of $\Phi(S)$ with Weyl group $\bar{W}_\mathcal{E}$.

An order on (X, Φ) related to the action of \mathcal{E} can be defined as follows.

Definition 2.4. A linear order $>$ on X which satisfies

$$(2.2) \quad \text{if } \chi > 0 \text{ and } \chi \notin X_0, \text{ then } \chi^\sigma > 0 \text{ for all } \sigma \in \mathcal{E}$$

is called a \mathcal{E} -linear order. A fundamental system of Φ with respect to a \mathcal{E} -linear order is called a \mathcal{E} -fundamental system of Φ .

A \mathcal{E} -linear order on X induces linear orders on $Y = X/X_0$ and X_0 , and conversely, given linear orders on X_0 and on Y , these uniquely determine a \mathcal{E} -linear order on X , which induces the given linear orders (i.e., if $\chi \notin X_0$, then define $\chi > 0$ if and only if $\pi(\chi) > 0$).

2.5. Fix a \mathcal{E} -linear order $>$ on X , let Δ be a \mathcal{E} -fundamental system of Φ and let Δ_0 be a fundamental system of Φ_0 with respect to the induced order on X_0 . Define $\bar{\Delta}_\mathcal{E} = \pi(\Delta - \Delta_0)$. This is called a *restricted fundamental system* of Φ relative to S .

There is also a natural (Weyl) group associated with the set of restricted roots, which is related to $W^\mathcal{E}/W_0$. Since W_0 is a normal subgroup of $W^\mathcal{E}$, every $w \in W^\mathcal{E}$ induces an automorphism of $\bar{X}_\mathcal{E} = X/X_0 = Y$. Denote the induced automorphism by $\pi(w)$. Then $\pi(w\chi) = \pi(w)\pi(\chi)$ ($\chi \in X$). Define $\bar{W}_\mathcal{E} = \{\pi(w) \mid w \in W^\mathcal{E}\}$. We call this the *restricted Weyl group*, with respect to the action of \mathcal{E} on X . It is not necessarily a Weyl group in the sense of Bourbaki [3, Ch.VI,no.1]. However we can show the following.

Proposition 2.6. *Let $X, X_0, \Phi, \Delta, W_0, W^\mathcal{E}, \bar{W}_\mathcal{E}$ be defined as above and let $S = \{t \in T \mid \chi(t) = e \text{ for all } \chi \in X_0\}$ be the annihilator of X_0 . Then $\bar{W}_\mathcal{E} \cong W^\mathcal{E}/W_0 \cong N_G(S)/Z_G(S)$.*

Proof. This result follows using the same arguments as in [25, Corollary 2.1.3] and [25, Proposition 2.1.4]. \square

Remark 2.7. In the case that S is a maximal k -split, θ -split or (θ, k) -split torus, then $\bar{\Phi}_\mathcal{E}$ is actually a root system with Weyl group $\bar{W}_\mathcal{E}$.

2.8. From Proposition 2.6 it follows that $W^\mathcal{E}$ acts on the set of \mathcal{E} -fundamental systems of Φ . There is also a natural action of \mathcal{E} on this set. If Δ is a \mathcal{E} -fundamental system of Φ , and $\sigma \in \mathcal{E}$, then the \mathcal{E} -fundamental system $\Delta^\sigma = \{\alpha^\sigma \mid \alpha \in \Delta\}$ gives the same restricted basis as Δ , i.e. $\bar{\Delta}^\sigma = \bar{\Delta}$. Let $w_\sigma \in W_0$ be the unique element $w_\sigma \in W_0$ such that $\Delta^\sigma = w_\sigma\Delta$ and define a new action of \mathcal{E} on (X, Φ) as follows:

$$(2.3) \quad \chi^{[\sigma]} = w_\sigma^{-1}\chi^\sigma, \quad \chi \in X, \quad \sigma \in \mathcal{E}.$$

It is easily verified that $\chi \rightarrow \chi^{[\sigma]}$ is an automorphism of the triple (X, Φ, Δ) and that $\chi^{[\sigma][\tau]} = \chi^{[\sigma\tau]}$ for all $\sigma, \tau \in \mathcal{E}$, $\chi \in X$. This action of \mathcal{E} on Ψ is essentially determined by Δ, Δ_0 and $[\sigma]$. Following Tits [29] the quadruple $(X, \Delta, \Delta_0, [\sigma])$ is called an *index of \mathcal{E}* or an \mathcal{E} -index. We will also use the name \mathcal{E} -diagram, following the notation in Satake [25, 2.4].

2.9. As in [29] we make a diagrammatic representation of the index of \mathcal{E} by coloring black those vertices of the ordinary Dynkin diagram of Φ , which represent roots in $\Delta_0(\mathcal{E})$ and indicating the action of $[\sigma]$ on Δ by arrows.

In the cases of $\mathcal{E} = \Theta$ and $\mathcal{E} = \Gamma$ we get the well known θ -index and Γ -index, which are essential in the respective classifications. For k -involutions we do not use the Γ_θ -index, but we combine it with the Γ -index and θ -index to add additional information. These indices are defined as follows.

2.10. **(Γ, θ)-order.** Assume $A_0 \subset A$ is a maximal (θ, k) -split torus of G , S a maximal θ -split torus of G such that $A_0 \subset S \subset T$. By [15, 4.5] tori A_0 , A , S and T exist.

Definition. Let A_0 , A , S and T be as above. A linear order on $X = X^*(T)$ which is simultaneously a Γ -, θ - and Γ_θ -order is called a (Γ, θ) -order. A fundamental system of Φ with respect to a (Γ, θ) -order is called a (Γ, θ) -fundamental system of Φ .

An Γ_θ -order on (X, Φ) is a (Γ, θ) -order if and only if the following condition is satisfied:

(2.4)

If $\Phi_1 \subset \Phi_0(\Gamma_\theta)$ irreducible component then $\Phi_1 \subset \Phi_0(\theta)$ or $\Phi_1 \subset \Phi_0(\Gamma)$.

Remarks 2.11. (1) A (Γ, θ) -order, as above, is completely determined by the sextuple

$$(X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta), [\sigma], \theta^*).$$

We will call this sextuple an *index of* (Γ, θ) or an (Γ, θ) -index. This terminology follows again Tits [29]. We will also use the name (Γ, θ) -diagram, following the notation in Satake [25, 2.4].

(2) We can make a diagrammatic representation of the (Γ, θ) -index by coloring black those vertices of the ordinary Dynkin diagram of Φ , which represent roots in $\Delta_0(\Gamma, \theta)$ and giving the vertices of $\Delta_0(\Gamma) \cup \Delta_0(\theta)$ which are not in $\Delta_0(\Gamma) \cap \Delta_0(\theta)$ a label k or θ if $\alpha \in \Delta_0(\Gamma) - \Delta_0(\Gamma) \cap \Delta_0(\theta)$ or $\alpha \in \Delta_0(\theta) - \Delta_0(\Gamma) \cap \Delta_0(\theta)$ respectively. The actions of $[\sigma]$ and θ^* are indicated by arrows.

(3) The above index of (Γ, θ) determines the indices of both Γ and θ and vice versa.

We conclude this section with some additional results for the action of the involution on (X, Φ) .

2.12. θ -orders on quasi k -split tori. For θ -stable maximal quasi k -split tori we will need both the action of θ and $-\theta$. The action of θ is as defined in 2.1. The action of $-\theta$ can be defined in a similar manner. We will discuss this action in the following. We will use the same notation as in 2.1. In particular let A be a θ -stable maximal quasi k -split torus of G and write $X = X^*(A)$ and $\Phi = \Phi(A)$. Similar as in 2.2 let

$$\begin{aligned} X_0(\theta) &= \{\chi \in X \mid \theta(\chi) = \chi\} & \text{and} & & \Phi_0(\theta) &= \Phi \cap X_0(\theta) \\ X_0(-\theta) &= \{\chi \in X \mid \theta(\chi) = -\chi\} & \text{and} & & \Phi_0(-\theta) &= \Phi \cap X_0(-\theta). \end{aligned}$$

Let π be the natural projection from X to $X/X_0(\theta)$ and π^- the natural projection from X to $X/X_0(-\theta)$. As in [10] we define a θ -order (resp. $-\theta$ -order) on Φ by choosing orders on $X_0(\theta)$ and $X/X_0(\theta)$ (resp. $X_0(-\theta)$ and $X/X_0(-\theta)$). To be more precise:

Definition 2.13. Let \succ be a linear order on X . The order \succ is called a θ^+ -order if it has the following property:

$$(2.5) \quad \text{if } \chi \in X, \chi \succ 0, \text{ and } \chi \notin X_0(\theta), \text{ then } \theta(\chi) \prec 0.$$

The order \succ is called a θ^- -order if it has the following property:

$$(2.6) \quad \text{if } \chi \in X, \chi \succ 0, \text{ and } \chi \notin X_0(\theta), \text{ then } \theta(\chi) \succ 0.$$

Similar as in [9], a θ^+ -order on X will also be called a θ -order on X . Note that a θ^- -order on X is a θ -order on X for the involution $-\theta$ of (X, Φ) .

A basis Δ of Φ with respect to a θ^+ -order (resp. θ^- -order) on X will be called a θ^+ -basis (resp. θ^- -basis) of Φ . If Δ is a basis of Φ with respect to a θ^+ -order on X , then we write $\Delta_0(\theta) = \Delta \cap \Phi_0(\theta)$ and $\bar{\Delta}_\theta = \pi(\Delta - \Delta_0(\theta))$. Similarly if Δ is a basis of Φ with respect to a θ^- -order on X , then we write $\Delta_0(-\theta) = \Delta \cap \Phi_0(-\theta)$ and $\bar{\Delta}_{-\theta} = \pi(\Delta - \Delta_0(-\theta))$. Clearly $\Delta_0(\theta)$ (resp. $\Delta_0(-\theta)$) is a basis of $\Phi_0(\theta)$ (resp. $\Phi_0(-\theta)$). A similar property holds for $\bar{\Delta}_\theta$ and $\bar{\Delta}_{-\theta}$ (see [9, 2.4]).

2.14. A characterization of θ on a θ^+ -basis of Φ . Let Δ_1 be a θ^+ -basis of Φ . As in [9, 2.8] we can write $\theta = -\text{id} \cdot \theta_1^* \cdot w_0(\theta)$, where $w_0(\theta)$ is the longest element of $W(\Phi_0(\theta))$ with respect to $\Delta_0(\theta)$ and $\theta_1^* \in \text{Aut}(X, \Phi, \Delta_1, \Delta_0(\theta)) = \{\phi \in \text{Aut}(X, \Phi) \mid \phi(\Delta_1) = \Delta_1 \text{ and } \phi(\Delta_0(\theta)) = \Delta_0(\theta)\}$, $(\theta_1^*)^2 = \text{id}$. For more details see [9, §2]. This is called a characterization of θ on its $(+1)$ -eigenspace.

Similarly we get a characterization of θ on a θ^- -basis of Φ as follows. Let Δ_2 be a θ^- -basis of Φ . Then $\theta = \theta_2^* \cdot w_0(-\theta)$, where $w_0(-\theta)$ is the longest element of $W(\Phi_0(-\theta))$ with respect to $\Delta_0(-\theta)$ and $\theta_2^* \in \text{Aut}(X, \Phi, \Delta_2, \Delta_0(-\theta)) =$

$\{\phi \in \text{Aut}(X, \Phi) \mid \phi(\Delta_2) = \Delta_2 \text{ and } \phi(\Delta_0(-\theta)) = \Delta_0(-\theta)\}$, $(\theta_2^*)^2 = \text{id}$. This is called a characterization of θ on its (-1) -eigenspace.

From the characterization of involutions of root systems in [10], we can easily deduce that not both θ_1^* and θ_2^* can be non trivial.

Lemma 2.15. *Let Δ_1 , Δ_2 , θ_1^* and θ_2^* be as above. If Φ is irreducible, then $\theta_1^* = \text{id}$ or $\theta_2^* = \text{id}$.*

3. Quasi parabolic k -subgroups

To characterize the H -conjugacy classes of θ -stable maximal quasi k -split tori we will first need to prove a number of results about these tori. The following result will be useful in studying the H -conjugacy classes of these tori.

Lemma 3.1. *Let A_1 , A_2 be θ -stable maximal quasi k -split tori of G . Then A_1 and A_2 are H -conjugate if and only if they are H^0 -conjugate.*

Proof. Assume $h \in H$ such that $hA_1h^{-1} = A_2$. Then $hA_1^-h^{-1} = A_2^-$. Let $S_i \supset A_i$ ($i = 1, 2$) be maximal θ -split tori. Since all maximal θ -split tori of G are conjugate under H^0 , there exist $h_1 \in H^0$ such that $h_1S_1h_1^{-1} = S_2$. So we may assume $S = S_1 = S_2$. Moreover since hSh^{-1} and S are also conjugate under H^0 , we may assume that $h \in N_G(S)$. By [22, 4.7] there exists $h_2 \in N_G(S) \cap H^0$ such that $h_2A_1^-h_2^{-1} = A_2^-$. Let $x = h_2h^{-1} \in N_G(A_2^-)$. Since $N_G(A_2^-)$ has representatives in H^0 we may assume that $x \in H \cap Z_G(A_2^-)$. It suffices to show now that A_2 and $A_3 = xA_2x^{-1}$ are conjugate under $H^0 \cap Z_G(A_2^-)$.

Let $T_i \supset A_i$ ($i = 2, 3$) be maximal tori of $Z_G(A_2^-)$ with T_i^- maximal θ -split in $Z_G(A_2^-)$. There exists $h_3 \in H^0 \cap Z_G(A_2^-)$ such that $h_3T_3h_3^{-1} = T_2$. So we may assume $G = Z_G(T_2^-)$, and $x \in N_H(T_2^+)$. Since T_2^+ is a maximal torus of H , it follows that x has a representative in H^0 , what proves the result. \square

Most results about θ -stable maximal quasi k -split tori can be proved using the natural class of parabolic subgroups related to these tori. These parabolic subgroups are defined as follows.

Definition 3.2. A parabolic subgroup P in G is called a *quasi parabolic k -subgroup* if there exist $x \in \mathcal{V}$ such that xPx^{-1} is a parabolic k -subgroup.

The quasi parabolic k -subgroups can be characterized as follows.

3.3. Let A be a θ -stable maximal quasi k -split torus of G , $\Phi(G, A)$ the set of roots of A in G and $X_*(A)$ the set of one parameter subgroups of A . By chambers, facets of $X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$, we mean those with respect to the hyperplanes $\ker(\alpha)$, $\alpha \in \Phi(G, A)$. The quasi parabolic k -subgroups of G containing

A are in bijective correspondence with the facets of $X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$. Given a facet F , the corresponding quasi parabolic k -subgroup $P(F)$ of G containing A is determined by

$$\Phi(P(F), A) = \{\alpha \in \Phi(G, A) \mid \langle x, \alpha \rangle \geq 0, \quad x \in F\}.$$

For $\lambda \in X_*(A)$, let $F(\lambda)$ denote the facet containing λ . For simplicity, we write $P(\lambda)$ for the quasi parabolic k -subgroup $P(F(\lambda))$ of G containing A .

Similar as in [15, 2.4] we can show the following.

Lemma 3.4. *Every minimal quasi parabolic k -subgroup P of G contains a θ -stable maximal quasi k -split torus, unique up to conjugation by an element from $H \cap R_u(P)$.*

Proof. Let $x \in \mathcal{V}$ such that xPx^{-1} is a parabolic k -subgroup. By [15, 2.4] there exists a θ -stable maximal k -split torus A of G , which is contained in xPx^{-1} . Since $x \in \mathcal{V}$, the torus $x^{-1}Ax \subset P$ is θ -stable.

Suppose now that A_1 and A_2 are θ -stable maximal quasi k -split tori in P . Let A be a θ -stable maximal k -split torus and $g_1, g_2 \in \mathcal{V}$ such that $A_1 = g_1Ag_1^{-1}$ and $A_2 = g_2Ag_2^{-1}$. Let $P_0 \supset A$ be a parabolic k -subgroup. After multiplying g_1 with a suitable element of $N_G(A)$ we may assume that $P = g_1P_0g_1^{-1} = g_2P_0g_2^{-1}$. It follows that $x = g_1g_2^{-1} \in P$ and $xA_2x^{-1} = A_1$. Since P is the semidirect product of $Z_G(A_2)$ and $R_u(P)$ we may assume that $x \in R_u(P)$. However, since $\theta(A_2) = A_2$ it follows that $x^{-1}\theta(x) \in N_G(A_2) \cap U = e$ and hence $x \in H \cap R_u(P)$. \square

We want to characterize the θ -stable maximal quasi k -split tori A with A^+ (resp A^-) maximal. These correspond to θ -stable respectively θ -split quasi parabolic k -subgroups. The θ -stable quasi parabolic k -subgroup of G can be characterized as follows.

Lemma 3.5. [15, 3.3] *Let P be a θ -stable quasi parabolic k -subgroup of G and M a θ -stable Levi k -subgroup of P . Let A be a θ -stable maximal quasi k -split torus of M and F the facet with $P = P(F)$. Then we have the following conditions:*

- (i) $\theta(F) = F$.
- (ii) *There is a $\lambda \in X_*(A_+)$ such that $P = P(\lambda)$ and $M = Z_G(\lambda)$.*

Proof. The first statement is immediate from the observation that $P(F) = \theta(P(F)) = P(\theta(F))$.

(ii). Let $\delta \in X_*(A) \cap F$. From (i) it follows that $\theta(\delta) \in F$. But then $\lambda = \delta + \theta(\delta) \in X_*(A_+) \cap F$, $P(\lambda) = P(F) = P$ and $Z_G(\lambda) = M$. This proves the result. \square

Proposition 3.6. *Let P be a minimal quasi parabolic k -subgroup and M a θ -stable Levi subgroup of P . Let A be a θ -stable maximal quasi k -split torus of M . Then P is θ -stable if and only if*

- (i) $\Phi(A)$ has no real roots.
- (ii) P induces a θ^- -order on $\Phi(A)$.

Proof. Assume P is θ -stable. Let Δ be the basis of Φ corresponding to the order induced by P . If $\alpha \in \Phi(A)^+$ is a real root then $\theta(\alpha) = -\alpha \in \Phi(A)^-$, what contradicts the fact that P is θ -stable. Similarly, it follows from the fact that P is θ -stable that $\theta(\Phi^+) = \Phi^+$ and consequently also $\theta(\Delta) = \Delta$. The result now follows from 2.14.

Conversely assume (i) and (ii) hold. Let Δ be the basis of $\Phi(A)$ related to the order induced by P . Since $\Phi(A)$ has no real roots, it follows from 2.14 that $\theta(\Delta) = \Delta$, what proves the result. \square

We still need to prove that there exists a θ -stable minimal quasi parabolic k -subgroup of G . For this we use singular roots:

Definition 3.7. Let A be a θ -stable maximal quasi k -split torus of G . For $\alpha \in \Phi(A)$ let $A_\alpha = \{a \in A \mid s_\alpha(a) = a\}^0$, $G_\alpha = Z_G(A_\alpha)$ and $\overline{G}_\alpha = [G_\alpha, G_\alpha]$. A root $\alpha \in \Phi(A)$ with $\theta(\alpha) = \pm\alpha$ is called θ -singular (resp. θ -compact) if $\overline{G}_\alpha \not\subset H$ (resp. $\overline{G}_\alpha \subset H$).

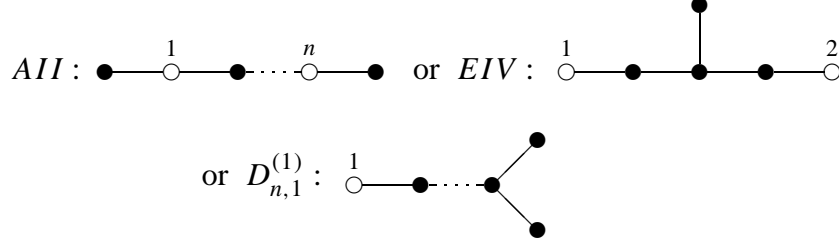
For the θ -singular roots we can show the following:

Proposition 3.8. *Let A be a θ -stable maximal quasi k -split torus. Then we have the following.*

- (1) *If $\lambda \in \Phi(A)$ is real, then there exist a $x \in G_\lambda$ such that $A_1 = xAx^{-1}$ is a θ -stable maximal quasi k -split torus with $\dim A^- > \dim A_1^-$.*
- (2) *If $\lambda \in \Phi(A)$ is θ -singular imaginary, then there exist a $x \in G_\lambda$ such that $A_1 = xAx^{-1}$ is a θ -stable maximal quasi k -split torus with $\dim A^- < \dim A_1^-$.*

Proof. We may assume that $G = G_\lambda$ and A is θ -split. Let $T \supset A$ be a θ -stable maximal torus with T^+ a maximal torus of $Z_G(A) \cap H$. If T^+ is not a maximal torus of H , then there exists $\alpha \in \Phi(T)$, $\theta(\alpha) = -\alpha$. Then, since T^+ is maximal in $Z_G(A) \cap H$, we get $s_\alpha|_A \neq \text{id}$. But then, since $\text{rank } \Phi(A) = 1$, also $s_\alpha|_A = s_\lambda$. By [10, 4.6] α is θ -singular and there exists $g \in Z_G((\ker \alpha)^0)$ such that $T_1 = gTg^{-1}$ is θ -stable with $\dim T_1^+ = \dim T^+ + 1$. Then also $A_1 = gAg^{-1}$ is θ -stable and $\dim A_1^+ = \dim A^+ + 1$, what proves the result. So assume T^+ is a maximal torus of H . Let $w_0 \in \Phi(T)$ be a maximal θ -singular involution of $W(T)$. Then $T_{w_0}^- \subset T^+$ and $T_{w_0}^+ \supset A$. We may restrict to $Z_G(T_{w_0}^-)$. So assume

that $\Phi(T)$ has no θ -singular roots. Then the θ -index of (G, θ) must be one of the following:



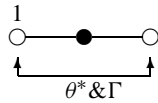
In each of these cases one easily shows that there exists $w_1 \in W(T_1)$ such that $w_1(T_1^-) \subset T_1^+$. Let $n \in N_G(T_1)$ be a representative of w_1 . Then, since $A \subset T_1^-$, we get $nAn^{-1} \subset nT_1^-n^{-1} \subset T_1^+$, what proves (1).

(2) follows with a similar argument as in (1). \square

Remarks 3.9. (1) If A is a θ -stable maximal quasi k -split torus, $\lambda \in \Phi(A)$ a θ -singular root and $x \in G_\lambda$ as in 3.8, then the tori A and $A_1 = xAx^{-1}$ are conjugate under the inner automorphism $c_\lambda = \text{Int}(x)$ of G_λ . In the case of maximal tori of real semisimple groups these automorphisms are also called Cayley transformations. For more details see [6].

(2) In general the root system $\Phi(A)$ of a θ -stable maximal quasi k -split torus A is reduced. It would be quite natural to expect that if $\lambda, 2\lambda \in \Phi(A)$ and λ is θ -singular, then also 2λ is θ -singular. This is true for real roots, as can be seen from 3.8. However for imaginary θ -singular roots this is in general not true, as can be seen from the following example.

Example 3.10. Assume (Γ, θ) -index of (G, θ) is:



Then G has a maximal k -split torus A , which is also θ -split. Let $T \supset A$ be a θ -stable maximal torus of G . The root system $\Phi(A)$ is of type BC_1 and the long root α is contained in $\Phi(A) \cap \Phi(T)$. This root is θ -singular, since H contains a maximal torus T_1 of G . Consequently T_1 contains a θ -stable maximal quasi k -split torus A_1 and the long root $\beta \in \Phi(A_1) \cap \Phi(T_1)$ is θ -singular. On the other hand, since $\Phi(T_1)$ contains θ -compact roots, there exists $w \in W(T_1)$ such that $w(\beta)$ is θ -compact and consequently $w(\beta)$ is not θ -singular. However, $\frac{1}{2}w(\beta)$ is still θ -singular, since $G_{\frac{1}{2}w(\beta)} = G$. Note that $\overline{G}_{w(\beta)}$ is of type A_1 .

Corollary 3.11. *There exists a θ -stable minimal quasi parabolic k -subgroup of G .*

Proof. let A be a θ -stable maximal quasi k -split torus of G . Let $\Phi_1 = \{\alpha \in \Phi(A) \mid \theta(\alpha) = -\alpha\}$ be the set of real roots, $\Delta_1 \subset \Phi_1$ a basis and w_1 the longest involution of $W(\Phi_1)$ with respect to Δ_1 . By [10, 2.7] there exist $\alpha_1, \dots, \alpha_n \in \Phi_1$ strongly orthogonal roots, such that $w_1 = s_{\alpha_1} \dots s_{\alpha_n}$. Since each of the α_i , ($i = 1, \dots, n$) is θ -singular, it follows from Proposition 3.8, with an easy induction, that there exists $x \in Z_G(A_{w_1}^-)$ such that $A_1 = xAx^{-1}$ is θ -stable, $\dim A_1^- = \dim A - n$ and A_1 has no real roots. From Proposition 3.6 it follows that there exists $\lambda \in X_*(A_1)$ such that $P(\lambda)$ is θ -stable. \square

Definition 3.12. Let A be a θ -stable maximal quasi k -split torus of G . We will say that A^+ is maximal if $\Phi(A)$ has no real roots. Similarly we will say that A^- is maximal if $\Phi(A)$ has no θ -singular imaginary roots.

The above results characterize the θ -stable maximal quasi k -split tori A with A^+ maximal. To characterize the θ -stable maximal quasi k -split tori with A^- maximal we use θ -split quasi parabolic k -subgroups. Similar as in [15, §4] we can prove the following results.

Lemma 3.13. *Let P be a θ -split quasi parabolic k -subgroup of G and A a θ -stable maximal quasi k -split torus of P . Then there exists $\lambda \in X_*(A^-)$ such that $P = P(\lambda)$ and $P \cap \theta(P) = Z_G(\lambda)$.*

Proof. Since A is θ -stable, $A \subset \theta(P)$. Let F be the facet of $X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$ such that $P = P(F)$. Note that $\theta(P) = P(-F)$. Hence $\theta(F) = -F$. Now choose $\tau \in X_*(A) \cap F$. Since $\tau, -\theta(\tau) \in F$ and F is a convex cone, $\lambda = \tau - \theta(\tau) \in F$. Then λ has the desired property. \square

Proposition 3.14. *Let P be a θ -split quasi parabolic k -subgroup of G and A a θ -stable maximal quasi k -split torus of P . Then the following conditions are equivalent:*

- (i) P is a minimal θ -split quasi parabolic k -subgroup of G .
- (ii) A^- is maximal and $Z_G(A^-) = P \cap \theta(P)$.

Proof. (i) \Rightarrow (ii). Assume A^- is not maximal. Let $\lambda \in \Phi(A)$ be a θ -singular imaginary root. Since P is θ -split there exists a θ -split torus S such that $P \cap \theta(P) = Z_G(S)$. Let $T \supset A$ be a maximal torus with T^- maximal θ -split in $P \cap \theta(P)$. So $T^- \supset S$. From Proposition 3.8 it follows that there exist $x \in Z_G(T^- \cdot \ker(\lambda)^0)$ such that $A_1 = xAx^{-1}$ is θ -stable with $\dim A_1^- = \dim A^- + 1$. Since $T^- \supset S$ it follows that $A_1 \subset P \cap \theta(P)$. Let δ be a nontrivial element of $X_*(A_1^-)$ and $P(\delta)$ the parabolic subgroup of G containing A_1 given

by $\Phi(P(\delta), A_1) = \{\alpha \in \Phi(A_1) \mid \langle \delta, \alpha \rangle \geq 0\}$. Then $P(\delta)$ is a proper parabolic subgroup of P and since $\theta(\delta) = -\delta$ it follows that $P(\delta)$ is θ -split. Clearly this is a contradiction. So A^- is maximal. From the same argument it follows that $A^- \subset S$. So $P \cap \theta(P) \subset Z_G(A^-)$. From Lemma 3.13 it follows that there exists $\delta \in X_*(A^-)$ such that $Z_G(\delta) = P \cap \theta(P)$. Since clearly $Z_G(A^-) \subset Z_G(\delta)$ we get $P \cap \theta(P) = Z_G(A^-)$.

(ii) \Rightarrow (i). Assume $P_1 \supset A$ a θ -split quasi parabolic k -subgroup with $P_1 \subset P$. Then $P_1 \cap \theta(P_1) \subset P \cap \theta(P) = Z_G(A^-)$. By Lemma 3.13 there exists $\delta \in X_*(A^-)$ such that $P_1 = P(\delta)$ and $P_1 \cap \theta(P_1) = Z_G(\delta)$. Since clearly $Z_G(A^-) \subset Z_G(\delta)$ it follows that $P_1 \cap \theta(P_1) = P \cap \theta(P) = Z_G(A^-)$. So $P_1 = P$. \square

Proposition 3.15. *let P be a minimal θ -split quasi parabolic k -subgroup of G and P_0 a minimal quasi parabolic k -subgroup of G contained in P . Then $H^0 P = H^0 P_0$ is open in G .*

Proof. Since $\Phi(A, Z_G(A^+))$ has no θ -singular roots it follows that $H^0 Z_G(A^-) = H^0 Z_G(A)$. Since $P = Z_G(A^-)P_0$ we get $H^0 P = H^0 P_0$, which by [32, Theorem 1.3.] is open. \square

Proposition 3.16. *All minimal θ -split quasi parabolic k -subgroups of G are conjugate under H^0 .*

Proof. Let P_1, P_2 be minimal θ -split quasi parabolic k -subgroups of G and let $P_0 \subset P_1, P'_0 \subset P_2$ be minimal quasi parabolic k -subgroups of G . Since both P_0 and P'_0 are conjugate to a minimal parabolic k -subgroups of G , there exists $g \in G$ such that $gP'_0g^{-1} = P_0$. By Proposition 3.15 both $H^0 P'_0$ and $H^0 P_0 = H^0 gP'_0g^{-1}$ are open in G . This yields that $H^0 P'_0$ and $H^0 gP'_0g^{-1}$ are the same open orbit of P'_0 in $H^0 \backslash G$. Hence $g \in H^0 P'_0$. Write $g = h.p, h \in H^0, p \in P'_0$. Then $hP'_0h^{-1} = P_0$. So we may assume that $P_0 \subset P_1 \cap hP_2h^{-1}$.

Let A be a θ -stable maximal quasi k -split torus of P_0 and $\delta_1, \delta_2 \in X_*(A^-)$ such that $P_1 = P(\lambda_1), P(\lambda_2) = hP_2h^{-1}$. Since $P(\lambda_1) \cap P(\lambda_2) \supset P_0$ it follows with the same argument as in [32, 1.2. Prop. 4] that $P_1 \cap hP_2h^{-1} = P(\lambda_1 + \lambda_2)$. Clearly $P(\lambda_1 + \lambda_2)$ is a θ -split quasi parabolic k -subgroup of G . By the minimality condition of P_1 we get $P_1 = P(\lambda_1 + \lambda_2) \subset hP_2h^{-1}$. By symmetry, we also have that $h^{-1}P_1h \subset P_2$. Thus $hP_2h^{-1} = P_1$. \square

For minimal θ -split parabolic k -subgroups one can give a similar result using [15, 4.11]. These parabolic k -subgroups are not H_k -conjugate, as can be seen from [15, Example 4.12]. For the maximal quasi k -split tori we can prove now a similar result:

Corollary 3.17. *All θ -stable maximal quasi k -split tori of G with A^- maximal are conjugate under H^0 .*

Proof. Let A_1, A_2 be θ -stable maximal quasi k -split tori with A_1^- and A_2^- maximal. Let $P_1 \supset A_1$ and $P_2 \supset A_2$ be minimal θ -split quasi parabolic k -subgroups of G . By Proposition 3.16 There exists $h \in H^0$ such that $hP_2h^{-1} = P_1$. So we may assume $P_1 = P_2$. Let $P_0 \supset A_1$ and $P'_0 \supset A_2$ be minimal quasi parabolic k -subgroups contained in P_1 . Since $H^0P_0 = H^0P'_0$ open in G , it follows that P_0 and P'_0 are H^0 -conjugate. So we may assume $P_0 = P'_0$ contains A_1 and A_2 . But then by Lemma 3.4 A_1 and A_2 are conjugate under $H \cap R_u(P_0)$. Finally from Lemma 3.1 it follows that A_1 and A_2 are conjugate under H^0 . \square

4. Standard k -split tori

The H_k -conjugacy classes of θ -stable maximal k -split tori are determined by the image and fibers of the map $\zeta : \mathcal{A}_k^\theta/H_k \rightarrow \mathcal{A}^\theta/H$ as in (1.2). The image of ζ consists of the H -conjugacy classes of θ -stable maximal k -split tori. To be able to give a characterization of these in terms of conjugacy classes of involutions in a Weyl group, we will first show in this section that every conjugacy class contains a so called standard torus. A characterization of the H -conjugacy classes of these standard tori will be given in section 6.

4.1. We use the notation of section 1. In particular let G be a connected reductive k -group, θ an involution of G defined over k and H a k -open subgroup of G_θ . Given a θ -stable torus A , we reserve the notation A^+ and A^- for A_θ^+ and A_θ^- respectively. For other involutions of A , we shall keep the subscript. As in 1.10 let \mathcal{A}_k^θ denote the set of θ -stable maximal k -split tori of G and \mathcal{A}^θ the image of ζ , i.e. the set of θ -stable quasi k -split tori of G , which are H -conjugate to a θ -stable maximal k -split torus.

For quasi k -split tori we defined singular roots with respect to the involution (see 1.18). For k -split tori we have to combine this with the k -structure of the group itself. We will first define this in the following.

4.2. Let A be a θ -stable maximal k -split torus. For $\alpha \in \Phi(A)$ let $A_\alpha = \{a \in A_1 \mid s_\alpha(a) = a\}^0$, $G_\alpha = Z_G(A_\alpha)$ and $\overline{G}_\alpha = [G_\alpha, G_\alpha]$. If α is either real or imaginary, then G_α is θ -stable.

The θ -singular roots can be divided in those which are singular with respect to the k -structure and those which are not. These are defined now as follows.

Definition 4.3. A real θ -singular root $\alpha \in \Phi(A)$ is called (θ, k) -singular (resp. θ -singular anisotropic) if $\overline{G}_\alpha \cap H$ is isotropic (resp. $\overline{G}_\alpha \cap H$ is anisotropic).

Similarly a imaginary θ -singular root is called (θ, k) -singular (resp. θ -singular anisotropic) if \overline{G}_α has a non-trivial (θ, k) -split torus (resp. \overline{G}_α has no non-trivial (θ, k) -split tori).

Similar as for quasi k -split tori we have the following result.

Proposition 4.4. *Let A be a θ -stable maximal k -split torus of G . Then we have the following.*

- (1) A^+ is a maximal k -split torus of H if and only if $\Phi(A)$ has no (θ, k) -singular real roots.
- (2) A^- is a maximal (θ, k) -split torus of G if and only if $\Phi(A)$ has no (θ, k) -singular imaginary roots.

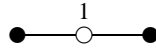
Proof. We will prove (1). The proof of (2) is similar.

Assume first that A^+ is a maximal k -split torus of H . If $\alpha \in \Phi(A)$ is a (θ, k) -singular real root, then $\overline{G}_\alpha \cap H$ contains a k -split torus, say S . Since α is real, $A^+ \subset \ker(\alpha) \subset Z(G_\alpha)$. So $S.A^+$ is a k -split torus of $G_\alpha \cap H$, what contradicts the maximality of A^+ .

Conversely assume $\Phi(A)$ has no (θ, k) -singular real roots. By passing to $Z_G(A^+)$, we can assume that $A^+ = \{e\}$. Since $\Phi(A)$ has no (θ, k) -singular real roots, it follows that $G_\alpha \cap H$ is anisotropic for all $\alpha \in \Phi(A)$. But since A is maximal k -split, we get that H is anisotropic and hence does not contain any k -split tori. \square

Remark 4.5. From Proposition 3.8 it follows that every real root of a θ -stable maximal k -split torus A is θ -singular. Unfortunately not all real roots are (θ, k) -singular. In section 9 we will show that the set of θ -singular roots in $\Phi(A)$ can be determined using only the (Γ, θ) -index of the k -involution θ . However whether a θ -singular root is (θ, k) -singular or not will depend on other properties of the k -involution, like the quadratic elements. We illustrate this in the following example.

Example 4.6. Let $k = \mathbb{R}$. Assume $G_{\mathbb{R}} = SU^*(4, \mathbb{R})$ and θ is of type A_{II} . Then the (Γ, θ) -index of (G, θ) is:



There are two k -involutions related to this (Γ, θ) -index. Let θ_1 be the standard involution of type $A_3^1(II)$ and $\theta_2 = \theta_1 \text{Int}(\epsilon_1)$ the involution of type $A_3^1(II)(\epsilon_1)$. We use here the same notation as in [9]. See also 9.19.

In both these cases G has a θ -split maximal k -split torus A and $\Phi(A) = \Phi(A^-)$ is of type A_1 . In the first case the Lie algebra of $(G_{\theta_1})_k$ is $\mathfrak{sp}(2)$ and

$(G_{\theta_1})_k^0$ is compact. So all roots of $\Phi(A)$ are θ -singular, but not (θ, k) -singular. In the second case the Lie algebra of $(G_{\theta_2})_k$ is $\mathfrak{sp}(1, 1)$ and $(G_{\theta_2})_k^0$ is isotropic. So all roots of $\Phi(A)$ are (θ, k) -singular.

Note that in this example $\Phi(T) \cap \Phi(T^-) = \emptyset$, where $T \supset A$ a maximal torus of G . So the reflections s_λ , $\lambda \in \Phi(A)$ do not come from reflections in $\Phi(T) \cap \Phi(T^-)$.

Similar as for the conjugacy classes of θ -stable maximal tori, there is only one conjugacy class with A^+ (resp. A^-) maximal. From [15, 10.3] we get the following.

Lemma 4.7. *Let A_1, A_2 be θ -stable maximal k -split tori with A_i^- ($i = 1, 2$) a maximal (θ, k) -split torus of G . Then A_1, A_2 are conjugate under H^0 . They are conjugate under H_k if and only if A_1^- and A_2^- are conjugate under H_k .*

Proof. From [15, 10.3] it follows that A_1^- and A_2^- are conjugate under $(H^0 Z_G(A_1))_k$. Say $h \in H^0$ and $z \in Z_G(A_1)$ such that $hzA_1^-z^{-1}h^{-1} = A_2^-$. Then A_2 and $A_3 = hA_1h^{-1}$ are θ -stable maximal k -split tori containing A_2^- . Since A_3^+ and A_2^+ are maximal k -split tori of $Z_G(A_2^-) \cap H$, they are conjugate under $(Z_G(A_2^-) \cap H)_k$, what proves the result. \square

This characterizes the θ -stable maximal k -split tori A with A_θ^- maximal. For θ -stable maximal k -split tori A with A_θ^+ maximal we can prove a similar result to Lemma 4.7.

Lemma 4.8. *Let A_1, A_2 be θ -stable maximal k -split tori containing a maximal k -split torus of H . Then A_1, A_2 are conjugate under H^0 . They are conjugate under H_k if and only if A_1^- and A_2^- are conjugate under H_k .*

Proof. Since all maximal k -split tori of H are conjugate under H_k , we may assume that $A_1^+ = A_2^+$. But then, since A_1^- and A_2^- are maximal (θ, k) -split in $Z_G(A_1^+)$, the result follows from [15, 10.3]. \square

Definition 4.9. For $A_1, A_2 \in \mathcal{A}_0^\theta$, the pair (A_1, A_2) is called *standard* if $A_1^- \subset A_2^-$ and $A_1^+ \supset A_2^+$. In this case, we also say that A_1 is *standard* with respect to A_2 .

One can always choose a minimal and a maximal element in \mathcal{A}_0^θ standard with respect to each other.

Lemma 4.10. *Let A be a θ -stable maximal k -split torus with A^- maximal (θ, k) -split. There exists a θ -stable maximal k -split torus S standard with respect to A and such that S^+ a maximal k -split torus of H .*

Proof. This result follows from Proposition 4.4 with an easy induction. \square

Any θ -stable maximal k -split torus is standard with respect to one containing a maximal (θ, k) -split torus (resp. a maximal k -split torus of H).

Lemma 4.11. *Let A_1 be a θ -stable maximal k -split torus. Then there exist a standard pair (A, S) of θ -stable maximal k -split tori, with A^- maximal (θ, k) -split, S^+ a maximal k -split torus of H and A_1 is standard with respect to A and S .*

Proof. Let $A_0 \supset A_1^-$ be a θ -stable maximal k -split torus with A_0^- a maximal (θ, k) -split torus of G . Then both A_0 and A_1 are contained in $Z_G(A_1^-)$ and A_1^+ is a maximal k -split torus of $Z_G(A_1^-) \cap H$. Since all maximal k -split tori of H are conjugate under H_k , there exists $h \in (Z_G(A_1^-) \cap H)_k$ such that $hA_0^+h^{-1} \subset A_1^+$. Then $A = hA_0h^{-1}$ is standard with respect to A_1 .

As for the second statement it suffices to consider $Z_G(A_1^+)$. Then A_1^- is a maximal (θ, k) -split torus of $Z_G(A_1^+)$. From Lemma 4.10 it follows that there exists a θ -stable maximal k -split torus S of $Z_G(A_1^+)$ with S^+ a maximal k -split torus of $Z_G(A_1^+) \cap H$ and S standard with respect to A_1 . \square

Notation 4.12. Let $A_0 \in \mathcal{A}_0^\theta$ (resp. $S_0 \in \mathcal{A}_0^\theta$) be θ -stable maximal k -split tori of G , such that A_0^- (resp. S_0^+) is a maximal (θ, k) -split torus of G (resp. a maximal k -split torus of H). We can choose S_0 to be standard with respect to A_0 . In the following we fix such a standard pair (S_0, A_0) .

It remains to show that any θ -stable maximal k -split torus is H -conjugate with a θ -stable maximal k -split torus standard with respect to A_0 and S_0 .

Proposition 4.13. *Let (S_0, A_0) be a standard pair as above and let $A_1 \in \mathcal{A}_0^\theta$. Then A_1 is H -conjugate with a θ -stable maximal k -split torus, which is standard with respect to A_0 and S_0 .*

Proof. As in Lemma 4.11 let S be a θ -stable maximal k -split torus with S^+ a maximal k -split torus of H and such that A_1 is standard with respect to S . Since S^+ and S_0^+ are conjugate under H_k we may assume that $S^+ = S_0^+$. Then there exists $h_1 \in Z_G(S^+) \cap H$ such that $h_1Sh_1^{-1} = S_0$. Let $A_2 = h_1A_1h_1^{-1}$. By [15, 10.3] there exists $z_1 \in Z_{Z_G(S_0^+)}(S_0)$ such that $z_1h_1 \in G_k$. From this it follows that A_2 is a maximal quasi k -split torus in $Z_G(A_2^+S_0^-)$ with A_2^- maximal. But then by Corollary 3.17 there exists $h_2 \in Z_G(A_2^+S_0^-) \cap H$ such that $h_2A_2^-h_2^{-1}$ is maximal (θ, k) -split in $Z_G(A_2^+S_0^-)$. Let $A_3 = h_2A_2h_2^{-1}$. This is a θ -stable maximal k -split torus, standard with respect to S_0 .

To prove that A_3 is H -conjugate with a θ -stable maximal k -split torus standard with respect to A_0 and S_0 , we may restrict to $Z_G(S_0^-)$. So assume that $S_0^- = \{e\}$. Let $A \supset A_3^-$ be a θ -stable maximal k -split torus with A^- a maximal (θ, k) -split torus of G and such that A is standard with respect to A_3 and S_0 . By Lemma 4.8 there exists $h_3 \in H$ such that $h_3 A h_3^{-1} = A_0$. Since A_0^+ and A^+ are k -split tori of H and $h_3 A^+ h_3^{-1} = A_0^+$, it follows that there exists $h_4 \in H_k$ such that $h_4 A^+ h_4^{-1} = A_0^+$. Since both A^+ and A_0^+ are contained in S_0 and since all maximal k -split tori of H are conjugate under H_k , it follows that we may assume that $h_4 \in N_{H_k}(S_0)$. Now $A_4 = h_4 A_3 h_4^{-1}$ is standard with respect to S_0 and $A_4^+ \supset A_0^+$.

Consider now $Z_G(A_4^+)$. Then $C = A_0 \cap Z_G(A_4^+)$ is a maximal (θ, k) -split torus of $Z_G(A_4^+)$. Since A_4^- is also a maximal (θ, k) -split torus of $Z_G(A_4^+)$, there exists $h_5 \in Z_G(A_4^+) \cap H$ such that $A_5 = h_5 A_4 h_5^{-1}$ is standard with respect to A_0 . This proves the result. \square

Similar as in the case that the base field k is algebraically closed (see [10]), a standard pair (A_1, A_2) of θ -stable maximal k -split tori of G gives rise to an involution in $W(A_1)$ (resp. $W(A_2)$).

Lemma 4.14. *Let (A_1, A_2) be a standard pair of θ -stable maximal k -split tori of G . Then we have the following conditions:*

- (i) *There exists $g \in Z_G(A_1^- A_2^+)$ such that $g A_1 g^{-1} = A_2$.*
- (ii) *If $n_1 = \theta(g)^{-1} g$ and $n_2 = \theta(g) g^{-1}$, then $n_1 \in N_G(A_1)$ and $n_2 \in N_G(A_2)$.*
- (iii) *Let w_1 and w_2 be the images of n_1 and n_2 in $W(A_1)$ and $W(A_2)$ respectively. Then $w_1^2 = \text{id}$, $w_2^2 = \text{id}$, and $(A_1)_{w_1}^+ = (A_2)_{w_2}^+ = A_1^- A_2^+$, which characterizes w_1 and w_2 .*

Proof. Since A_1 and A_2 are maximal k -split tori of $Z_G(A_1^- A_2^+)$, the first statement is clear. For (ii) and (iii), consider first n_1 . Since $g \in Z_G(A_1^- A_2^+)$ and $A_1 = A_1^- A_2^+ (A_1^+ \cap g^{-1} A_2^- g)$ it suffices to look at $A_1^+ \cap g^{-1} A_2^- g$. So let $x \in A_1^+ \cap g^{-1} A_2^- g$ and write $x = g^{-1} t g$ with $t \in A_2^-$. Then $n_1 x n_1^{-1} = \theta(g^{-1} t^{-1} g) = \theta(x)^{-1} = x^{-1}$. It follows that

$$\text{Int}(n_1)|_{A_1^- A_2^+} = \text{id}, \quad \text{Int}(n_1)|_{A_1^+ \cap g^{-1} A_2^- g} = -\text{id}$$

which implies that $n_1 \in N_G(A_1)$, $w_1^2 = \text{id}$ and $(A_1)_{w_1}^+ = A_1^- A_2^+$.

The assertion for n_2 and w_2 follows with a similar argument. \square

Remark 4.15. By (iii) of Lemma 4.14, w_1 and w_2 are independent of the choice of the element $g \in Z_G(A_1^- A_1^+)$ with $g A_1 g^{-1} = A_2$.

The above leads to the following definition of standard involutions:

Definition 4.16. Let $A_1, A_2, w_1 \in W(A_1)$ and $w_2 \in W(A_2)$ be as in Lemma 4.14. We call w_1 (resp. w_2) the A_2 -standard involution (resp. A_1 -standard involution) of $W(A_1)$ (resp. $W(A_2)$). Moreover we will call an involution $w \in W(A_2)$ a k -standard involution, if there exists a θ -stable maximal k -split torus A_3 standard with respect to A_2 such that w is the A_3 -standard involution in $W(A_2)$.

Remark 4.17. To show that the H -conjugacy classes in \mathcal{A}_0^θ correspond to conjugacy classes of the k -standard involutions, we need to prove first a similar result for \mathcal{A}^θ/H , the H -conjugacy classes of θ -stable maximal quasi k -split tori. Namely if $A_1, A_2 \in \mathcal{A}_0^\theta$ standard with respect to A_0 and $h \in H$ such that $hA_1h^{-1} = A_2$, then $A_3 = hA_0h^{-1}$ and A_0 are θ -stable maximal quasi k -split tori with A_3^- and A_0^- maximal. To prove that the A_1 -standard involution and the A_2 -standard involution are conjugate under $W(A_0)$ we will need to show that A_3 and A_0 are actually maximal quasi k -split tori of $Z_G(A_2^-)$. This will be shown in the next section, where we have a closer look at the H -conjugacy classes of θ -stable maximal quasi k -split tori.

5. Standard quasi k -split tori

In this section we show that, similar as in the case of k -split tori, every H -conjugacy class of θ -stable maximal quasi k -split tori contains a standard torus. The conjugacy classes in \mathcal{A}^θ/H are not only of importance for a characterization of the orbits in V (and hence also for a characterization of the double cosets $H \backslash G/P$), but also for a classification of the subset \mathcal{A}_0^θ/H of \mathcal{A}^θ/H . The characterization of \mathcal{A}^θ/H is more complicated than that of \mathcal{A}_0^θ/H .

5.1. Here we set the notations. Let G be a connected reductive k -group, θ an involution of G defined over k . Let H be a k -open subgroup of G_θ . As in 1.8 let \mathcal{A}^θ denote the set of θ -stable maximal quasi k -split tori of G . In the remainder of this section let A_0 denote a θ -stable maximal k -split torus with A_0^- a maximal (θ, k) -split torus of G and $T \supset A_0$ a maximal torus of G , such that T_θ^- is a maximal θ -split torus of G . As in 1.5 we write \mathcal{V} for $\tau^{-1}(N_G(A_0))$.

Remark 5.2. The H -conjugacy classes of θ -stable maximal quasi k -split tori correspond with twisted involutions in the Weyl group. This can be seen as follows. Let $A_1, A_2 \in \mathcal{A}^\theta$ be θ -stable maximal quasi k -split tori of G and let $g \in G$ be such that $A_2 = gA_1g^{-1}$. Since both A_1 and A_2 are θ -stable it follows that $n = g^{-1}\theta(g) \in N_G(A_1)$. The corresponding Weyl group element $w =$

$\varphi(g) \in W(A_1)$ is a twisted involution. Let Δ be a basis of $\Phi(A_1)$, $w_0 \in W(A_1)$ such that $w_0(\Phi(A_1)^+) = \theta(\Phi(A_1)^+)$ and $\theta' = \theta w_0$. From Proposition 1.27 it follows that there exists a subset Π of Δ and $\alpha_1, \dots, \alpha_k \in \Delta$ such that

$$(5.1) \quad w' = w w_0 = s_1 \dots s_k w_\Pi^0 \theta'(s_k) \dots \theta'(s_1)$$

and $l(w') = 2k + l(w_\Pi^0)$. We can reduce to w_Π^0 by removing the twisting factor $s_1 \dots s_k$ as follows. Choose $n_1, \dots, n_k \in N_G(A_1)$ with images s_1, \dots, s_k in $W(A_1)$ respectively. Set $u = g n_1 \dots n_k$, $m = \tau(u) = u^{-1} \theta(u)$, $w_1 = \varphi(u) \in W(A_1)$. Then $w_1 w_0 = w_\Pi^0$.

In the case of H -conjugacy classes of θ -stable maximal tori the conjugacy classes of the tori corresponded to conjugacy classes of the involutions w_Π^0 (see [10, 3.7]). Unfortunately for θ -stable maximal quasi k -split tori this is no longer true and consequently the involutions w_Π^0 can correspond to more than one H -conjugacy class of θ -stable maximal quasi k -split tori. Although the conjugacy classes of the involutions w_Π^0 do not always characterize the H -conjugacy classes of θ -stable maximal quasi k -split tori, they provide an important first invariant in the characterization of the H -conjugacy class of these tori. The additional conjugacy classes can also be characterized in terms of Weyl group elements of a maximal torus containing A_1 .

The involutions w_Π^0 correspond to tori in standard position. In the following we first analyze this correspondence. Similar as in 4.9 we define standard pairs of θ -stable maximal quasi k -split tori as follows.

Definition 5.3. For $A_1, A_2 \in \mathcal{A}^\theta$, the pair (A_1, A_2) is called *almost standard* if $A_1^- \subset A_2^-$. An almost standard pair (A_1, A_2) is called *standard* if $A_1^+ \supset A_2^+$. In these cases, we also say that A_1 is *almost standard* (resp. *standard*) with respect to A_2 .

The conjugacy of the θ -stable maximal quasi k -split tori A with A^- maximal (see Corollary 3.17) enables us to show that any θ -stable maximal quasi k -split torus is conjugate to one almost standard with respect to A_0 .

Proposition 5.4. *Let A_1 be a θ -stable maximal quasi k -split torus of G . Then there exists $h \in H^0$ such that $h A_1^- h^{-1} \subset A_0^-$.*

Proof. We use induction with respect to $\dim A_0^- - \dim A_1^-$. If $\dim A_0^- - \dim A_1^- = 1$, then since A_1^- is not maximal there exists $\lambda \in \Phi(A_1)$ a θ -singular imaginary root. By Proposition 3.8 there exists $g \in Z_G((\ker \lambda)^0)$ such that $A_2 = g A_1 g^{-1}$ is θ -stable and $\dim A_2^- = \dim A_1^- + 1 = \dim A_0^-$. But then A_2^- is maximal, hence by Corollary 3.17 there exists $h \in H^0$ such that $h A_2 h^{-1} = A_0$. Since $A_1^- \subset (\ker \lambda)^0 \subset A_2$ it follows that $h A_1^- h^{-1} \subset A_0^-$.

Assume now that $\dim A_0^- - \dim A_1^- = k > 0$. Let $\lambda \in \Phi(A_1)$ be a θ -singular imaginary root and $g \in Z_G((\ker \lambda)^0)$ such that $A_2 = gA_1g^{-1}$ is θ -stable and $\dim A_2^- = \dim A_1^- + 1 = \dim A_0^-$. Since $\dim A_0^- - \dim A_2^- = k - 1$ it follows from the induction hypothesis that there exists $h \in H^0$ such that $hA_2^- \subset A_0^-$. Since $A_1^- \subset A_2^-$ the result follows. \square

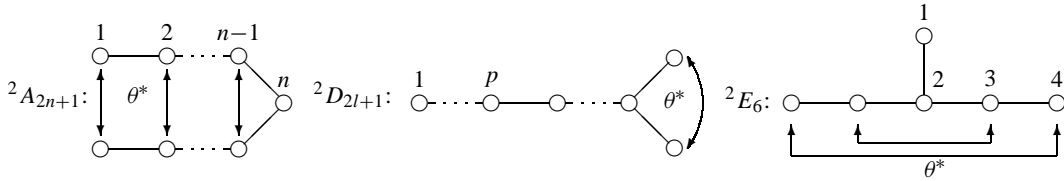
Notations 5.5. If A_1 is a torus and $S \subset A_1$, then we will say that $\alpha \in \Phi(A_1)$ is *perpendicular to S* if $\alpha(S) = e$. If $w \in W(A_1)$ is an involution, then S is called the *orthogonal complement of w* if $S = (A_1)_w^+$. An involution $w \in W(A_1)$ is called a *maximal involution* if it is the longest involution of $W(A_1)$ with respect to a basis of $\Phi(A_1)$.

The following result characterizes the orthogonal complement in A_0 of a θ -stable maximal quasi k -split torus almost standard with respect to A_0 .

Lemma 5.6. *Let A_1 be a θ -stable maximal quasi k -split torus almost standard with respect to A_0 . There exists an involution $w \in W(A_0)$ with $(A_0)_w^- \subset A_0^-$ and such that $A_1^- = ((A_0)_w^- \cap A_0^-)^0$.*

Proof. If θ exchanges two irreducible component of $\Phi(A_0)$, then both A_0^- and A_0^+ are maximal, since $\Phi(A_0)$ has no real or imaginary roots. So we may assume that $\Phi(A_0)$ is irreducible.

If $\Phi(A_1)$ contains a real root α , then we can reduce to $Z_G(\alpha^\vee)$ and then the result follows with induction. So assume $\Phi(A_1)$ contains no real roots. From Proposition 3.8 it follows with an easy induction that there exist strongly orthogonal θ -singular imaginary roots $\alpha_1, \dots, \alpha_n \in \Phi(A_1)$ ($n = \dim A_0^- - \dim A_1^-$) and $x \in Z_G((A_1)_w^+)$ such that $A_2 = xA_1x^{-1}$ is θ -stable and $\dim A_2^- = \dim A_1^- + n = \dim A_0^-$. Here $w = s_{\alpha_1} \cdots s_{\alpha_n}$. By Corollary 3.17 there exists $h \in H^0$ such that $hA_2h^{-1} = A_0$. Note that $\alpha_1, \dots, \alpha_n \in \Phi(A_1)$ map under x to a set of strongly orthogonal real roots in $\Phi(A_2)$. Since $A_1^- \subset (A_0)_w^+$ it follows that $\Phi(A_0)$ contains n strongly orthogonal real roots β_1, \dots, β_n and $\beta_i(hA_1^-h^{-1}) = e$ ($i = 1, \dots, n$). If $A_1^- = \{e\}$, then we are done, so assume $A_1^- \neq \{e\}$. Let Δ_1 be a θ^+ -basis of $\Phi(A_1)$. Since $\Phi(A_1)$ has no real roots and $A_1^- \neq \{e\}$ it follows that the θ -index of $\theta|\Phi(A_1)$ is one of the following:



In each of these cases $\Phi(A_1^-)$ is a root system and if w_0 is the longest involution of $W(A_1)$ with respect to a θ^+ -basis of $\Phi(A_1)$, then $A_1^- = (A_1)_{w_0}^+$ and $A_1^+ = (A_1)_{w_0}^-$. So in particular there are r strongly orthogonal roots $\gamma_1, \dots, \gamma_r$ in $\Phi(A_1)$ ($r = \dim A_1^+$) perpendicular to A_1^- . Note that $\Phi(A_2)$ contains n real roots and $r - n$ imaginary roots perpendicular to A_1^- . Since A_2 and A_0 are H^0 -conjugate and $\Phi(A_1^-) = \{\alpha | A_2^- | \alpha \in \Phi(A_2)\} = \{\beta | A_1^- | \beta \in \Phi(A_0)\}$ it follows that $\Phi(A_0)$ contains n strongly orthogonal real roots perpendicular to A_1^- , what proves the result. \square

This result enables us to prove the following.

Proposition 5.7. *Let A_0 be as above and let $A_1 \in \mathcal{A}^\theta$ be a θ -stable maximal quasi k -split torus, almost standard with respect to A_0 . Then we have the following conditions:*

- (i) *There exists $g \in Z_G(A_1^-)$ such that $gA_1g^{-1} = A_0$.*
- (ii) *If A_1 is standard with respect to A_0 , then there exists $g \in Z_G(A_1^-A_0^+)$ such that $gA_1g^{-1} = A_0$.*

Proof. If $\Phi(A_1)$ contains a real root α , then we can reduce to $Z_G(\alpha^\vee)$ and then the result follows with induction. So assume $\Phi(A_1)$ contains no real roots. Let $\Phi_1 = \{\alpha \in \Phi(A_0) | \theta(\alpha) = -\alpha\}$. Since $\Phi(A_1)$ contains no real roots it follows from Lemma 5.6 that there is a maximal involution w_0 in $W(\Phi_1)$ such that $(A_0^-)_{w_0}^+ = A_1^-$. On the other hand from Proposition 3.8 it follows that there exist strongly orthogonal θ -singular imaginary roots $\alpha_1, \dots, \alpha_n \in \Phi(A_1)$ ($n = \dim A_0^- - \dim A_1^-$) and $x \in Z_G((A_1)_w^+)$ such that $A_2 = xA_1x^{-1}$ is θ -stable and $\dim A_2^- = \dim A_1^- + n = \dim A_0^-$. Here $w = s_{\alpha_1} \cdots s_{\alpha_n}$. By Corollary 3.17 there exists $h \in H^0$ such that $hA_2h^{-1} = A_0$. Note that $w \in W(A_1)$ map under hx to an involution $w' \in W(A_0)$ with $(A_0^-)_{w'} \subset A_0^-$ and $(A_0^-)_{w'}^+ = hA_1^-h^{-1}$. It follows that $w' \in W(\Phi_1)$ is a maximal involution. Since all maximal involutions are conjugate and since $W(\Phi_1)$ has representatives in $N_{H^0}(A_0)$, it follows that there exists $h_1 \in H^0$ such that $h_1hA_1^-h^{-1}h_1^{-1} = A_1^-$. But since $W(A_1^-)$ has representatives in H^0 the result follows.

(ii). Let $g \in G$ such that $gA_1g^{-1} = A_0$. By (i) we may assume that $g \in Z_G(A_1^-)$. So by restricting to $[Z_G(A_1^-), Z_G(A_1^+)]$ we may assume $A_1 \subset H$ and $A_0^+ \subset A_1^+$. If $\alpha \in \Phi(A_0)$ imaginary, then α corresponds with a similar root in $\Phi(A_1)$. Multiplying g , if necessary, on the right with an element of $N_G(A_0)$, we may assume that $g \in Z_G(\alpha^\vee)$ and then the result follows by induction.

So assume $\Phi(A_0)$ contains no imaginary roots. In this case there are n strongly orthogonal real roots $\alpha_1, \dots, \alpha_n \in \Phi(A_0)$ ($n = \dim A_0^-$). Since $\Phi(A_0^+) =$

$\{\alpha|A_0^+ \mid \alpha \in \Phi(A_1)\} = \{\beta|A_0^+ \mid \beta \in \Phi(A_0)\}$ there also need to be n strongly orthogonal imaginary roots $\beta_1, \dots, \beta_n \in \Phi(A_1)$ with $\beta_i(A_0^+) = e$ ($i = 1, \dots, n$). Since $\Phi(A_0)$ has no compact roots, we can choose β_1, \dots, β_n θ -singular. Let $w = s_{\beta_1} \cdots s_{\beta_n}$. Then $(A_1)_w^+ = A_0^+$. Similarly as above let $x \in Z_G(A_0^+)$, such that $A_2 = xA_1x^{-1}$ is θ -stable with $\dim A_2^- = \dim A_1^- + n = \dim A_0^-$. By Corollary 3.17 there exists $h \in H^0$ such that $hA_2h^{-1} = A_0$. Then $hx \in N_G(A_0^+)$. Since A_0^+ is the orthogonal complement of a maximal involution w of $\Phi(A_1)$, it follows that $\Phi(A_0^+)$ is a root system with Weyl group $W(A_0^+) = N_G(A_0^+)/Z_G(A_0^+)$. Assume that $w_1 \in W(A_0^+)$ is the Weyl group element corresponding to $hx \in N_G(A_0^+)$. If $\lambda \in \Phi(A_0^+)$, then take $\alpha \in \Phi(A_1)$ with $\alpha|A_0^+ = \lambda$. Now let $\beta = w(\alpha)$. If $\alpha \perp \beta$ then $s_\lambda = s_\alpha s_\beta$. If $\langle \alpha, \beta \rangle \neq 0$, then $s_\lambda = s_\alpha s_\beta s_\alpha$. In both cases $s_\lambda((A_0)_w^-) = (A_0)_w^-$, so there exists $w_2 \in W(A_1)$ such that $w_2(A_0^+) = A_0^+$ and $w_2|A_0^+ = w_1$. Let $n \in N_G(A_1)$ be a representative. Then $g = hxn^{-1} \in Z_G(A_2^+)$ and $gA_1g^{-1} = A_0$. \square

Corollary 5.8. *Any θ -stable maximal quasi k -split torus is H^0 -conjugate to one standard with respect to A_0 .*

Proof. Let A_1 be a θ -stable maximal quasi k -split torus of G . From Proposition 5.4 it follows that we may assume that A_1 is almost standard with respect to A_0 . Similar as in the proof of Lemma 5.6 there exist strongly orthogonal θ -singular imaginary roots $\alpha_1, \dots, \alpha_n \in \Phi(A_1)$ ($n = \dim A_0^- - \dim A_1^-$) and $x \in Z_G((A_1)_w^+)$ such that $A_2 = xA_1x^{-1}$ is θ -stable and $\dim A_2^- = \dim A_1^- + n = \dim A_0^-$. Here $w = s_{\alpha_1} \cdots s_{\alpha_n}$. By Corollary 3.17 there exists $h \in H^0$ such that $hA_2h^{-1} = A_0$. Let $A_3 = hA_1h^{-1}$. Since $A_1^- \subset (A_1)_w^+ \subset A_2^-$ and $A_2^+ \subset A_1^+$ it follows that $A_3^- = hA_1^-h^{-1} \subset A_0^-$ and $A_3^+ = hA_1^+h^{-1} \supset hA_2^+h^{-1} = A_0^+$. This proves the result. \square

This result reduces the study of H^0 -conjugacy classes of θ -stable maximal quasi k -split tori to H^0 -conjugacy classes of θ -stable maximal quasi k -split tori standard with respect to A_0 . As in Lemma 4.14 we can associate an involution with a θ -stable maximal quasi k -split torus, standard with respect to A_0 .

Lemma 5.9. *Let A_0 be as above and let $A_1 \in \mathcal{A}^\theta$ be a θ -stable maximal quasi k -split torus, standard with respect to A_0 and $g \in Z_G(A_1^- A_0^+)$ such that $gA_1g^{-1} = A_0$. Then we have the following conditions:*

- (i) *If $n = \theta(g)g^{-1}$, then $n \in N_G(A_0)$.*
- (ii) *Let w be the image of n in $W(A_0)$. Then $w^2 = e$ and $(A_0)_w^+ = A_1^- A_0^+$, which characterizes w .*

This result follows using a similar argument as in Lemma 4.14.

Remark 5.10. By (ii) of Lemma 5.9, w is independent of the choice of the element $g \in Z_G(A_1^- A_0^+)$ such that $gA_1g^{-1} = A_0$.

This leads to the following definition of standard involutions for the θ -stable maximal quasi k -split torus, standard with respect to A_0 :

Definition 5.11. Let $A_1 \in \mathcal{A}^\theta$ and $w \in W(A_0)$ be as in Lemma 5.9. We call w the A_1 -standard involution of $W(A_0)$. Moreover we will call an involution $w_0 \in W(A_0)$ a standard involution, if there exists a θ -stable maximal quasi k -split torus A_1 standard with respect to A_0 such that w_0 is the A_1 -standard involution in $W(A_0)$.

For a θ -stable k -torus A_0 of G , write $W(A_0, H)$ for $N_H(A_0)/Z_H(A_0)$. We have now the following result.

Proposition 5.12. *Assume that $A_1, A_2 \in \mathcal{A}^\theta$ such that they are standard with respect to A_0 . Let w_1 and w_2 be the A_1 -standard and A_2 -standard involutions in $W(A_0)$ respectively. If A_1 and A_2 are conjugate under H , then w_1 and w_2 are conjugate under $W(A_0, H)$.*

Proof. Assume $h \in H^0$ such that $hA_1h^{-1} = A_2$. Then $A_3 = hA_0h^{-1}$ is θ -stable with A_3^- maximal. It suffices to show that A_3 and A_0 are conjugate under $H^0 \cap Z_G(A_1^- A_0^+)$.

Let $\alpha \in \Phi(A_1)$ with $\theta(\alpha) = -\alpha$ and β the corresponding root in $\Phi(A_2)$. If A_1^- does not contain any real roots, then A_i^+ ($i = 1, 2$) is maximal and in this case both w_1 and w_2 are maximal involutions in $\Phi_1 = \{\alpha \in \Phi(A_0) \mid \theta(\alpha) = -\alpha\}$ and hence conjugate under $W(\Phi_1) \subset W(A_0, H)$.

Since both α and β correspond to roots in $\Phi(A_0)$, there exists $w \in W(A_0, H^0)$ such that $w(\alpha) = \beta$. Let $h_1 \in H^0$ be a representative of w and $A_4 = h_1A_1h_1^{-1}$. Then $hh_1^{-1}A_4h_1h^{-1} = hA_1h^{-1} = A_2$, $hh_1^{-1}A_0h_1h^{-1} = A_3$ and $hh_1^{-1} \in Z_H(\beta^\vee)$. Using induction it follows that there exists $h_2 \in Z_G(A_2^- A_0^+) \cap H^0$ such that $h_2A_3h_2^{-1} = A_0$. Then $g = h_2h \in N_H(A_0)$ maps w_1 to w_2 . \square

Remarks 5.13. (1) The converse of this result unfortunately does not hold, since the θ -stable maximal quasi k -split tori A_1 with A_1^+ maximal are not conjugate under H . We can still use these standard involutions to characterize the conjugacy classes of θ -stable maximal quasi k -split tori, but a second condition will be needed. This will be discussed in the next section.

(2) That the θ -stable maximal quasi k -split tori A_1 with A_1^+ maximal are not necessarily conjugate under H can be seen as follows. If T is a θ -stable maximal

torus with T^+ a maximal torus of H , then $N_G(T) \neq N_H(T)$. This means that T can contain several H -conjugacy classes of θ -stable maximal quasi k -split tori. These are basically conjugates of the A_0^+ part of these tori. An example of this is included in 3.10, where there are several H -conjugacy classes of maximal quasi k -split tori contained in H .

A consequence of this all is that there is not a unique minimal element in the set of H -conjugacy classes of θ -stable maximal quasi k -split tori. For the minimal element all we can show is the following:

Lemma 5.14. *Let S be a θ -stable maximal quasi k -split torus with S^+ maximal. There exists a θ -stable maximal torus T of G with $T \supset S$ and T^+ is a maximal torus of H .*

This result is immediate from Corollary 3.11.

5.15. Using the results in this section we can now give a description of the involutions w_Π^0 in the characterization of the twisted involutions as in Proposition 1.27.

Let A_0 be a θ -stable maximal k -split torus of G with A_0^- a maximal (θ, k) -split torus and A_1 a θ -stable maximal quasi k -split torus of G . There exists $g \in G$ such that $A_1 = gA_0g^{-1}$. Let $w = \tau(g)Z_G(A_0) \in W(A_0)$. Fix a basis Δ_0 of $\Phi_0 = \Phi_0(-\theta) = \{\alpha \in \Phi(A_0) \mid \theta(\alpha) = -\alpha\}$. We can extend Δ_0 to a θ^- -basis Δ of $\Phi(A_0)$. Let w_0 be the opposition involution of $W(\Phi_0)$ with respect to Δ_0 . Then by 2.14 it follows that $\theta = \theta^*w_0$. So $\theta' = \theta w_0 = \theta^*$ and hence $\theta'(\Delta) = \Delta$. Write $w' = ww_0 = s_{\alpha_1} \dots s_{\alpha_k} w_\Pi^0 \theta'(s_{\alpha_k}) \dots \theta'(s_{\alpha_1})$ as in Proposition 1.27. Here Π is a subset of Δ and $\alpha_1, \dots, \alpha_k \in \Delta$. As in 5.2 we can remove the twisting factor $s_1 \dots s_k$. So assume $w' = ww_0 = w_\Pi^0$.

Proposition 5.16. *Let g, w, w' and w_0 be as above. Then there exists $w_1 \in W(A_0)$ such that $w_1 w_\Pi^0 \theta'(w_1)^{-1} = w_{\Pi_1}^0$, with $\Pi_1 \subset \Delta_0$ and $w_{\Pi_1}^0$ the longest involution of Φ_{Π_1} with respect to Π_1 .*

Proof. By Corollary 5.8 there exists $h \in H^0$ such that $A_2 = hA_1h^{-1}$ is standard with respect to A_0 . Let $g_1 \in Z_G(A_2^- A_0^+)$ such that $g_1 A_0 g_1^{-1} = A_2$. Then $h^{-1}g_1$ and g both map A_0 to A_1 , namely $h^{-1}g_1 A_0 g_1^{-1} h = g A_0 g^{-1} = A_1$. So there exists $n \in N_G(A_0)$ such that $h^{-1}g_1 = gn$. Let $w_1 \in W(A_0)$ be the Weyl group element corresponding to n and let $w_2 = \tau(h^{-1}g_1)Z_G(A_0) = \tau(g_1)Z_G(A_0) \in W(A_0)$. Then $w_2 = w_1^{-1}w\theta(w_1)$ and $(A_0)_{w_2}^- \subset A_0^-$. Since $\theta' = w_0^{-1}\theta = \theta w_0$ and $w' = w_\Pi^0$, we get

(5.2)

$$w_2 = w_1^{-1}w\theta(w_1) = w_1^{-1}ww_0\theta'w_1\theta = w_1^{-1}w'\theta'w_1\theta'w_0^{-1} = w_1^{-1}w_\Pi^0\theta'(w_1)w_0^{-1}.$$

By [10, 2.14] there exists $w_3 \in W(\Phi_0)$ such that $w_3 w_2 w_3^{-1} w_0 = w_{\Pi_1}^0$ with $\Pi_1 \subset \Delta_0$. Together with (5.2) this gives:

$$\begin{aligned} w_{\Pi_1}^0 &= w_3 w_2 w_3^{-1} w_0 = w_3 w_1^{-1} w_{\Pi}^0 \theta' w_1 \theta' w_0^{-1} w_3^{-1} w_0 \\ &= w_3 w_1^{-1} w_{\Pi}^0 \theta' w_1 w_3^{-1} \theta w_0 = w_3 w_1^{-1} w_{\Pi}^0 \theta' (w_1 w_3^{-1}). \end{aligned}$$

This proves the result. \square

6. Characterization of \mathcal{A}_0^θ/H

In this section we use the results of the previous section to get a description of the H -conjugacy classes in \mathcal{A}_0^θ . Throughout this section let $A_0 \in \mathcal{A}_0^\theta$ be a θ -stable maximal k -split torus of G with A_0^- a maximal (θ, k) -split torus of G . From Proposition 5.12 we get the following result.

Proposition 6.1. *Assume that $A_1, A_2 \in \mathcal{A}_0^\theta$ such that they are standard with respect to A_0 . Let w_1 and w_2 be the A_1 -standard and A_2 -standard involutions in $W(A_0)$ respectively. If A_1 and A_2 are H -conjugate then w_1 and w_2 are conjugate under $W(A_0, H)$.*

Remark 6.2. Ideally one would like the converse of this statement to hold, but this is not clear, since an element of $W(A_0, H)$ would map A_1 to a quasi k -split torus, which is not necessarily k -split. However with an additional condition we can show the converse. For this we first note the following.

6.3. We give first another description of the group $W(A_0, H)$. Let $X = X^*(A_0)$, $X_0(\theta)$ and $\Phi_0(\theta)$ be as in 2.2. Write

$$(6.1) \quad W_1(\theta) = W_1(A_0, \theta) = \{w \in W(A_0) \mid w(X_0(\theta)) \subset X_0(\theta)\}$$

and $W_0(\theta) = W_0(A_0, \theta) = W(\Phi_0(\theta))$. Then by [25, 2.1.3] we have

$$(6.2) \quad W(A_0^-) \simeq W_1(\theta)/W_0(\theta).$$

The group $W(A_0, H)$ corresponds with $W_1(\theta)$ due to the following result.

Proposition 6.4. *Let $A_0 \in \mathcal{A}_0^\theta$ be a θ -stable maximal k -split torus of G with A_0^- a maximal (θ, k) -split torus of G . Then we have the following.*

- (i) Any $w \in W(A_0^-)$ has a representative in $(H^0 Z_G(A_0))_k \cap N_G(A_0^-)$.
- (ii) $N_G(A_0^-) = N_{H^0}(A_0^-) Z_G(A_0^-)$.

Proof. (i) Let $n \in N_{G_k}(A_0^-)$ and P a minimal θ -split parabolic k -subgroup of G . Then $P_1 = n P n^{-1}$ is also a minimal θ -split parabolic k -subgroup of G containing A_0 . By [15, 4.9] there exists $x \in (H^0 P)_k$ such that $x P x^{-1} = P_1$. Let P_0 be a minimal parabolic k -subgroup of P containing A_0 . By [15, 4.8] $H^0 P_0 = H^0 P$.

On the other hand $(H^0 P)_k = (H^0 Z_G(A_0))_k U_k$, where $U = R_u(P_0)$ (see [15, 10.2]). It follows that $x = hzu$ with $h \in H^0$, $z \in Z_G(A_0)$ and $u \in U_k$. If we take $g = hz \in (H^0 Z_G(A_0))_k$, then $gPg^{-1} = P_1$ and $gA_0^-g^{-1}$ is (θ, k) -split. Moreover $gA_0^-g^{-1} \subset P_1 \cap \theta(P_1) = Z_G(A_0^-)$, so $gA_0^-g^{-1}A_0^-$ is a (θ, k) -split torus of G . Since A_0^- is maximal (θ, k) -split it follows that $gA_0^-g^{-1} = A_0^-$, what proves the result.

(ii) follows immediately from (i). \square

Remark 6.5. Ideally one would like $W(A_0^-)$ to have representatives in H_k , but as the proof of this result indicates, this will unfortunately not always be true. However in many cases, including the standard pairs for $k = \mathbb{R}$, one can show that $W(A_0^-)$ has representatives in H_k . It is an interesting open question to give necessary and sufficient conditions such that $W(A_0^-)$ has representatives in H_k .

Corollary 6.6. *Let $A_0 \in \mathcal{A}_0^\theta$ be a θ -stable maximal k -split torus of G with A_0^- a maximal (θ, k) -split torus of G . Then $W_0(A_0, H) = W_1(A_0, \theta) = W_1(\theta)$.*

Proof. Clearly $W_0(A_0, H) \subset W_1(A_0, \theta)$. As for the other inclusion, it suffices by Proposition 6.4 to show that $W_0(\theta) = W_0(A_0, \theta) \subset W_0(A_0, H)$. But this follows from the fact that every root $\alpha \in \Phi_0(A_0, \theta)$ is compact and therefore $G_\alpha \subset H$. \square

The study of $W_1(\theta)$ -conjugacy classes of involutions in $W(A_0) \cap W(A_0^-)$ can be reduced to conjugacy classes under the action of the full Weyl group. Before we show this we need first a little more notation. Let $A \in \mathcal{A}_0^\theta$ be a θ -stable maximal k -split torus of G . Write $E = X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$, $\Phi = \Phi(A)$, $W = W(A)$ and for $\sigma \in \text{Aut}(\Phi)$ denote the eigenspace of σ for the eigenvalue ξ , by $E(\sigma, \xi)$. Write $\Phi(\sigma)$ for $\Phi_0(-\sigma)$. We have now the following result:

Proposition 6.7. *Let θ be an involution of Φ such that $\bar{\Phi}_\theta$ is a root system with Weyl group \bar{W}_θ . If $w_1, w_2 \in W$ are involutions with $E(w_i, -1) \subset E(\theta, -1)$ ($i = 1, 2$), then the following are equivalent:*

- (i) w_1 and w_2 are conjugate under W .
- (ii) w_1 and w_2 are conjugate under $W_1(\theta)$.
- (iii) $w_1\theta$ and $w_2\theta$ are conjugate under W .
- (iv) $w_1\theta$ and $w_2\theta$ are conjugate under $W_1(\theta)$.

Proof. (i) \Leftrightarrow (ii) follows from [10, 2.16].

(ii) \Leftrightarrow (iv) follows from the fact that $W_1(\theta)$ maps A_0^- onto itself, since $\bar{W}_\theta \cong W_1(\theta)/W_0(\theta)$. Since (iv) \Rightarrow (iii) is clear, it remains to show (iii) \Rightarrow (iv).

Assume $w \in W$ such that $ww_1\theta w^{-1} = w_2\theta$. If $\theta \in W$ then the result follows from (i) \Rightarrow (ii). So assume $\theta \notin W$. We will use induction on the rank of Φ .

If $\text{rank } \Phi = 1$, then $\theta = -\text{id}$, $E(w_1, -1) = E(w_2, -1) = E$ and the result is clear.

Assume now that the result is proved for root systems of rank $\leq n$ and assume that Φ has rank $n + 1$. Since $w w_1 \theta w^{-1} = w_2 \theta$, we have $w(\Phi(w_1 \theta)) = \Phi(w_2 \theta)$. If $\Phi(w_1 \theta) = \emptyset$, then w_1 and w_2 are opposition involutions of $W(\theta)$ with respect to bases of $\Phi(\theta)$ and hence are conjugate under $W(\theta)$ (see [10, 2.13]). So assume now that $\Phi(w_1 \theta) \neq \emptyset$. Let $\alpha \in \Phi(w_1 \theta)$ and $\beta = w(\alpha) \in \Phi(w_2 \theta)$. Since $E(w_i, -1) \subset E(\theta, -1)$ ($i = 1, 2$), it follows that both $\alpha, \beta \in \bar{\Phi}_\theta$. Moreover, since $\beta = w(\alpha)$, both α and β are contained in one irreducible component of Φ . Hence by [10, 2.15] contained in one irreducible component of $\bar{\Phi}_\theta$. Since $W(\bar{\Phi}_\theta) \cong W_1(\theta)/W_0(\theta)$, it follows that there exists $\tilde{w} \in W_1(\theta)$ such that $\tilde{w}(\alpha) = \beta$.

Let $r_1 = s_\beta \tilde{w} w_1 \theta \tilde{w}^{-1} = s_\beta \tilde{w} w_1 \tilde{w}^{-1} \theta$, $r_2 = s_\beta w_2 \theta$, $w_0 = w \tilde{w}^{-1}$ and write

$$\Phi_\beta = \Phi_0(s_\beta) = \{\gamma \in \Phi \mid (\beta, \gamma) = 0\}.$$

Now $r_1(\beta) = r_2(\beta) = w_0(\beta) = \beta$, so by [4, 2.5.5] $w_0 \in W(\Phi_\beta) = W_0(s_\beta)$. Since $\theta(\beta) = -\beta$ both $r_1 \theta = s_\beta \tilde{w} w_1 \tilde{w}^{-1} \theta$ and $r_2 \theta = s_\beta w_2 \theta$ are contained in $W(\Phi_\beta)$. Since $w_0 r_1 w_0^{-1} = r_2$ it follows with induction that there exists $\tilde{r} \in W_1(\theta) \cap W_0(s_\beta)$ such that $\tilde{r} r_1 \tilde{r}^{-1} = r_2$. Now $r = \tilde{r} \tilde{w} \in W_1(\theta)$ and $r w_1 \theta r^{-1} = w_2 \theta$, which proves the result. \square

We can now prove the following characterization of the conjugacy classes in \mathcal{A}_0^θ/H in the case that $W(A_0^-)$ has representatives in H_k .

Theorem 6.8. *Let $A_0 \in \mathcal{A}_0^\theta$ be a θ -stable maximal k -split torus of G with A_0^- a maximal (θ, k) -split torus of G and assume $W(A_0^-)$ has representatives in H_k . Let $A_1, A_2 \in \mathcal{A}_0^\theta$ be standard with respect to A_0 and w_1, w_2 the A_1 -standard and A_2 -standard involutions in $W(A_0)$. Then the following are equivalent.*

- (i) A_1 and A_2 are H -conjugate.
- (ii) A_1 and A_2 are H_k -conjugate.
- (iii) w_1 and w_2 are conjugate under $W(A_0, H_k)$.
- (iv) w_1 and w_2 are conjugate under $W(A_0, H)$.
- (v) w_1 and w_2 are conjugate under $W(A_0)$.

Proof. (iv) \Leftrightarrow (v) follows from Proposition 6.7 and (iii) \Rightarrow (iv) is clear.

(iv) \Rightarrow (iii). Since w_1 and w_2 are contained in $W(A_0) \cap W(A_0^-)$, there exist $w \in W(A_0^-)$ such that $w w_1 w^{-1} = w_2$. Let $h \in H_k \cap N_G(A_0^-)$ be a representative of w . Since $h A_0 h^{-1}$ is a θ -stable maximal k -split torus of G containing A_0^- , it is conjugate to A_0 under $H_k \cap Z_G(A_0^-)$ (see Lemma 4.8). So we may assume that $h \in N_{H_k}(A_0)$, what proves (iii).

(i) \Rightarrow (iv) follows from Proposition 6.1 and (ii) \Rightarrow (i) is clear. So it remains to show (iii) \Rightarrow (ii). Let $w \in W(A_0, H_k)$ be such that $ww_1w^{-1} = w_2$ and let $h \in N_{H_k}(A_0)$ be a representative of w . Then $A_3 = hA_1h^{-1}$ is a θ -stable maximal k -split torus of G with $A_3^- = A_2^-$. Restricting to $Z_G(A_2^-)$ it follows that A_2^+ and A_3^+ are maximal k -split tori of $Z_G(A_2^-) \cap H$ and hence A_2 and A_3 are conjugate under $(Z_G(A_2^-) \cap H)_k$. But then A_1 and A_2 are H_k -conjugate, what proves the result. \square

Remark 6.9. Note that the condition “ $W(A_0^-)$ has representatives in H_k ” is only needed for the proof of (iv) \Rightarrow (iii) and (iii) \Rightarrow (ii).

The equivalence of (i) and (ii) means that the orbit map $\zeta : \mathcal{A}_k^\theta/H_k \rightarrow \mathcal{A}^\theta/H$ as in (1.2) is one-to-one.

Corollary 6.10. *Let $A_0 \in \mathcal{A}_0^\theta$ be a θ -stable maximal k -split torus of G with A_0^- a maximal (θ, k) -split torus of G and assume $W(A_0^-)$ has representatives in H_k . Then the map $\zeta : \mathcal{A}_k^\theta/H_k \rightarrow \mathcal{A}^\theta/H$ as in (1.2) is one-to-one.*

Remarks 6.11. (1) This result is essential in the classification of \mathcal{A}_k/H_k in the case that $k = \mathbb{R}$. In this case the condition “ $W(A_0^-)$ has representatives in H_k ” is satisfied for many pairs (G_k, H_k) including the standard pairs. For the remaining pairs one has to include the action of the quadratic elements, what leads to non-trivial fibers of ζ . These fibers can essentially be described by $W(A_0, H)/W(A_0, H_k)$. For more details, see [13].

(2) The above results provides a sound criterion when elements in \mathcal{A}_0^θ are H -conjugate in the case that $W(A_0^-)$ has representatives in H_k . In that case it actually gives even a characterization of the classes in \mathcal{A}_k^θ/H_k . To complete the characterization of H -conjugacy classes of \mathcal{A}_0^θ , it reduces to single out those $w \in W(A_0)$ which are the A -standard involutions for some $A \in \mathcal{A}_0^\theta$. A characterization of these involutions will be given in section 8. First we give a characterization of the H -conjugacy classes in \mathcal{A}^θ .

7. Characterization of \mathcal{A}^θ/H

Although the A -standard involutions do not characterize the H -conjugacy classes of θ -stable maximal quasi k -split tori, they provide an important invariant in the characterization of the H -conjugacy classes in \mathcal{A}^θ . In this section we will characterize the different H -conjugacy classes of θ -stable maximal quasi k -split tori, which correspond with the same A -standard involution. For this we will first show that the A -standard involution can be split in 2 parts. The first part comes from an involution related to a unique conjugacy class of maximal tori containing A and the second part is a twisted involution in the Weyl

group of such a maximal torus. The analyses of this second part will lead to a characterization of the H -conjugacy classes in \mathcal{A}^θ corresponding with the same A -standard involution.

7.1. Let A be a θ -stable maximal quasi k -split torus of G . We will consider the following H -conjugacy classes of maximal tori, associated with A . Let S_A, T_A be θ -stable maximal tori of $Z_G(A)$ with S_A^+ a maximal torus of $Z_G(A) \cap H$ and T_A^- a maximal θ -split torus of $Z_G(A)$. The choice of S_A and T_A is unique up to conjugacy of $Z_G(A) \cap H$ (see [10]).

Definition 7.2. Let A be a θ -stable maximal quasi k -split torus of G and S_A, T_A as above. The pair (A, S_A) is called a θ^+ -pair and the pair (A, T_A) is called a θ^- -pair.

Instead of H -conjugacy classes of θ -stable maximal quasi k -split tori one can restrict to H -conjugacy of θ^+ -pairs or θ^- -pairs.

Proposition 7.3. Let $A_1, A_2 \in \mathcal{A}_\theta$ be θ -stable maximal quasi k -split tori of G and $S_1 = S_{A_1}, S_2 = S_{A_2}$ (resp. $T_1 = T_{A_1}, T_2 = T_{A_2}$) maximal tori of G , such that (A, S_{A_1}) and (A, S_{A_2}) (resp. (A, T_{A_1}) and (A, T_{A_2})) are θ^+ -pairs (resp. θ^- -pairs). Then the following are equivalent.

- (1) A_1 and A_2 are H -conjugate.
- (2) (A_1, S_{A_1}) and (A_2, S_{A_2}) are H -conjugate.
- (3) (A_1, T_{A_1}) and (A_2, T_{A_2}) are H -conjugate.

Proof. (2) \Rightarrow (1) and (3) \Rightarrow (1) are clear. For (1) \Rightarrow (2) and (1) \Rightarrow (3), we may assume that $A_1 = A_2$. Then (1) \Rightarrow (3) follows from the fact that all maximal tori of $Z_G(A_1)$ containing a maximal θ -split torus are conjugate under $Z_G(A_1) \cap H$ (see [32, §1]).

Similarly (1) \Rightarrow (2) follows from the fact that all maximal tori of $Z_G(A_1)$ containing a maximal torus of $Z_G(A_1) \cap H$ are conjugate under $Z_G(A_1) \cap H$ (see [22]). \square

Remark 7.4. The H -conjugacy classes of θ -stable maximal tori were determined in [10], so essentially we have reduced the problem now to a problem about Weyl groups.

Unfortunately Proposition 7.3 does not give any indication as to which θ -stable maximal tori T contain a θ -stable maximal quasi k -split torus A such that (A, T) is a θ^+ (or θ^-) pair. For a characterization of these θ -stable maximal tori, see Proposition 7.10. From now on a θ -stable maximal torus containing a θ -stable maximal quasi k -split torus A , such that (A, T) is a θ^+ (or θ^-) pair will

be called a θ^+ (or θ^-)-admissible maximal torus. Denote the set of all θ^+ pairs (A, T) by \mathcal{A}^+ and the set of all θ^- pairs (A, T) by \mathcal{A}^- .

In the remainder of this subsection we characterize the H -conjugacy classes of pairs in \mathcal{A}^+ . For \mathcal{A}^- one can give a similar characterization.

7.5. Standard pairs. In the following let A_0 denote a θ -stable maximal k -split torus with A_0^- a maximal (θ, k) -split torus of G and $T_0 \supset A_0$ a maximal torus of G , such that (A_0, T_0) is a θ^+ pair. Similar as in [10, 3.2] we call a θ -stable maximal torus T_1 -standard with respect to T_0 if $T_1^- \subset T_0^-$ and $T_1^+ \supset T_0^+$. The corresponding involution $w_1 \in W(T_0)$ is called the T_1 -standard involution (see [10, 3.5]).

For pairs in \mathcal{A}^+ there are several different notions of standard pairs:

Definition 7.6. For $(A_1, T_1) \in \mathcal{A}^+$, the pair (A_1, T_1) is called *almost standard* with respect to (A_0, T_0) if the following conditions are satisfied:

- (1) T_1 is standard with respect to T_0 .
- (2) A_1 is standard with respect to A_0 .

The pair (A_1, T_1) is called *standard* with respect to (A_0, T_0) if it is almost standard with respect to (A_0, T_0) and satisfies additionally:

- (3) $w(A_0) \subset A_0$, where $w \in W(T_0)$ is the T_1 -standard involution.

The pair (A_1, T_1) is called *maximal standard* with respect to (A_0, T_0) if it is standard with respect to (A_0, T_0) and satisfies additionally:

- (4) $(A_1^+ \cap T_0^+)^0 = A_0^+$.
- (5) $(A_0^- \cap T_1^-)^0 = A_1^-$.

A θ -stable maximal torus T_1 is called *standard* with respect to (A_0, T_0) if it satisfies condition (1) and (3).

In order to be able to prove that every pair in \mathcal{A}^+ is H -conjugate with a pair standard with respect to (A_0, T_0) , we need the following result.

Lemma 7.7. *Let A_0 be a θ -stable maximal k -split torus with A_0^- a maximal (θ, k) -split torus of G and $T_1 \supset A_0$ a maximal k -torus of G , such that T_1^- is a maximal θ -split torus of G . There exists a θ -stable maximal torus T_2 standard with respect to T_1 such that T_2^+ is a maximal torus of H and the T_2 -standard involution $w \in W(T_1)$ stabilizes A_0 , i.e. $w(A_0) = A_0$.*

Proof. Let $T_0 \subset Z_G(A_0)$ with T_0^+ a maximal torus of $Z_G(A_0) \cap H$. Since both T_1 and T_0 are contained in $Z_G(A_0)$ we may assume that T_0 is standard with respect to T_1 . Clearly the T_0 -standard involution in $W(T_1)$ leaves A_0 pointwise fixed. Let $G_1 = [Z_G(T_0^+), Z_G(T_0^-)]$ and let $\tilde{T}_1 = T_1 \cap G_1$, $\tilde{A}_0 = A_0 \cap G_1$. It

suffices now to prove the above result for the triple $(G_1, \tilde{T}_1, \tilde{A}_0)$. So from now on assume that $T_1 \supset A_0$ is θ -split and $\Phi_0(\Gamma) = \{\alpha \in \Phi(T_1) \mid \sum_{\sigma \in \Gamma} \alpha^\sigma = 0\} = \emptyset$. It suffices to show now that there is a maximal involution of $W(T_1)$, which leaves A_0 stable.

The Galois group Γ acts on $(X^*(T_1), \Phi(T_1))$ as in 2.1. If Γ acts trivially, then $A_0 = T_1$ and we are done. So assume Γ acts non-trivially. Using induction we may assume that the (Γ, θ) -index is absolutely irreducible. Then, since $\Phi_0(\Gamma) = \emptyset$ and G is θ -split we only have 3 cases:

- (1) $[\sigma]$, $(\sigma \in \Gamma)$ permutes the irreducible components.
- (2) $\Phi(T_1)$ is irreducible, $[\sigma]$, $(\sigma \in \Gamma)$ is a diagram automorphism and $\theta^* = \text{id}$.
- (3) $\Phi(T_1)$ is irreducible, $[\sigma]$, $(\sigma \in \Gamma)$ is a diagram automorphism and $\theta^* \neq \text{id}$.

In (1) we can take the same longest involution in the Weyl group of each irreducible component. In (2) and (3) it follows from [9, 2.17] that the longest involution with respect to the basis commutes with the actions of θ^* and $[\sigma]$, $(\sigma \in \Gamma)$, what proves the result. \square

Proposition 7.8. *Let $(A_1, T_1) \in \mathcal{A}^+$. Then we have the following.*

- (i) *There exists $h \in H^0$ such that (hA_1h^{-1}, hT_1h^{-1}) is almost standard with respect to (A_0, T_0) .*
- (ii) *If (A_1, T_1) is almost standard with respect to (A_0, T_0) , then there exists $n \in N_G(T_0) \cap H$ such that (nA_1n^{-1}, nT_1n^{-1}) is standard with respect to (A_0, T_0) .*

Proof. (i). By Corollary 5.8 we may assume that A_1 is standard with respect to A_0 . Since both T_1 and T_0 are contained in $Z_G(A_1^- A_0^+)$, we can reduce to $Z_G(A_1^- A_0^+)$. So assume that $A_1 \subset H$ and $A_0 = A_0^-$. Since $A_1 \subset H$ it follows that T_1^+ is a maximal torus of H . By [10, 3.3] there exists $h \in H^0$ such that $h^{-1}T_0h$ is standard with respect to T_1 . But then hT_1h^{-1} is standard with respect to T_0 and (hA_1h^{-1}, hT_1h^{-1}) is almost standard with respect to (A_0, T_0) .

(ii). Assume (A_1, T_1) is almost standard with respect to (A_0, T_0) . Similarly as in (i) we can reduce to $Z_G(A_1^- A_0^+)$. So assume $A_1 \subset H$ and $A_0 = A_0^-$. Note that T_1^+ is a maximal torus of H . From Lemma 7.7 it follows that there exists a maximal torus T_3 standard with respect to (A_0, T_0) . Since T_1 and T_3 are H -conjugate and both are standard with respect to T_0 , the T_1 -standard and T_3 -standard involutions in $W(T_0)$ are conjugate under $W(T_0, H)$. Say $h \in N_H(T_0)$ a representative. Then (hA_1h^{-1}, hT_1h^{-1}) is standard with respect to (A_0, T_0) . \square

From Proposition 7.8 we can derive a nice criterium for when a θ -stable maximal torus T_1 contains a θ -stable maximal quasi k -split torus A_1 , such that $(A_1, T_1) \in \mathcal{A}^+$. First we note the following.

Lemma 7.9. *Let T_1 be a θ -stable maximal torus standard with respect to (A_0, T_0) . There exists a θ -stable maximal quasi k -split torus $A_1 \subset T_1$ such that $(A_1, T_1) \in \mathcal{A}^+$ is maximal standard with respect to (A_0, T_0) .*

Proof. Let $w \in W(T_0)$ be the T_1 -standard involution and $g \in Z_G((T_0)_w^+)$ such that $gT_0g^{-1} = T_1$. Let $A_1 = gA_0g^{-1}$. Then $(A_1, T_1) \in \mathcal{A}^+$ is maximal standard with respect to (A_0, T_0) . \square

Combined with Proposition 7.8 this leads to the following:

Proposition 7.10. *Let T_1 be a θ -stable maximal torus standard with respect to T_0 . Then T_1 contains a θ -stable maximal quasi k -split torus A_1 , such that $(A_1, T_1) \in \mathcal{A}^+$ if and only if there exists $n \in N_G(T_0) \cap H$ such that nT_1n^{-1} is standard with respect to (A_0, T_0) .*

Proof. If there exists $n \in N_G(T_0) \cap H$ such that nT_1n^{-1} is standard with respect to (A_0, T_0) , then the results follows from Lemma 7.9.

So assume that T_1 contains a θ -stable maximal quasi k -split torus A_1 , such that $(A_1, T_1) \in \mathcal{A}^+$. By Corollary 5.8 there exists $h \in H^0$ such that hA_1h^{-1} is standard with respect to A_0 . Similar as in Proposition 7.8 there exists $h_1 \in Z_G(hA_1^-h^{-1}A_0^+) \cap H^0$ such that $T_2 = h_1hT_1h^{-1}h_1^{-1}$ is standard with respect to T_0 . But then by [10, 3.6] T_2 and T_1 are conjugate under $N_H(T_0)$, what proves the result. \square

Corollary 7.11. *Let (A_0, T_0) be as above. If $(A_1, T_1) \in \mathcal{A}^+$ is standard with respect to (A_0, T_0) , then there exists $n \in N_G(T_1) \cap Z_G(A_1^-A_0^+)$ such that (nA_1n^{-1}, T_1) is maximal standard with respect to (A_0, T_0) .*

Proof. We may restrict to $Z_G(A_1^-A_0^+)$. So assume $A_1^-A_0^+ = \{e\}$. Since T_1 is standard with respect to T_1 , there exist by Lemma 7.9 a θ -stable maximal quasi k -split torus $A_2 \subset T_1$ such that (A_2, T_1) is maximal standard with respect to (A_0, T_0) . Since both A_1 and A_2 are maximal quasi k -split tori, there exists $x \in G$ such that $xA_1x^{-1} = A_2$. Let $T_2 = xT_1x^{-1}$. Since both T_1 and T_2 contain A_2 , there exists $x \in Z_G(A_2)$ such that $xT_2x^{-1} = T_1$. But then $n = xg \in N_G(T_1)$ and $nA_1n^{-1} = A_2$. \square

7.12. A θ -stable maximal torus T_1 standard with respect to (A_0, T_0) can contain θ -stable maximal quasi k -split tori A_1 and A_2 such that $(A_1, T_1) \in \mathcal{A}^+$ and $(A_2, T_1) \in \mathcal{A}^+$, but A_1 and A_2 are not H -conjugate. An example of this, for

$k = \mathbb{R}$, is given in 4.6. here (G, θ) is of type $A_3^1(II)$ and there exists a maximal quasi k -split torus A_1 , which is contained in $H \cap T_0$ and (A_1, T_0) is automatically standard with respect to (A_0, T_0) .

As in Lemma 4.14 we can associate an involution with a pair $(A_1, T_1) \in \mathcal{A}^+$, standard or maximal standard with respect to (A_0, T_0) . This follows from the following result.

Proposition 7.13. *Let (A_0, T_0) be as above and let $(A_1, T_1) \in \mathcal{A}^+$ be a pair standard with respect to (A_0, T_0) . Then we have the following conditions:*

- (i) *There exists $g \in Z_G(T_1^- T_0^+)$ such that $gT_0g^{-1} = T_1$ and (gA_0g^{-1}, T_1) is maximal standard with respect to (A_0, T_0) .*
- (ii) *Let $A_2 = gA_0g^{-1}$. There exists $x \in Z_G(A_1^- A_0^+) \cap N_G(T_1)$ such that $xA_2x^{-1} = A_1$.*
- (iii) *$q = x^{-1}\theta(x) \in N_G(A_2) \cap N_G(T_1)$ and $p = \theta(g^{-1})x^{-1}\theta(x)\theta(g) \in N_G(A_0) \cap N_G(T_0)$.*
- (iv) *$n = g^{-1}\theta(g) \in N_G(A_0) \cap N_G(T_0)$ and $m = g^{-1}x^{-1}\theta(x)\theta(g) = g^{-1}\theta(g)p = np \in N_G(A_0) \cap N_G(T_0)$.*
- (v) *If w is the image of m in $W(A_0) \cap W(T_0)$ and w_1 the image of n in $W(A_0) \cap W(T_0)$, then w is the A_1 -standard involution of $W(A_0)$ and w_1 is the T_1 -standard involution of $W(T_0)$. In particular $(A_0)_w^+ = A_1^- A_0^+$ and $(A_0)_{w_1}^+ = A_2^- A_0^+$.*
- (vi) *Let w_2 be the image of p in $W(A_0) \cap W(T_0)$. Then $w_2^2 = e$, $(A_0)_{w_2}^+ = A_1^- A_0^+ (A_0)_{w_1}^- = (A_0)_w^+ (A_0)_{w_1}^-$ and $w = w_1 w_2 = w_2 w_1$.*
- (vii) *Let w_3 be the image of q in $W(A_1) \cap W(T_1)$. Then $w_3^2 = e$, $(A_0)_{w_3}^+ = A_1^- A_0^+$ and $(A_0)_{w_3}^- = (A_0)_{w_2}^- \subset A_1^-$.*

Proof. (i). Since T_1 is standard with respect to T_0 , there exists $g \in Z_G(T_1^- T_0^+)$ such that $gT_0g^{-1} = T_1$ and by Lemma 7.9 (gA_0g^{-1}, T_1) is maximal standard with respect to (A_0, T_0) .

(ii). Since A_2 is maximal standard with respect to (A_0, T_0) we have $A_1^- \subset A_0^- \cap T_1^- = A_2^-$. So both A_1 and A_2 are contained in $Z_G(A_1^- A_0^+)$ and are standard with respect to A_0 . From Proposition 5.7 it follows that they are maximal quasi k -split tori of $Z_G(A_1^- A_0^+)$. But then it follows from Corollary 7.11 that there exists $x \in Z_G(A_1^- A_0^+) \cap N_G(T_1)$ such that $xA_2x^{-1} = A_1$.

(iii). The first statement is clear. Since $A_0 = g^{-1}A_2g = \theta(g^{-1})A_2\theta(g)$, the map $\text{Int}(\theta(g)^{-1})$ lifts $x^{-1}\theta(x) \in N_G(A_2) \cap N_G(T_1)$ to $p = \theta(g^{-1})x^{-1}\theta(x)\theta(g) \in N_G(A_0) \cap N_G(T_0)$.

(iv). Since (A_2, T_1) is maximal standard with respect to (A_0, T_0) it follows that $n = g^{-1}\theta(g) \in N_G(A_0) \cap N_G(T_0)$. Similarly the second statement follows

from the observation that $A_1 = xgA_0g^{-1}x^{-1}$ and $xg \in Z_G(A_1^-A_0^+)$. Hence $m = g^{-1}x^{-1}\theta(x)\theta(g) = g^{-1}\theta(g)p \in N_G(A_0) \cap N_G(T_0)$.

(v) follows from Proposition 5.7 and [10, 3.4].

(vi). Since $A_1^- \subset A_2^-$ it follows that $(A_0)_{w_1}^- \subset (A_0)_w^-$. So w and w_1 commute and $w_2 = ww_1$ is an involution with $(A_0)_{w_2}^+ = A_1^-A_0^+(A_0)_{w_1}^- = (A_0)_w^+(A_0)_{w_1}^-$.

Finally (vii) is immediate from (iii) and (vi). \square

Remark 7.14. From [10, 3.4] it follows that, w_1 is independent of the choice of the element $g \in Z_G(T_1^-T_0^+)$ such that $gT_1g^{-1} = T_0$. Similarly from Lemma 5.9 it follows that w is independent of the choice of g and x as above. Finally from (vi) of Proposition 7.13 it follows that w_2 is independent of the choice of $x \in Z_G(A_1^-A_0^+) \cap N_G(T_1)$.

From Proposition 7.13 it follows that the standard involutions in $W(A_0)$ can be split in two parts. These are defined as follows.

Definition 7.15. Let $(A_1, T_1) \in \mathcal{A}^+$ be standard with respect to (A_0, T_0) and let w and $w_1 \in W(A_0) \cap W(T_0)$ be as in Proposition 7.13. We will call w the (A_1, T_1) -standard involution of $W(A_0) \cap W(T_0)$ and w_1 the maximal (A_1, T_1) -standard involution of $W(A_0) \cap W(T_0)$. We will also call these the standard (resp. maximal standard) involutions of $W(A_0) \cap W(T_0)$.

The maximal standard involutions of $W(A_0) \cap W(T_0)$ also characterize the maximal tori related to pairs in \mathcal{A}^+ .

Corollary 7.16. *Let T_1 be a θ -stable maximal torus standard with respect to T_0 and $w_1 \in W(T_0)$ the T_1 -standard involution. Then T_1 contains a θ -stable maximal quasi k -split torus A_1 , such that $(A_1, T_1) \in \mathcal{A}^+$ if and only if w_1 is conjugate under $W(T_0, H)$ with a maximal standard involution of $W(A_0) \cap W(T_0)$.*

Proof. Assume first that there exists a θ -stable maximal quasi k -split torus A_1 , such that $(A_1, T_1) \in \mathcal{A}^+$. By Proposition 7.10 and Proposition 7.8(ii), there exist $h \in N_G(T_0) \cap H$ such that hT_1h^{-1} is standard with respect to (A_0, T_0) . If $w \in W(T_0, H)$ is the Weyl group element corresponding to h , then $w_2 = ww_1w^{-1}$ is a maximal standard involution of $W(A_0) \cap W(T_0)$.

The converse statement is clear. \square

The question which remains is how many H -conjugacy classes correspond to one A_1 -standard involution in $W(A_0)$. First we show the following:

Theorem 7.17. *Let $w \in W(A_0)$ be a standard involution. If (A_1, T_1) , $(A_2, T_2) \in \mathcal{A}^+$ are pairs standard with respect to (A_0, T_0) , such that w is both the A_1 -standard involution and the A_2 -standard involution, then we have the following:*

- (i) There exists $h \in Z_G((A_0)_w^+) \cap H^0$ such that $hT_2h^{-1} = T_1$.
- (ii) If h is as in (i) and $A_3 = hA_2h^{-1}$, then (A_3, T_1) is standard with respect to (A_0, T_0) .
- (iii) A_1 and A_3 are H^0 -conjugate if and only if there exists $h \in N_H(T_1)$ such that $hA_3h^{-1} = A_1$.

Proof. (i). Since w is the A_1 -standard involution and the A_2 -standard involution, we have $(A_0)_w^+ = A_1^-A_0^+ = A_2^-A_0^+$. But then T_1 and T_2 are both maximal tori in $Z_G((A_0)_w^+)$ with T_1^+ and T_2^+ maximal tori of $Z_G((A_0)_w^+) \cap H$ and these are conjugate under $Z_G((A_0)_w^+) \cap H^0$.

(ii) is clear from (i).

(iii). If $h_1 \in H^0$ such that $h_1A_3h_1^{-1} = A_1$, then $T_3 = h_1T_1h_1^{-1}$ and T_1 are maximal tori in $Z_G(A_1)$ with T_3^+ and T_1^+ maximal tori of $Z_G(A_1) \cap H$. So there exists $h_2 \in Z_G(A_1) \cap H$ such that $h_2T_3h_2^{-1} = h_2h_1T_1h_1^{-1}h_2^{-1}$ and $h = h_2h_1 \in N_H(T_1)$.

The converse follows immediately from Lemma 3.1. \square

If $(A_1, T_1) \in \mathcal{A}^+$ is a pair standard with respect to (A_0, T_0) and $M = Z_G(A_1^-A_0^+)$, then we will write

$$W(T_1, M) = N_G(T_1) \cap M/T_1,$$

$$W(T_1, M, H) = W(T_1, M) \cap W(T_1, H) \text{ and}$$

$$W(A_1, T_1, M) = \{w \in W(T_1, M) \mid w(A_1) = A_1\}.$$

We get now the following result.

Corollary 7.18. *Let $w \in W(A_0)$ be a standard involution, $(A_1, T_1) \in \mathcal{A}^+$ a pair standard with respect to (A_0, T_0) , such that w is the (A_1, T_1) -standard involution and let $M = Z_G(A_1^-A_0^+)$. The number of H -conjugacy classes of θ -stable maximal quasi k -split tori, which have w as the corresponding standard involution is less or equal to:*

$$|W(T_1, M, H) \setminus W(T_1, M) / W(A_1, T_1, M)|$$

Proof. Let $(A_2, T_2) \in \mathcal{A}^+$ be another pair standard with respect to (A_0, T_0) and such that $A_2^- = A_1^-$. By the above result we may assume that $T_2 = T_1$. By Proposition 5.7 there exists $g \in M$, such that $gA_2g^{-1} = A_1$. Since gT_1g^{-1} and T_1 are maximal tori of $Z_G(A_1)$, there exists $x \in Z_G(A_1)$ such that $xgT_1g^{-1}x^{-1} = T_1$. So $xg \in N_G(T_1) \cap Z_G(A_1^-A_0^+)$. Clearly $W(A_1, T_1, M)$ acts on the right and $W(T_1, M, H)$ on the left, what proves the result. \square

Remarks 7.19. (1) It is an open question whether two θ -stable maximal quasi k -split tori A_1 and A_2 , which are H -conjugate and correspond to the same standard

involutions $w \in W(A_0)$ are also conjugate under $Z_G(A_1^- A_0^+) \cap H$. The element of H could give an outer automorphism for $\Phi(A_1, Z_G(A_1^- A_0^+))$.

(2) Note that if $A_1 \in \mathcal{A}_0^\theta$, standard with respect to A_0 , then there exists a maximal torus T_1 , such that $(A_1, T_1) \in \mathcal{A}^+$ and is conjugate with a pair standard with respect to (A_0, T_0) .

(3) Note that in the case that $W(A_0^-)$ has representatives in H_k , then the $W(A_0, H)$ -conjugacy classes of standard involutions completely characterize the conjugacy classes.

7.20. The question which remains is to give a description of those H -conjugacy classes of θ -stable maximal quasi k -split tori, which correspond to the same standard involution. A conjecture with a detailed description of these H -conjugacy classes of θ -stable maximal quasi k -split tori will be discussed in 8.34. However the standard and maximal standard involutions of $W(A_0) \cap W(T_0)$ provide an important first step. In many cases these involutions actually suffice to characterize the H -conjugacy classes (see for example [13]). A classification of the standard and maximal standard involutions of $W(A_0) \cap W(T_0)$, for a number of base fields, will be discussed in the next sections.

8. Involutions and singular roots

In the previous chapters we defined various types of standard involutions characterizing the H -conjugacy classes of maximal (quasi) k -split tori. In this section we will give other descriptions of these involutions in terms of orthogonal reflections.

8.1. Throughout this section let A_0 be a θ -stable maximal k -split torus with A_0^- a maximal (θ, k) -split torus of G and $T_0 \supset A_0$ a maximal torus of G , such that (A_0, T_0) is a θ^+ pair. Denote the set of θ -stable maximal quasi k -split tori standard with respect to A_0 by \mathcal{A}_1 . For $A \in \mathcal{A}_1$ and $\alpha \in \Phi(A)$ let $A_\alpha = \{a \in A_1 \mid s_\alpha(a) = a\}^0$, $G_\alpha = Z_G(A_\alpha)$ and $\overline{G}_\alpha = [G_\alpha, G_\alpha]$. If α is either real or imaginary, then G_α is θ -stable. Similarly if $w \in W(A)$ satisfies $w^2 = e$ and $w\theta = \theta w$, then we set $G_w = Z_G(A_w^+)$ and $\overline{G}_w = [G_w, G_w]$. Let n be a preimage of w in $N_G(A)$. Then $n \in Z_G(A_w^+)$ and $A_w^- \cap Z(G_w)$ is finite. As a consequence, A_w^- is a θ -stable maximal k -split torus of $[G_w, G_w]$.

8.2. Recall that a k -involution τ of a connected reductive k -group M is called (τ, k) -split if there exists a τ -split maximal k -split torus of M .

In 3.7 and 4.3 we defined θ -singular and (θ, k) -singular roots. We can lift these definitions to involutions in the Weyl group by defining the following.

Definition 8.3. Let $A \in \mathcal{A}^\theta$ be standard with respect to A_0 and $w \in W(A)$. Then w is called θ -singular if

- (1) $w^2 = e$.
- (2) $\theta w = w\theta$.
- (3) The involution $\theta|_{[G_w, G_w]}$ is (θ, k) -split.
- (4) $[G_w, G_w] \cap H$ contains a maximal quasi k -split torus of $[G_w, G_w]$.

An involution $w \in W(A)$ is called (θ, k) -singular if it satisfies (1), (2), (3) and if

- (5) $[G_w, G_w] \cap H$ contains a maximal k -split torus of $[G_w, G_w]$.

A root $\alpha \in \Phi(A)$ is called θ -singular (resp. (θ, k) -singular) if the corresponding reflection $s_\alpha \in W(A)$ is θ -singular (resp. (θ, k) -singular).

Remark 8.4. Note that these definitions of θ -singular (resp. (θ, k) -singular) roots coincide with those defined in 3.7 and 4.3.

Lemma 8.5. *Let A be a θ -stable maximal k -split torus with A^- a maximal (θ, k) -split torus of G and $w \in W(A)$ a (θ, k) -singular (resp. θ -singular) involution. Then we have the following conditions:*

- (i) $A_w^- = (A_w^-)^-$ is a θ -split maximal k -split torus of $[G_w, G_w]$.
- (ii) $A_w^+ = A^+(A^-)_w^+$.
- (iii) w is k -standard (resp. standard).

Proof. We show the assertion for the case that w is (θ, k) -singular. The result for the case that w is θ -singular follows with a similar argument.

(i). From $\theta w = w\theta$ it follows that $A^- = (A^-)_w^+(A^-)_w^-$. Since A^- is a maximal (θ, k) -split torus of G , $(A^-)_w^-$ is also a maximal θ -split torus of $[G_w, G_w]$. But since $\theta|_{[G_w, G_w]}$ is (θ, k) -split, it follows that $(A^-)_w^-$ is a maximal k -split torus of $[G_w, G_w]$. Hence $(A^-)_w^- = A_w^-$.

(ii) is immediate from (i) and the observation that $A_w^+ = (A_w^+)^+(A_w^+)^-$.

(iii). Let \tilde{S} be a maximal torus of $[G_w, G_w] \cap H$. Then $A_1 = \tilde{S}A_w^+$ is a maximal k -split torus of G and by (ii) A_1 is standard with respect to A . Clearly w is the A_1 -standard involution in $W(A)$. \square

Analogous to [10], we have the following characterization of the k -standard involutions in $W(A_0)$.

Proposition 8.6. *Let A_0 be a θ -stable maximal k -split torus of G with A_0^- a maximal (θ, k) -split torus of G . Then there is a one to one correspondence between the $W(A_0, H)$ -conjugacy classes of k -standard (resp. standard) involutions in $W(A_0)$ and the $W(A_0)$ -conjugacy classes of (θ, k) -singular (resp. θ -singular) involutions of $W(A_0)$.*

Proof. Let $w \in W(A_0)$ be a k -standard involution and $A \in \mathcal{A}_1$ a θ -stable maximal k -split torus, standard with respect to A_0 . Then $(A_0)_w^+ = A^- A_0^+$ and $(A_0)_w^- \subset A_0^-$. Clearly $(A_0)_w^+$ is θ -stable and so $w\theta = \theta w$. On the other hand $A_0^-, A^+ \subset G_w$ and their images in $G_w/Z(G_w)$ are maximal k -split tori of $G_w/Z(G_w)$. It follows that $\theta|[G_w, G_w]$ is (θ, k) -split and $[G_w, G_w] \cap H$ contains a maximal k -split torus of $[G_w, G_w]$. So w is (θ, k) -singular.

Conversely if $w \in W(A_0)$ is (θ, k) -singular, then it follows from Lemma 8.5 that w is k -standard. So we have shown that there is a correspondence between the $W(A_0, H)$ -conjugacy classes of k -standard involutions and (θ, k) -singular involutions of $W(A_0)$. That these conjugacy classes correspond with the $W(A_0)$ -conjugacy classes of these involutions follows now from Proposition 6.7.

The correspondence between the standard involutions and the θ -singular involutions follows with a similar argument. One only has to replace maximal k -split tori of $[G_w, G_w] \cap H$ by maximal quasi k -split tori of $[G_w, G_w] \cap H$. \square

In the case that $W(A_0^-)$ has representatives in H_k , the above leads to the following result.

Corollary 8.7. *Let A_0 be as above and assume that $W(A_0^-)$ has representatives in H_k . Then there exists a one to one correspondence between the H_k -conjugacy classes of θ -stable maximal k -split tori and the $W(A_0)$ -conjugacy classes of (θ, k) -singular involutions of $W(A_0)$.*

Proof. This result follows from Theorem 6.8 and the above result. \square

8.8. These results also lead to a further characterization of the maps $\phi : V/W \rightarrow \mathfrak{L}/W$ and $\phi_k : V_k/W \rightarrow \mathfrak{L}/W$ as in 1.14. Fix a basis Δ_0 of $\Phi_0 = \Phi_0(-\theta)$ and extend it to a θ^- -basis Δ of $\Phi(A_0)$. Let w_0 be the opposition involution of $W(\Phi_0)$ with respect to Δ_0 and let $\theta' = \theta w_0$. Then $\theta'(\Delta) = \Delta$ and $\mathfrak{L}_{\theta'} = \mathfrak{L}_{\theta} w_0$. We have now the following:

Lemma 8.9. *Each twisted orbit in $\varphi(V)/W \subset \mathfrak{L}_{\theta}/W$ contains a θ -singular involution.*

Proof. This result is immediate from Proposition 5.16, using the bijection $\iota : \mathfrak{L}_{\theta}/W \rightarrow \mathfrak{L}_{\theta'}/W$ as in (1.6). \square

For the θ -singular involutions in a twisted W -orbit we can show the following:

Proposition 8.10. *Let $w_1, w_2 \in \varphi(V)$ be θ -singular involutions. Then w_1 and w_2 are in the same twisted W -orbit if and only if w_1 and w_2 are W -conjugate.*

Proof. Assume first that w_1 and w_2 are W -conjugate. Since A_0^- is a maximal (θ, k) -split torus, we have $(A_0^-)_{w_i}^- \subset A_0^-$ ($i = 1, 2$) and hence $w_1, w_2 \in W(A_0) \cap W(A_0^-)$. By Proposition 6.7 w_1 and w_2 are conjugate under $W_1(A_0)$. Let $w \in W_1(A_0)$ such that $ww_1w^{-1} = w_2$. Since $W_1(A_0) = W(A_0, H)$ (see Corollary 6.6) it follows that $\theta(w) = w$ and hence $ww_1\theta(w^{-1}) = ww_1w^{-1} = w_2$.

For the converse statement assume $w \in W$ such that $ww_1\theta(w^{-1}) = w_2$. Then by Lemma 1.29 we have $w((A_0^-)_{w_1}^+) = (A_0^-)_{w_2}^+$ and with a similar argument we also have $w((A_0^-)_{w_1}^- A_0^+) = (A_0^-)_{w_2}^- A_0^+$. So $ww_1\theta w^{-1} = w_2\theta$. Then by Proposition 6.7 w_1 and w_2 are W -conjugate, what proves the result. \square

In the case that $W(A_0^-)$ has representatives in H_k , we get now the following result:

Corollary 8.11. *Let A_0 be as above and assume that $W(A_0^-)$ has representatives in H_k . Then the maps $\phi_0 = \phi|V_0 : V_0/W \rightarrow \mathfrak{L}/W$ and $\phi_k : V_k/W \rightarrow \mathfrak{L}/W$ are one-to-one.*

Proof. The result is immediate from Corollary 8.7, Proposition 8.10 and Corollary 6.10. \square

This result is important for studying the Bruhat type order on V_k and V . See for example [8] and [23].

Next we will prove that every singular involution can be written as a product of orthogonal singular reflections. For this we need the following results.

Lemma 8.12. *Let A be a θ -stable maximal k -split (resp. quasi k -split) torus of G and let $\Psi = \{\alpha_1, \dots, \alpha_r\}$ be a set of strongly orthogonal roots of $\Phi(A)$. Let G_Ψ denote the closed subgroup of G generated by the $G_{s_{\alpha_i}}$ ($i = 1, \dots, r$). Then:*

$$[G_\Psi, G_\Psi] \cong \prod_{i=1}^r [G_{s_{\alpha_i}}, G_{s_{\alpha_i}}]$$

Moreover, if $\alpha_1, \dots, \alpha_r$ are (θ, k) -singular (resp. θ -singular), then $\theta|[G_\Psi, G_\Psi]$ is (θ, k) -split and $[G_\Psi, G_\Psi] \cap H$ contains a maximal k -split (resp. quasi k -split) torus of $[G_\Psi, G_\Psi]$.

Proof. Since $[G_{s_\alpha}, G_{s_\beta}] = e$ for $\alpha, \beta \in \Phi(A)$ strongly orthogonal, the first assertion is clear.

If $\alpha_1, \dots, \alpha_r$ are θ -singular, then it follows from Proposition 3.8 with an easy induction that $\theta|[G_\Psi, G_\Psi]$ is (θ, k) -split and $[G_\Psi, G_\Psi] \cap H$ contains a maximal quasi k -split torus of $[G_\Psi, G_\Psi]$.

The result for $\alpha_1, \dots, \alpha_r$ (θ, k) -singular follows with a similar argument. \square

Proposition 8.13. *Let A be a θ -stable maximal k -split (resp. quasi k -split) torus of G , standard with respect to A_0 , with $\dim A^- = r$, $\dim A^+ = s$. Then the following are equivalent:*

- (i) θ is (θ, k) -split and H contains a maximal k -split (resp. quasi k -split) torus of G .
- (ii) There exist strongly orthogonal subsets $\{\alpha_1, \dots, \alpha_r\}$ and $\{\beta_1, \dots, \beta_s\}$ of θ -singular roots of $\Phi(A)$, such that $\theta(\alpha_i) = -\alpha_i$ ($i = 1, \dots, r$), $\theta(\beta_j) = \beta_j$ ($j = 1, \dots, s$) and $-\text{id} = s_{\alpha_1} \dots s_{\alpha_r} s_{\beta_1} \dots s_{\beta_s}$.

Proof. We show the assertion for the case that A is k -split. The result for the case that A is quasi k -split follows with a similar argument. The result is proved by induction in several steps.

(1) If both A^+ and A^- are nontrivial, then the derived subgroup of $Z_G(A^-)$ satisfies the conditions. Hence the induction argument works. So it suffices to prove the result in the cases that $A = A^-$ or $A = A^+$.

(2) Assume first that $A = A^-$. Let S be a maximal k -split torus of G contained in H . Then S is standard with respect to A and $-\text{id}$ is the S -standard involution in $W(A)$. So in particular $\theta|_A = -\text{id} \in W(A)$. Let $\alpha_1, \dots, \alpha_r \in \Phi(A)$ be strongly orthogonal such that $-\text{id} = s_{\alpha_1} \dots s_{\alpha_r}$. By Proposition 3.8 each α_i ($i = 1, \dots, r$) is (θ, k) -singular.

(3) Assume next that $A = A^+$. Let A_0 be a θ -split maximal k -split torus of G . Then $-\text{id} \in W(A_0)$ and $-\text{id} = s_{\alpha_1} \dots s_{\alpha_r}$ with $\alpha_1, \dots, \alpha_r \in \Phi(A_0)$ strongly orthogonal (θ, k) -singular roots. Let $\Psi = \{\alpha_1, \dots, \alpha_r\}$ and G_Ψ the subgroup of G generated by the $G_{s_{\alpha_i}}$ ($i = 1, \dots, r$). From Lemma 8.12 it follows now that G_Ψ satisfies (i). Let S be a maximal k -split torus of $G_\Psi \cap H$. Then $\Phi(S, G_\Psi) \cong \Phi(A_0, G_\Psi)$ is of type $A_1 + \dots + A_1$ (r times) and each root of $\Phi(S, G_\Psi)$ is (θ, k) -singular, because $\theta|_{[G_{s_{\alpha_i}}, G_{s_{\alpha_i}}]} \neq \text{id}$ for ($i = 1, \dots, r$). Say $\Phi(S, G_\Psi) = \{\pm\gamma_1, \dots, \pm\gamma_r\}$. Since S and A are H_k -conjugate, $\gamma_1, \dots, \gamma_r$ are mapped into a set of strongly orthogonal (θ, k) -singular roots of $\Phi(A)$, which proves the result.

(ii) \Rightarrow (i) follows immediately from Lemma 8.12. \square

Corollary 8.14. *Let A be a θ -stable maximal k -split (resp. quasi k -split) torus of G and $w \in W(A)$ a (θ, k) -singular (resp. θ -singular) involution. Then there exists strongly orthogonal subsets $\{\alpha_1, \dots, \alpha_r\}$ and $\{\beta_1, \dots, \beta_s\}$ of θ -singular roots of $\Phi(A)$, such that $\theta(\alpha_i) = -\alpha_i$ ($i = 1, \dots, r$), $\theta(\beta_j) = \beta_j$ ($j = 1, \dots, s$) and $w = s_{\alpha_1} \dots s_{\alpha_r} s_{\beta_1} \dots s_{\beta_s}$.*

Proof. Consider $G_w = Z_G(T_w^+)$. Then the result follows from Proposition 8.13. \square

Remark 8.15. These results generalize similar results of Matsuki [19] for $k = \mathbb{R}$. The (θ, k) -singular involutions correspond to the \mathfrak{q} -orthogonal systems in [19].

The θ -singular involutions in $W(A_0)$ can now also be characterized as follows.

Theorem 8.16. *Let A_0 be a θ -stable maximal k -split torus of G with A_0^- a maximal (θ, k) -split torus of G and $w \in W(A_0)$, $w^2 = e$. Then the following are equivalent:*

- (i) w is θ -singular
- (ii) $(A_0)_w^- \subset A_0^-$

Proof. (ii) \Rightarrow (i). From $(A_0)_w^- \subset A_0^-$ it follows that w and θ commute. Since $(A_0)_w^-$ is a θ -split maximal k -split torus of $[G_w, G_w]$ it suffices to show that $[G_w, G_w] \cap H$ contains a maximal quasi k -split torus of $[G_w, G_w]$. But this follows from Proposition 8.13.

(i) \Rightarrow (ii) follows from Lemma 8.5. □

Corollary 8.17. *Let A_0 be a θ -stable maximal k -split torus of G with A_0^- a maximal (θ, k) -split torus of G . Then we have the following.*

- (i) Every involution in $W(A_0) \cap W(A_0^-)$ is θ -singular.
- (ii) $\alpha \in \Phi(A_0)$ is θ -singular if and only if α is a real root.

This result is immediate from Theorem 8.16.

8.18. Characterization of the maximal standard involutions. Let $(A, T) \in \mathcal{A}^+$ be standard with respect to (A_0, T_0) . The θ -singular roots of $\Phi(A)$ can be split in three classes related to the maximal torus T . For $\alpha \in \Phi(A)$ let $A_\alpha = \{a \in A_1 \mid s_\alpha(a) = a\}^0$, $G_\alpha = Z_G(A_\alpha)$, $\overline{G}_\alpha = [G_\alpha, G_\alpha]$ and $T^\alpha = T \cap \overline{G}_\alpha$.

Definition 8.19. A real root $\alpha \in \Phi(A)$ is called T -stable θ -singular (resp. (θ, T) -singular) if $T^\alpha \cap H$ contains a maximal quasi k -split torus of \overline{G}_α (resp. $T^\alpha \cap H$ contains no quasi k -split tori).

Similarly an imaginary θ -singular root $\alpha \in \Phi(A)$ is called T -stable θ -singular (resp. (θ, T) -singular) if T^α contains a θ -split maximal quasi k -split torus of \overline{G}_α (resp. $T^\alpha \cap H$ contains no quasi (θ, k) -split tori).

Proposition 8.20. *Let $w \in W(A_0) \cap W(T_0)$. Then w is a maximal standard involution of $W(A_0) \cap W(T_0)$ if and only if there exist orthogonal (θ, T_0) -singular roots $\alpha_1, \dots, \alpha_\ell \in \Phi(A_0)$ such that $w = s_{\alpha_1} \dots s_{\alpha_\ell}$.*

Proof. Assume first that w is a maximal standard involution of $W(A_0) \cap W(T_0)$. Let T_1 be a θ -stable maximal torus standard with respect to T_0 , such that w is

the T_1 -standard involution in $W(T_0)$ and $g \in Z_G((T_0)_w^-)$ such that $T_1 = gT_0g^{-1}$. Then $A_1 = gA_0g^{-1}$ is a θ -stable maximal quasi k -split torus standard with respect to A_0 . So if $\alpha_1, \dots, \alpha_\ell \in \Phi(A_0) \cap \Phi(T_0)$ such that $w = s_{\alpha_1} \dots s_{\alpha_\ell}$, then $\alpha_1, \dots, \alpha_\ell$ are (θ, T_0) -singular.

The converse statement follows with a similar argument as in Proposition 8.13. \square

8.21. Characterization of \overline{G}_α . A question which remains is how one can determine the possible pairs $(\overline{G}_\alpha, \theta|_{\overline{G}_\alpha})$ for a θ -singular root $\alpha \in \Phi(A)$ and which of these pairs correspond to (θ, T) -singular (resp. T -stable θ -singular) roots. We will discuss this in the following.

Let $(A, T) \in \mathcal{A}^+$ be a θ^+ -pair. In spite of the fact that A^- is not maximal (θ, k) -split and T^- is not maximal θ -split the restriction to \overline{G}_α leads to a (Γ, θ) -index, as follows from the following result.

Proposition 8.22. *Let $(A, T) \in \mathcal{A}^+$ be a θ^+ -pair, standard with respect to (A_0, T_0) and let $T_1 \supset A$ be a maximal k -torus of G , standard with respect to T , such that (A, T_1) is a θ^- -pair. Let $\alpha \in \Phi(A)$ be a θ -singular real root, $\overline{A} = A \cap \overline{G}_\alpha$, $\overline{T} = T \cap \overline{G}_\alpha$ and $\overline{T}_1 = T_1 \cap \overline{G}_\alpha$. Then we have the following.*

- (i) $\overline{A} = \overline{A}^-$ is a θ -split maximal k -split torus of \overline{G}_α .
- (ii) $\overline{T} = \overline{T}_1$
- (iii) \overline{T}^- is a maximal θ -split torus of \overline{G}_α .

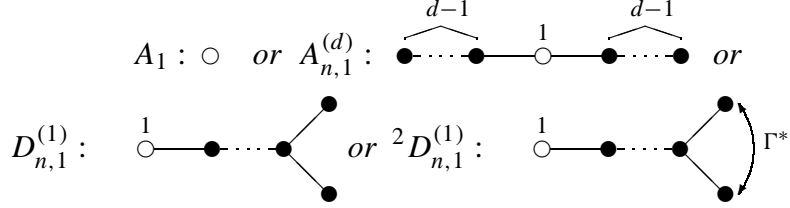
Proof. Since α is a real root $\overline{A} \subset A^-$. But, since A is standard with respect to A_0 the torus A^- is (θ, k) -split. So (i) is clear.

(ii). Let $w \in W(T_0)$ be the T_1 -standard involution. Since $T_w^- \subset G_\alpha = Z_G((\ker \alpha)^0)$, it follows that $\overline{T} = T \cap \overline{G}_\alpha = T_1 \cap \overline{G}_\alpha$, what proves (ii). Finally (iii) is immediate from (ii). \square

8.23. The (Γ, θ) -indices for the pairs $(\overline{G}_\alpha, \theta|_{\overline{G}_\alpha})$ have restricted rank one and moreover the underlying Γ -index has rank one. A list of the possible pairs follows easily from the classification of (Γ, θ) -indices in [7], but unfortunately there are many different cases. We can restrict this to a much smaller number of cases by taking representatives for the θ -singular reflections in the set of indecomposable roots $\Phi' = \{\alpha \in \Phi(A) \mid 2\alpha \neq \Phi(A)\}$ instead of $\Phi(A)$. (Note that if α is a real θ -singular root in $\Phi(A)$ with $\alpha, 2\alpha \in \Phi(A)$, then α is θ -singular if and only if 2α is θ -singular.) For the roots in Φ' we can prove the following.

Proposition 8.24. *Let $(A, T) \in \mathcal{A}^+$ be a θ^+ -pair, standard with respect to (A_0, T_0) , $\Phi' = \{\alpha \in \Phi(A) \mid 2\alpha \neq \Phi(A)\}$ and $\alpha \in \Phi'$ a θ -singular real root.*

If $\overline{T} = T \cap \overline{G}_\alpha$, $\overline{A} = A \cap \overline{G}_\alpha$, then the k -irreducible isotropic component of the Γ -index of \overline{G}_α is one of the following:



or s copies of one of these, when the Γ -index is not absolutely irreducible.

Proof. It suffices to prove the result in the case that the Γ -index is absolutely irreducible. So assume that $\Phi(\overline{T})$ is irreducible. If $\alpha \in \Phi(\overline{A}) \cap \Phi(\overline{T})$, then $\overline{G}_\alpha = SL_2(k)$ and the Γ -index is of type A_1 .

If $\Phi(\overline{A}) \cap \Phi(\overline{T}) = \emptyset$, then it follows from the classification of Γ -indices in [29], that the Γ -index can only be of type $A_{n,1}^{(d)}$ with $d = \frac{n+1}{2}$, $D_{n,1}^{(1)}$ with $n-1 = 2m \geq 2$ or ${}^2D_{n,1}^{(1)}$ with $n > 2$. This proves the result. \square

Remark 8.25. There are many more restricted rank one Γ -indices, but most could be omitted due to the fact that we chose $\alpha \in \Phi'$. This result gives also a big reduction of the number of possible rank one (Γ, θ) -indices. All one needs to do is to combine the above Γ -indices with the possible involutions for these. It also follows from this which θ -singular roots are (θ, T) -singular or T -stable θ -singular. If we use the notation for the (Γ, θ) -indices as in [7] (see also 9.19 and Table 1), then we have the following result.

Corollary 8.26. *Let $(A, T) \in \mathcal{A}^+$ be a θ^+ -pair, standard with respect to (A_0, T_0) , $\Phi' = \{\alpha \in \Phi(A) \mid 2\alpha \neq \Phi(A)\}$ and $\alpha \in \Phi'$ a θ -singular real root. Assume that the (Γ, θ) -index \mathcal{D} of $(\overline{G}_\alpha, \theta|\overline{G}_\alpha)$ is absolutely irreducible. Then we have the following:*

- (i) *If α is T -stable θ -singular, then \mathcal{D} is of type: $A_{3,1}^{(2)}(I)$, $A_{3,1}^{(2)}(II)$, $A_{n,1}^{(d)}(I)$ with $d = \frac{n+1}{2} > 2$, $A_{n,1}^{(2d)}(II)$ with $2d = \frac{n+1}{2} > 2$, $D_{n,1}^{(1)}(I_a^q)$ with $1 \leq q < n-1 = 2m$, $D_{n,1}^{(1)}(I_b)$ with $n-1 = 2m \geq 2$, ${}^2D_{n,1}^{(1)}(I_a^q)$ with $1 \leq q < n-1$ or ${}^2D_{n,1}^{(1)}(I_b)$ with $n > 2$.*
- (ii) *If α is (θ, T) -singular, then \mathcal{D} is of type: $A_{1,1}^{(1)}(I)$.*

Proof. This result is immediate from Proposition 8.22, Proposition 8.24 and the classification of (Γ, θ) -indices in [7]. \square

Remark 8.27. The above results 8.24 and 8.26 only hold for real θ -singular roots. In 3.10 one can find an example that this is in general not true for imaginary θ -singular roots. For imaginary θ -singular roots it is actually essential to look at the set of indecomposable roots instead of the set Φ' as above. The long roots are important in determining the fibers of the map ϕ as in 1.14. We discuss this in some more detail in the remainder of this section.

8.28. A conjecture about the fibers of ϕ . The question which remains unanswered is how many H -conjugacy classes of θ -stable maximal quasi k -split tori, correspond to a single θ -singular involution. By Lemma 8.9 and Proposition 8.10 these are exactly the fibers of the map $\phi|_{\varphi(V)/W}$. In the following we discuss a conjecture, which gives a detailed description of these conjugacy classes.

Let $w \in W(A_0)$ be a θ -singular involution and $(A, T) \in \mathcal{A}^+$ a pair standard with respect to (A_0, T_0) , such that w is the corresponding (A, T) -standard involution of $W(A_0) \cap W(T_0)$. From Proposition 7.13 it follows that we can write $w = w_1 w_2 = w_2 w_1$ with w_1, w_2 involutions and w_1 a maximal standard involution. The involution w_1 is also the T -standard involution of $W(T_0)$.

Recall that, since T is standard with respect to T_0 , there also exists a θ -singular involution $w_3 \in W(T)$ and strongly orthogonal θ -singular imaginary roots $\alpha_1, \dots, \alpha_n \in \Phi(T)$ such that $w_3 = s_{\alpha_1} \cdots s_{\alpha_n}$. In [10] this involution is also called the T_0 -standard involution of $W(T)$. We define now the following:

Definition 8.29. A pair $(A, T) \in \mathcal{A}^+$, standard with respect to (A_0, T_0) , is called a θ -singular standard pair if the T_0 -standard involution of $W(T)$ is contained in $W(A) \cap W(T)$.

Remark 8.30. If the T_0 -standard involution $w \in W(T)$ is contained in $W(A) \cap W(T)$ and $w = s_{\alpha_1} \cdots s_{\alpha_n}$ with $\alpha_1, \dots, \alpha_n \in \Phi(T)$ strongly orthogonal θ -singular imaginary roots, then also $\alpha_1, \dots, \alpha_n \in \Phi(A)$ are θ -singular imaginary roots with respect to A . Note that in this case $\alpha_1, \dots, \alpha_n$ is a maximal set of strongly orthogonal θ -singular imaginary roots in $\Phi(A) \cap \Phi(T)$.

Proposition 8.31. Let $w \in W(A_0)$ be a maximal standard involution and $(A_1, T), (A_2, T) \in \mathcal{A}^+$ pairs θ -singular standard with respect to (A_0, T_0) , such that w is the corresponding (A_1, T) -standard and (A_2, T) -standard involution of $W(A_0) \cap W(T_0)$. Then (A_1, T) and (A_2, T) are H -conjugate.

Proof. Note first that the pairs $(A_1, T), (A_2, T) \in \mathcal{A}^+$ are maximal standard pairs, since w is a maximal standard involution. Let $\alpha_1, \dots, \alpha_n$ be a maximal set of strongly orthogonal θ -singular imaginary roots in $\Phi(A_1) \cap \Phi(T)$, β_1, \dots, β_n a maximal set of strongly orthogonal θ -singular imaginary roots

in $\Phi(A_2) \cap \Phi(T)$ and $w_1 = s_{\alpha_1} \cdots s_{\alpha_n}$, $w_2 = s_{\beta_1} \cdots s_{\beta_n}$ the corresponding θ -singular involutions. We can extend $\alpha_1, \dots, \alpha_n$ with $\alpha_{n+1}, \dots, \alpha_m$ to a maximal set of strongly orthogonal θ -singular imaginary roots in $\Phi(T)$. Similarly we can extend β_1, \dots, β_n with $\beta_{n+1}, \dots, \beta_m$ to a maximal set of strongly orthogonal θ -singular imaginary roots in $\Phi(T)$. With an easy induction it follows then that $w_3 = s_{\alpha_{n+1}} \cdots s_{\alpha_m}$ and $w_4 = s_{\beta_{n+1}} \cdots s_{\beta_m}$ are conjugate under $W(T, H)$. So passing to $Z_G(T_{w_3}^-)$, we may assume that w_1 and w_2 are maximal θ -singular involutions in $W(T) \cap Z_G(T^-)$. But then it follows from [10, 3.7] that w_1 and w_2 are conjugate under $W(T, H)$. Finally, passing to $Z_G(A_{w_1}^-)$ the result follows from Corollary 3.17. \square

8.32. Let $(A, T) \in \mathcal{A}^+$ be a θ -singular standard pair. Under the action of $W(T) \cap Z_G(A^-)$ some of the the strongly orthogonal θ -singular imaginary roots of $\Phi(A) \cap \Phi(T)$ can be mapped to θ -compact roots of T . If $w_1 \in W(T)$ is such a Weyl group element, then the corresponding pair $(w_1(A), T)$ is no longer a θ -singular standard pair and consequently (A, T) and $(w_1(A), T)$ are not H -conjugate. To describe this phenomena we define now the following:

Definition 8.33. Let $w \in W(A_0)$ be a θ -singular involution and $(A_1, T) \in \mathcal{A}^+$ a pair standard with respect to (A_0, T_0) , such that w is the corresponding (A_1, T) -standard involution of $W(A_0) \cap W(T_0)$. Then we define the *θ -singular imaginary rank of (A_1, T)* as the number of roots in a maximal set of strongly orthogonal θ -singular imaginary roots $\alpha_1, \dots, \alpha_r \in \Phi(A_1) \cap \Phi(T)$. We denote this by $\text{rank}_\theta(A_1, T)$.

We have now the following conjecture for the pairs related to a θ -singular involution of $W(A_0)$:

Conjecture 8.34. Let $w \in W(A_0)$ be a θ -singular involution and $(A_1, T), (A_2, T) \in \mathcal{A}^+$ pairs standard with respect to (A_0, T_0) , such that w is the corresponding (A_1, T) -standard and (A_2, T) -standard involution of $W(A_0) \cap W(T_0)$. Then (A_1, T) and (A_2, T) are H -conjugate if and only if $\text{rank}_\theta(A_1, T) = \text{rank}_\theta(A_2, T)$

Remark 8.35. In the case that the θ -singular imaginary rank of a pair (A_1, T) is maximal, then (A_1, T) is a θ -singular standard pair. If then additionally $w \in W(A_0)$ is a maximal standard involution, then this conjecture follows from Proposition 8.31. So part of what this conjectures states is that the reflections in $\Phi(A)$ corresponding to T -stable θ -singular imaginary roots (see 8.19) remain θ -singular under the action of $W(T)$. So all one really has to look at are the imaginary roots $\alpha \in \Phi(A) \cap \Phi(T)$ for which $\frac{1}{2}\alpha \in \Phi(A)$.

9. Classification of the conjugacy classes of θ -singular involutions

In this section we classify the conjugacy classes of θ -singular involutions, what gives a partial classification of the H -conjugacy classes of maximal quasi k -split tori. We will use for this the classification of involutions in Weyl groups as in [10] and the classification of (Γ, θ) -indices as in [9] and [7].

9.1. Throughout this section let A_0 be a θ -stable maximal k -split torus with A_0^- a maximal (θ, k) -split torus of G and $T_0 \supset A_0$ a maximal torus of G , such that (A_0, T_0) is a θ^+ pair. Denote the set of θ -stable maximal quasi k -split tori standard with respect to A_0 by \mathcal{A}_1 .

Similar as in [10] we use the following notation. Let $E = X^*(A_0) \otimes_{\mathbb{Z}} \mathbb{R}$, $\Phi = \Phi(A_0)$ and $W = W(A_0)$. If $\sigma \in \text{Aut}(\Phi)$, then we denote the eigenspace of σ for the eigenvalue ξ , by $E(\sigma, \xi)$.

9.2. To classify the conjugacy classes of θ -singular involutions of $W = W(A_0)$ it suffices, by Theorem 8.16 and Corollary 8.17, to classify the $W(A_0)$ -conjugacy classes of involutions w with $(A_0)_w^- \subset A_0^-$, i.e. involutions of $W(A_0) \cap W(A_0^-)$. This means we need to determine the conjugacy classes of involutions $w \in W$ with $E(w, -1) \subset E(\theta, -1)$. So the first step is to get a classification of the eigenspaces $E(\theta, -1)$. This follows essentially from the classification of the pair $(\Phi(A_0), \theta|\Phi(A_0))$. A list of these pairs can be derived from the (Γ, θ) -indices. Basically $\Phi(A_0)$ follows from the Γ -index and $\theta|\Phi(A_0)$ can be derived from the (Γ, θ) -index. We represent the pairs $(\Phi(A_0), \theta|\Phi(A_0))$ with a θ -index as in 2.9. A list of these is included in Table 1.

9.3. As in 2.12 let $\Phi_0(-\theta) = \{\alpha \in \Phi \mid \theta(\alpha) = -\alpha\}$ and $W_0(-\theta) = W(\Phi_0(-\theta))$ the Weyl group of $\Phi_0(-\theta)$. Note that $W_0(-\theta) \subset W_1(\theta)$. We call an involution $w \in W$ a *maximal involution* if $E(w, -1)$ is maximal. By [10, 2.11] any maximal involution w of W is the opposition involution with respect to a basis Δ_0 of $\Phi_0(-\theta)$. So $w = w_{\Delta_0}^0$.

Similar as in [10, 2.12] we call an involution $w \in W$ a *θ -maximal involution* of W if $E(w, -1) \subset E(\theta, -1)$ and w is a maximal involution of $W_0(-\theta)$.

We can restrict now to a θ -maximal involution:

Proposition 9.4. *Let $w_0 \in W_0(-\theta)$ be a θ -maximal involution. Then we have the following.*

- (i) *If $w \in W$ an involution with $E(w, -1) \subset E(\theta, -1)$ then there exists $w_1 \in W_0(-\theta)$ such that $E(w_1 w w_1^{-1}, -1) \subset E(w_0, -1) \subset E(\theta, -1)$.*

- (ii) If $w_1, w_2 \in W$ such that $E(w_i, -1) \subset E(w_0, -1) \subset E(\theta, -1)$ ($i = 1, 2$), then w_1 and w_2 are conjugate under W if and only if w_1 and w_2 are conjugate under $W_1(w_0)$.

Proof. The first statement follows from [10, 2.14] and the second statement follows from [10, 2.18]. \square

Remark 9.5. From this result it follows that it suffices to determine the conjugacy classes under W of involutions in $W_0(-w_0)$ instead of in $W_0(-\theta)$. Here w_0 is a fixed θ -maximal involution of W .

9.6. The diagram of conjugacy classes of θ -singular involutions. There is a natural diagram associated with the W -conjugacy classes of θ -singular involutions of W . Denote the set of W -conjugacy (resp. $\text{Aut}(\Phi)$ -conjugacy) classes of involutions in W by \mathcal{W} (resp. \mathcal{W}^*) and the set of W -conjugacy (resp. $\text{Aut}(\Phi)$ -conjugacy) classes of involutions in $\text{Aut}(\Phi)$ by \mathcal{W}_a (resp. \mathcal{W}_a^*). The elements of \mathcal{W} (resp. \mathcal{W}^* , \mathcal{W}_a , \mathcal{W}_a^*) will be denoted by $[w]$, where w is an involution in the class $[w]$. An order on \mathcal{W} (resp. \mathcal{W}^* , \mathcal{W}_a , \mathcal{W}_a^*) is defined as follows.

Definition 9.7. Let $[w_1], [w_2] \in \mathcal{W}$ (resp. \mathcal{W}^* , \mathcal{W}_a , \mathcal{W}_a^*). We call $[w_1] > [w_2]$ if and only if $E(w_1, -1) \subset E(w_2, -1)$ for some representatives w_i of $[w_i]$ ($i = 1, 2$).

Notation 9.8. For an involution $\sigma \in \text{Aut}(\Phi)$ let $\mathcal{W}(\sigma)$ (resp. $\mathcal{W}^*(\sigma)$) denote the subset of those $[w] \in \mathcal{W}$ (resp. \mathcal{W}^*) with $[w] > [\sigma]$. If $\sigma = \theta$ then $\mathcal{W}(\theta)$ (resp. $\mathcal{W}^*(\theta)$) is the set of W -conjugacy (resp. $\text{Aut}(\Phi)$ -conjugacy) classes of θ -singular involutions in W . The diagram of $(\mathcal{W}, >)$ (resp. $(\mathcal{W}^*, >)$) will be denoted by $\mathcal{L}(\Phi)$ (resp. $\mathcal{L}(\Phi)$) and the diagram of $(\mathcal{W}(\sigma), >)$ (resp. $(\mathcal{W}^*(\sigma), >)$) by $\mathcal{L}(\sigma)$ (resp. $\mathcal{L}^*(\sigma)$).

From Proposition 9.4 we get now the following result.

Proposition 9.9. *Let $w_0 \in W_0(-\theta)$ be a θ -maximal involution. Then $\mathcal{W}(\theta) = \mathcal{W}(w_0)$ and $\mathcal{L}(\theta) = \mathcal{L}(w_0)$.*

9.10. To classify the W -conjugacy classes of θ -singular involutions, it suffices to determine the diagram $\mathcal{L}(w_0)$ for a θ -maximal involution $w_0 \in W_0(-\theta)$. This diagram is contained in $\mathcal{L}(\Phi)$, the diagram of W -conjugacy classes of all involutions. Similar as in [10] we will show that this diagram is completely determined by the type of the root system $\Phi_0(-w_0)$ if (G, θ) is not (θ, k) -split. Note that it also suffices to classify the diagram $\mathcal{L}(w_0)$ in the case that the (Γ, θ) -index is absolutely irreducible. The other cases easily follow from that, using a similar argument as in [7].

In the following we will first discuss the classification for the (θ, k) -split case and after that the classification for the case that (G, θ) is not (θ, k) -split.

9.11. The (θ, k) -split case. The classification of the W -conjugacy classes of θ -singular involutions in the case that (G, θ) is (θ, k) -split is a question of a classification of W -conjugacy classes of all involutions in W . This classification was given in [10, §7]. Before we state the result first some more notation.

Notation 9.12. For an involution $w \in W$ we will also write $\Phi(w) = \Phi_0(-w)$. If Φ is irreducible, then $\Phi(w)$ is of type $r \cdot A_i + X_\ell$, where either $X_\ell = \emptyset$ or one of $B_\ell (\ell \geq 1)$, $C_\ell (\ell \geq 1)$, $D_{2\ell} (\ell \geq 1)$, E_7 , E_8 , F_4 or G_2 (see [10, 7.7]).

Lemma 9.13. *Let Δ a basis of Φ . Every involution of W is conjugate with an involution $w_\Pi^0 \in W$ for a subset Π of Δ .*

Proof. Let Δ_1 be a $(-w)$ -basis of Φ . Then $w = w_\Pi^0$ for Π a subset of Δ_1 . Since Δ_0 and Δ_1 are conjugate under W the result follows. \square

We have now the following classification of involutions in W (see [10, 7.19]):

Theorem 9.14. [10, 7.19] *Let Φ be irreducible, Δ a basis of Φ , $w_\Pi^0 \in W$ an involution of type $r \cdot A_1 + X_\ell$ with X_k as in 9.12. Then we have the following:*

- (1) *The $\text{Aut}(\Phi)$ -conjugacy class of w is completely determined by the type of $\Phi(w)$ except in the cases E_7 , $r = 3$, $X_k = \emptyset$; F_4 , $r = 1$, $X_k = \emptyset$; G_2 , $r = 1$, $X_k = \emptyset$ where there are 2 conjugacy classes of the same type. In the cases of E_7 and F_4 the conjugacy classes are completely determined by the type of $\Phi(w)$ and $\Phi(-w)$.*
- (2) *The $\text{Aut}(\Phi)$ -conjugacy class of w and the W -conjugacy class of w are the same, except in the cases D_4 , $r = 2$, $X_k = \emptyset$ and $D_{2\ell}$ ($\ell \geq 3$), $r = \ell$, $X_k = \emptyset$, where there are respectively 3 and 2 W -conjugacy classes of the same type.*

Remark 9.15. A list of the possible types of $\Phi_\Pi = \Phi(w_\Pi^0)$ together with the type of the diagram $\mathcal{L}(\Phi)$ is given in [10, Table II]. The type of the diagram $\mathcal{L}(\Phi)$ is also given in Table 1.

9.16. The non (θ, k) -split case. To classify the W -conjugacy classes of the θ -singular involutions in the case that (G, θ) is not (θ, k) -split, we need to determine the diagram $\mathcal{L}(w_0)$ for a θ -maximal involution w_0 . The θ -maximal involutions of W are exactly the opposition involutions of $\Phi_0(-\theta) = \Phi \cap \bar{\Phi}_\theta$. The type of $\Phi(w_0) \subset \Phi_0(-\theta)$ for an opposition involution (i.e. θ -maximal involution) w_0 , follows from [10, Table II]. So it suffices to determine $\Phi_0(-\theta)$.

This can be deduced from the θ -index of $(\Phi, \theta|\Phi)$ as follows. Note first that the black nodes in the θ -index correspond to $\Delta_0(\theta)$ and hence

$$(9.1) \quad \Phi(w_0) \subset \Phi_0(-\theta) \subset \Delta_0(\theta)^\perp.$$

The type of $\Delta_0(\theta)^\perp$ follows from [10, Table I]. When $\theta^*|_{\Delta_0(\theta)^\perp} = \text{id}$, then $\Phi_0(-\theta) = \Delta_0(\theta)^\perp$, what is actually true in most cases. In the few remaining cases one has to incorporate the diagram automorphism θ^* . A list with the type of the root system $\Phi_0(-\theta)$ for each of the absolutely irreducible (Γ, θ) -indices is included in Table 1.

Without using the classification we can actually prove the following characterization of $\Phi_0(-\theta)$ and $\Phi(w_0)$ for a θ -maximal involution w_0 (see [10, 7.21]).

Lemma 9.17. [10, 7.21] *Let Φ be irreducible, $\theta \in \text{Aut}(\Phi)$ an involution such that $\bar{\Phi}_\theta$ is a root system with Weyl group \bar{W}_θ , Δ a $(-\theta)$ -basis of Φ , $\Delta_0 = \Delta \cap \Phi_0(-\theta)$ the corresponding basis of $\Phi_0(-\theta)$ and $w_0 = w_{\Delta_0}^0$. Then w_0 is a θ -maximal involution of Φ and $\Phi_0(-\theta)$ (resp. $\Phi(w_0)$) is of type $r \cdot A_1$ ($r \geq 0$) or of type X_ℓ with X_ℓ one of B_ℓ ($\ell \geq 1$), C_ℓ ($\ell \geq 1$), $D_{2\ell}$ ($\ell \geq 2$), E_7 , E_8 , F_4 , or G_2 .*

Since this result also follows from the classification, we omit the proof. However it is easily shown by itself, using the fact that both Φ and $\bar{\Phi}_\theta$ are irreducible.

Contrary to the (θ, k) -split case, the type of the root system $\Phi(w_\Pi^0)$ for an involution w_Π^0 , completely determines the conjugacy class in the case that (G, θ) is not (θ, k) -split:

Theorem 9.18. *Assume (G, θ) is not (θ, k) -split and let Φ be irreducible, Δ a $(-\theta)$ -basis of Φ , $\Delta_0 = \Delta \cap \Phi_0(-\theta)$ and $w_0 = w_{\Delta_0}^0$. Then we have the following.*

- (i) *Every involution $w \in W$ with $E(w, -1) \subset E(\theta, -1)$ is conjugate under $W_0(-\theta)$ with an involution w_Π^0 , where $\Pi \subset \Delta_0$.*
- (ii) *The W - (resp. $\text{Aut}(\Phi)$ -) conjugacy class of an involution $w_\Pi^0 \in W$ ($\Pi \subset \Delta_0$) is completely determined by the type of $\Phi(w_\Pi^0)$.*

Proof. From Proposition 9.4 it follows that there exists $w_1 \in W_0(-\theta)$ such that $E(w_1 w w_1^{-1}, -1) \subset E(w_0, -1) \subset E(\theta, -1)$. So we may assume $E(w, -1) \subset E(w_0, -1)$. Let Δ_1 be a $(-w)$ -basis of $\Phi_0(-\theta)$. Then $w = w_{\Pi_1}^0$ for Π_1 a subset of Δ_1 . Since Δ_0 and Δ_1 are conjugate under $W_0(-\theta)$ (i) follows.

(ii) follows from Theorem 9.14 and Lemma 9.17. \square

Remarks 9.19. (1) In Table 1 we list for each absolutely irreducible (Γ, θ) -index the type of the diagram of conjugacy classes $\mathcal{L}(w_0)$. Moreover we give the θ -index of $(\Phi(A_0), \theta|\Phi(A_0))$ and the type of the root system $\Phi(A_0) \cap \Phi(A_0^-)$.

In order to be able to refer to the absolutely irreducible (Γ, θ) -indices in this table, we will use the following notation, which combines the notation of Tits [29] for Γ -indices and the notation in [9] for θ -indices. For the Γ -indices we use the notation ${}^g X_{n,r}^t$. Here X denotes the type of Φ , i.e. one of A, B, \dots, G , n the rank of Φ , r the rank of $\bar{\Delta}_\Gamma$ and g the order of the action of Γ on the Dynkin diagram. In the case that $g = 1$ (i.e. the Dynkin diagram has no nontrivial automorphism) we will omit it in the notation. Finally t denotes either the degree of the division algebra, which occurs in the definition of the considered form or the dimension of the anisotropic kernel. To differentiate between these two cases we put t between parentheses when it stands for the degree of the division algebra. In fact the degree of the division algebra is only used if X is of classical type.

As for the θ -indices they can be described by the type in the Cartan notation together with the rank of the restricted root system $\bar{\Delta}_\theta$, see [9, Table II]. We will use a superindex to indicate the rank of $\bar{\Delta}_\theta$. So combining these two we will denote a (Γ, θ) -index by ${}^g X_{n,r}^t(\text{type } \theta^p)$, where ${}^g X_{n,r}^t$ is as above, type θ is the Cartan notation of the involution and p denotes the rank of $\bar{\Delta}_\theta$. For example ${}^2 A_{2n+1, n+1}^{(1)}(III^p)$ means that Φ is of type A_{2n+1} , θ is of type $AIII$, the action of Γ on the Dynkin diagram is the diagram automorphism, the degree of the division algebra is 1, $\text{rank } \bar{\Delta}_\Gamma = n + 1$ and $\text{rank } \bar{\Delta}_\theta = p$.

(2) If Φ is irreducible and $w_0 = w_\Delta^0$ the opposition involution of (Φ, Δ) , then $\Phi(w_0)$ is irreducible or of type $r \cdot A_1$ ($r = \text{rank } \Phi(w_0)$). Moreover $\mathcal{L}(w_0) = \mathcal{L}(-\text{id}) = \mathcal{L}(\Phi)$ and the diagrams $\mathcal{L}(\Phi)$ and $\mathcal{L}^*(\Phi)$ are the same, except for $D_{2\ell}$ ($\ell \geq 2$). In this case we will also write $D^*(2\ell)$ for $\mathcal{L}^*(\Phi)$.

If $\Phi(w_0)$ is of type $r \cdot A_1$, then we will also write $A(r)$ for the diagram $\mathcal{L}(\Phi)$. Since the diagrams for B_ℓ and C_ℓ ($\ell \geq 2$) are identical, we will denote both of them by $B(\ell)$. Similarly the diagram $\mathcal{L}(\Phi)$ for $D_{2\ell+1}$ ($\ell \geq 2$) is identical to the diagram $\mathcal{L}^*(\Phi)$ for $D_{2\ell}$ and will be denoted by $D^*(2\ell)$. For E_7 , E_8 and F_4 we will denote the diagram $\mathcal{L}(\Phi)$ by respectively $E(7)$, $E(8)$ and $F(4)$. Finally the diagram for G_2 is identical to the diagram for B_2 and will also be denoted by $B(2)$.

Table 1: Conjugacy classes of θ -singular involutions

Type G	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
$A_{n,n}^{(1)}(I)$			A_n	$A(\Gamma \frac{n}{2})$
$A_{2n+1, 2n+1}^{(1)}(II)$			\emptyset	$A(0)$
$A_{n,n}^{(1)}(III_a^p)$ $n - 2p \geq 1$			$p \cdot A_1$	$A(p)$
$A_{2n, 2n}^{(1)}(III_b)$			$n \cdot A_1$	$A(n)$
${}^2A_{2n, n}^{(1)}(III_b)$ $\sigma = \Gamma \& \theta$			BC_n	$B(n)$
${}^2A_{2n, n}^{(1)}(I)$ $\sigma = \Gamma$				

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
$A_{2n-1, 2n-1}^{(1)}(II_b)$			$n.A_1$	$A(n)$
${}^2A_{2n-1, n}^{(1)}(III_b)$ $\sigma = \Gamma \& \theta$			C_n	$B(n)$
$A_{2n+1, n}^{(2)}(I)$			A_n	$A(\Gamma \begin{smallmatrix} n \\ 2 \end{smallmatrix} \uparrow)$
$A_{2n+1, n}^{(2)}(II)$			A_n	$A(\Gamma \begin{smallmatrix} n \\ 2 \end{smallmatrix} \uparrow)$
$A_{2n+1, n}^{(2)}(III_a^{2p})$ $1 \leq 2p < n$			$p.A_1$	$A(p)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
$A_{4n-1, 2n-1}^{(2)}(II_b)$			$n \cdot A_1$	$A(n)$
$A_{4n+1, 2n}^{(2)}(II_b)$			$n \cdot A_1$	$A(n)$
$A_{n,p}^{(2d)}(II)$ $2d = \frac{n+1}{p+1} > 2$			A_p	$A(\lceil \frac{d}{2} \rceil)$
$A_{n,p}^{(d)}(I)$ $d = \frac{n+1}{p+1} > 2$			A_p	$A(\lceil \frac{d}{2} \rceil)$
$2A_{4n-1, 2n}^{(1)}(II)$			$n \cdot A_1$	$A(n)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
${}^2A_{4r+1, 2r+1}^{(1)}(II)$			n, A_1	$A(n)$
${}^2A_{2n, n}^{(1)}(III^p)$ $1 \leq p < n$			BC_p	$B(p)$
${}^2A_{2r+1, r+1}^{(1)}(III^p)$ $1 \leq p \leq n$			C_p	$B(p)$
${}^2A_{n, p}^{(1)}(I)$ $n - 2p > 1$			BC_p	$B(p)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
${}^2A_{2n+1,n}^{(I)}$			BC_n	$B(n)$
${}^2A_{4n+1,2n}^{(II)}$			$n.A_1$	$A(n)$
${}^2A_{2n+1,n}^{(III^p)}$ $1 \leq p \leq n$			BC_p	$B(p)$
${}^2A_{2n+1,n}^{(IIIb)}$			BC_n	$B(n)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
${}^2A_{2n+1, 2p}^{(1)}(II)$ $1 \leq 2p < n$			p, A_1	$A(p)$
${}^2A_{n, p}^{(1)}(III^q)$ $1 \leq p < q \leq \frac{1}{2}(n+1)$ $p < \frac{1}{2}(n-1)$			BC_p	$B(p)$
${}^2A_{n, q}^{(1)}(III^p)$ $1 \leq p \leq q \leq \frac{1}{2}(n-1)$			BC_p	$B(p)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
${}^2A_{n,p}^{(d)}(I)$ $\sigma^* = \Gamma^*$			BC_p	$B(p)$
${}^2A_{n,p}^{(d)}(IIb)$ $\sigma^* = \Gamma^* \& \theta^*$ $d \geq 2$ $2pd \leq n+1$ $d n+1$			BC_p	$B(p)$
${}^2A_{2n-1,p}^{(2d)}(II)$ $d \geq 1$ $2pd \leq n$ $d n$			BC_p	$B(p)$
${}^2A_{n,p}^{(d)}(IIIa)$ $d \geq 2$ $2pd \leq n+1$ $d n+1$ $q \geq pd$			BC_p	$B(p)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
${}^2A_{n,q}^{(d)}(II_a^p)$ $d \geq 2$ $2qd \leq n+1$ $d n+1$ $p = rd, r < q$			BC_p	$B(p)$
$B_{n,n}(I^p)$			BC_p	$B(p)$
$B_{n,n-1}(I^p)$ $p \leq n-1$			BC_p	$B(p)$
$B_{n,q}(I^p)$ $1 \leq p \leq q \leq n-2$			BC_p	$B(p)$
$B_{n,p}(I^q)$ $1 \leq p \leq q < n$			BC_p	$B(p)$
$C_{n,n}^{(1)}(I)$			C_n	$B(n)$
$C_{2n,2n}^{(1)}(II_b)$			$n.A_1$	$A(n)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
$C_{n,n}^{(1)}(II^p)$ $1 \leq p \leq \frac{1}{2}(n-1)$			p, A_1	$A(p)$
$C_{2n,n}^{(2)}(I)$			C_n	$B(n)$
$C_{2n,n}^{(2)}(II_b)$			C_n	$B(n)$
$C_{2n,n}^{(2)}(II^p)$ $1 \leq p \leq (n-1)$			C_p	$B(p)$
$C_{2n+1,n}^{(2)}(I)$			BC_n	$B(n)$
$C_{2n+1,n}^{(2)}(II^p)$ $1 \leq p \leq n$			BC_p	$B(p)$
$C_{n,p}^{(2)}(I)$ $p < \lfloor \frac{n-1}{2} \rfloor$			BC_p	$B(p)$
$C_{2n,p}^{(2)}(II_b)$ $p < n-1$			BC_p	$B(p)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
$C_{n,p}^{(2)}(II^q)$ $1 \leq p \leq q < \frac{1}{2}n$			BC_p	$B(p)$
$C_{n,q}^{(2)}(III^p)$ $1 \leq p \leq q < \frac{1}{2}n$			BC_p	$B(p)$
$D_{n,n}^{(1)}(I_b)$			D_n	$D^*(n)$ (n even) $D^*(n-1)$ (n odd)
$D_{n,n}^{(1)}(I^p)$ $p \neq n-1$			D_p	$D^*(p)$
$D_{2n,2n}^{(1)}(III_a)$			$n.A_1$	$A(n)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
$D_{2n+1, 2n+1}^{(1)}(IIb)$			n, A_1	$A(n)$
$D_{n,p}^{(1)}(I_a^q)$ $1 \leq p \leq q \leq n-2$ $n-p=2m \geq 2$			B_p	$B(p)$
$D_{n,p}^{(1)}(I_b^p)$ $1 \leq p < q \leq n-2$ $n-p=2m \geq 2$			B_p	$B(p)$
$D_{n,p}^{(1)}(Ib)$ $n-p=2m \geq 2$			B_p	$B(p)$
$D_{2n, 2p}^{(1)}(III_a)$ $p < n$			p, A_1	$A(p)$
$D_{2n, n}^{(2)}(I_a^{2p})$ $p \leq n-1$			BC_p	$B(p)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
$D_{2n,n}^{(2)}(I_b)$			C_n	$B(n)$
$D_{2n,n}^{(2)}(III_a)$			C_n	$B(n)$
$D_{2n,n}^{(2)}(III_a)'$			C_{n-1}	$B(n-1)$
$D_{2n+3,n}^{(2)}(I_a^{2,p})_{1 \leq p \leq n}$			BC_p	$B(p)$
$D_{2n+3,n}^{(2)}(I_b)$			B_n	$B(n)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
$D_{2n+3,n}^{(2)} (III_b)$			B_n	$B(n)$
$D_{2n,p}^{(2)} (I_a^q)$ $1 \leq 2p \leq q < 2n-1$			B_p	$B(p)$
$D_{2n,p}^{(2)} (I_a^{2p})$ $1 \leq 2p < 2q \leq 2n-1$			BC_p	$B(p)$
$D_{2n,p}^{(2)} (I_b)$ $n-p \geq 1$			B_p	$B(p)$
$D_{2n,p}^{(2)} (II_a)$ $n-p \geq 1$			B_p	$B(p)$
${}^2D_{n+1,n}^{(1)} (I^p)$ $p \neq n$			BC_p	$B(p)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
${}^2D_{2n+1, 2n}^{(1)}(II_b)$			$n \cdot A_1$	$A(n)$
${}^2D_{n, p}^{(1)}(I_d^p)$ $p \leq q \leq n-2$			B_p	$B(p)$
${}^2D_{n, p}^{(1)}(I_d^p)$ $1 \leq p < q \leq n-2$ $n-p=2m \geq 2$			B_p	$B(p)$
${}^2D_{n, p}^{(1)}(I_b)$ $n-p > 1$			B_p	$B(p)$
${}^2D_{2n+1, 2, 2p}^{(1)}(II_b)$ $p < n$			$p \cdot A_1$	$A(p)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
${}^2D_{2n+2,n}^{(2)} (I_a^{2p})$ $p \leq n$			BC_p	$B(p)$
${}^2D_{2n+2,n}^{(2)} (I_b)$			B_n	$B(n)$
${}^2D_{2n+1,n}^{(2)} (I_a^{2p})$ $1 \leq p < n$			BC_p	$B(p)$
${}^2D_{2n+1,n}^{(2)} (I_b)$			B_n	$B(n)$
${}^2D_{2n+1,n}^{(2)} (II_b)$			B_n	$B(n)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
${}^3D_{4,2}^2(I_b)$			G_2	$B(2)$
${}^6D_{4,2}^2(I_b)$			G_2	$B(2)$
${}^3D_{4,1}^0(I_b)$		\emptyset	BC_1	$A(1)$
${}^6D_{4,1}^0(I_b)$		\emptyset	BC_1	$A(1)$
${}^3D_{4,0}^{28}(I_b)$		\emptyset	\emptyset	$A(0)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
${}^6D_{4,0}^{28}(I_a)$		\emptyset	\emptyset	$A(0)$
${}^1E_{6,6}^0(I)$			E_6	$A(4)$
${}^1E_{6,6}^0(II)$			D_4	$A(4)$
${}^2E_{6,4}^{16}(I)$ $\sigma^* = \Gamma^*$			F_4	$F(4)$
${}^2E_{6,4}^{16}(II)$ $\sigma^* = \Gamma^* \& \theta^*$				

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
${}^1E_{6,6}^0(III)$			$2.A_1$	$A(2)$
${}^1E_{6,6}^0(IV)$			\emptyset	$A(0)$
${}^1E_{6,2}^{16}(II)$			G_2	$B(2)$
${}^1E_{6,2}^{16}(I)$			G_2	$B(2)$
${}^1E_{6,2}^{28}(I)$			A_2	$A(1)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
${}^1E_{6,2}^{28}(II)$			A_1	$A(1)$
${}^1E_{6,2}^{28}(III)$			A_1	$A(1)$
${}^2E_{6,2}^{16'}(IV)$			A_1	$A(1)$
${}^1E_{6,2}^{28}(IV)$			A_2	$A(1)$
${}^1E_{6,0}^{78}(I)$		\emptyset	\emptyset	$A(0)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
${}^1E_{6,0}^{78} (II)$		\emptyset	\emptyset	$A(0)$
${}^1E_{6,0}^{78} (III)$		\emptyset	\emptyset	$A(0)$
${}^1E_{6,0}^{78} (IV)$		\emptyset	\emptyset	$A(0)$
${}^2E_{6,4}^{16} (IV)$			A_1	$A(1)$
${}^2E_{6,4}^{16} (III)$			B_2	$B(2)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
${}^2E_{6,2}^{16'}(I)$			BC_2	$B(2)$
${}^2E_{6,2}^{16'}(II)$			BC_2	$B(2)$
${}^2E_{6,2}^{16'}(III)$			BC_2	$B(2)$
${}^2E_{6,2}^{16''}(I)$			G_2	$B(2)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
${}^2E_{6,2}^{16'}(II)$			G_2	$B(2)$
${}^2E_{6,1}^{29}(I)$		○	BC_1	$A(1)$
${}^2E_{6,1}^{29}(III)$		○	BC_1	$A(1)$
${}^2E_{6,1}^{29}(II)$		○	BC_1	$A(1)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
${}^2E_{6,1}^{29} (IV)$		○	BC_1	$A(1)$
${}^2E_{6,1}^{35} (I)$		○	BC_1	$A(1)$
${}^2E_{6,1}^{33} (II)$		○	BC_1	$A(1)$
${}^2E_{6,1}^{35} (III)$		○	BC_1	$A(1)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
$E_{7,7}^0(VI)$			D_4	$A(4)$
$E_{7,7}^0(VII)$			$3.A_1$	$A(3)$
$E_{7,4}^0(V)$			F_4	$F(4)$
$E_{7,4}^0(VI)$			F_4	$F(4)$
$E_{7,4}^0(VII)$			B_2	$B(2)$
$E_{7,3}^{28}(V)$			C_3	$B(3)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
$E_{7,3}^{28}(VI)$			C_2	$B(2)$
$E_{7,3}^{28}(VII)$			C_3	$B(3)$
$E_{7,2}^{31}(V)$			A_2	$A(1)$
$E_{7,0}^{133}(V)$		\emptyset	\emptyset	$A(0)$
$E_{7,0}^{133}(VI)$		\emptyset	\emptyset	$A(0)$
$E_{7,0}^{133}(VII)$		\emptyset	\emptyset	$A(0)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
$F_{8,8}^0$ (VIII)			E_8	$E(8)$
$E_{8,8}^0$ (IX)			D_4	$A(4)$
$F_{8,4}^{28}$ (VIII)			F_4	$F(4)$
$E_{8,4}^{28}$ (IX)			F_4	$F(4)$
$E_{8,0}^{248}$ (VIII)		\emptyset	\emptyset	$A(0)$
$E_{8,0}^{248}$ (IX)		\emptyset	\emptyset	$A(0)$
$F_{4,4}^0$ (I)			F_4	$F(4)$

Table 1: (continued)

Type (G, θ)	(Γ, θ) -index	$\theta \Phi(A_0)$	$\Phi_0(-\theta)$	$\mathcal{L}(w_0)$
$F_{4,4}^0(II)$			A_1	$A(1)$
$F_{4,1}^{21}(I)$			BC_1	$A(1)$
$F_{4,1}^{21}(II)$			BC_1	$A(1)$
$F_{4,0}^{32}(I)$			\emptyset	$A(0)$
$F_{4,0}^{52}(II)$			\emptyset	$A(0)$
$G_{2,2}^0(I)$			G_2	$B(2)$
$G_{2,0}^{14}(I)$			\emptyset	$A(0)$

(3) The possible types of the root systems $\Phi(w_{\Pi}^0)$ follow from [10, Table II] and the type of the θ -maximal involution as given in Table 1. In fact Table 1 gives the type of the diagram $\mathcal{L}(w_{\Delta_0}^0)$, which completely describes the conjugacy classes.

9.20. Concluding remarks. The classification of the W -conjugacy classes of the θ -singular involutions only depends on the (Γ, θ) -index and not on the specific k -involution θ . To classify the W -conjugacy classes of the (θ, k) -singular involutions all we need to do is to determine which of the conjugacy classes of the θ -singular involutions contain a (θ, k) -singular involution. This will not only depend on the (Γ, θ) -index, but also on other properties of the k -involution, like the associated quadratic elements (see [9] and [7] for more details on these). A classification of the W -conjugacy classes of the (θ, k) -singular involutions, for $k = \mathbb{R}$, will appear in the forthcoming part III of this series of papers. This includes a classification of the fibers of the map $\zeta : \mathcal{A}_k^\theta / H_k \rightarrow \mathcal{A}^\theta / H$ as in (1.2).

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