A CONJUGACY THEOREM FOR SYMMETRIC SPACES

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Abstract. In this paper we prove that two Cartan subspaces of a semisimple symmetric pair $(\mathfrak{g}, \tau)$ are conjugate under $G = \mathrm{Int}(\mathfrak{g})$ if and only if they are conjugate under $(G^\tau)_0$. Moreover we derive a double coset decomposition for G, which improves the results of Oshima and Matsuki.

1. Introduction

Let $G_\mathbb{C}$ be a semisimple complex Lie group and $G \subseteq G_\mathbb{C}$ a real form. Let $\mathfrak{t}_1, \ldots \mathfrak{t}_n$ be representatives of the G-conjugacy classes of Cartan subalgebras of $\mathfrak{g}$. Then the set

$$\bigcup_{i=1}^{n} G N_{G_\mathbb{C}}(\mathfrak{t}_i) G$$

contains an open and dense subset of $G_\mathbb{C}$. This was proved in [St86] using a result of Rothschild ([PR72]) which says that two Cartan subalgebras in $\mathfrak{g}$ are conjugate under $G_\mathbb{C}$ if and only if they are conjugate under G. In [HN93] and [La94] this double coset decomposition was a main tool for examining maximal subsemigroups of $G_\mathbb{C}$ that contain G nonisolated.

In [NO97] this decomposition was generalized to noncompactly causal semisimple symmetric spaces G/H. It was shown there, that

$$\bigcup_{i=1}^{n} H N_G(\mathfrak{a}_i) H$$

contains an open and dense subset of G. Here the $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are the representatives of H_0-conjugacy classes of Cartan subspaces in $\mathfrak{q}$, the (-1)-eigenspace of the involution in $\mathfrak{g}$, where H_0 is the connected component of H containing the identity. The essential fact used in the proof is that in noncompactly causal symmetric spaces the G-conjugacy classes of Cartan subspaces are the same as the H_0-conjugacy classes. In [NO97] this was established by classifying the H_0-conjugacy classes and required detailed information on the special structure theory of noncompactly causal spaces.

In this paper we prove for general connected semisimple symmetric spaces G/H that two Cartan subspaces are conjugate under G if and only if they are conjugate under H. As a consequence of our theorem we can generalize the double coset decomposition to all semisimple symmetric spaces.

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2. The Conjugacy Theorem

Definition 1. 1. *A symmetric space is a triple (G, H, τ) consisting of a Lie group G, an involutive automorphism τ of G and a subgroup H with $G_0^\tau \subseteq H \subseteq G^\tau$.*

2. *A symmetric Lie algebra is a pair $(\mathfrak{g}, \tau)$ of a Lie algebra $\mathfrak{g}$ and an involutive automorphism τ. We denote by $\mathfrak{h}$ respectively $\mathfrak{q}$ the $+1$ respectively -1 eigenspaces of τ.*
3. *We denote by $\mathfrak{g}$ the Lie algebra of G and we write τ also for the derivative of τ acting on $\mathfrak{g}$.*

We fix a symmetric space (G, H, τ). For the purpose of this paper we will assume that the group G is connected and semisimple. Denote the center of G by $Z(G)$. We will assume that $H/(H \cap Z(G))$ is noncompact. In particular τ is not a Cartan involution. We recall that for each such involution there exists a Cartan involution θ on $\mathfrak{g}$ that commutes with τ. We denote by $\mathfrak{k}$ (respectively $\mathfrak{p}$) the $+1$ (respectively -1) eigenspaces of θ. Then

$$\begin{aligned} \mathfrak{g} &= \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p} \\ &= \mathfrak{h}_\mathfrak{k} \oplus \mathfrak{h}_\mathfrak{p} \oplus \mathfrak{q}_\mathfrak{k} \oplus \mathfrak{q}_\mathfrak{p} \end{aligned}$$

where subscripts denote intersection, i.e. $\mathfrak{h}_\mathfrak{k} := \mathfrak{h} \cap \mathfrak{k}$ and analogously in the other cases.

Definition 2. *A subspace $\mathfrak{a}$ of $\mathfrak{g}$ is called a Cartan subspace, if:*

1. *$\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{q}$.*
2. *Every element of $\mathfrak{a}$ is semisimple.*

Lemma 3. [Lo69, p.153, Theorem 2.1] *Let θ_1 and θ_2 denote two Cartan involutions on $\mathfrak{g}$ commuting with τ. Then there exists an $h \in G^tau_0$ such that $\theta_1 = \mathrm{Ad}(h)\theta_2\mathrm{Ad}(h^{-1})$.*

Lemma 4. [OM80, Lemma 5] *Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{g}$. Then there exists a Cartan involution θ on $\mathfrak{g}$, which commutes with τ and leaves $\mathfrak{a}$ invariant.*

Lemma 5. *Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{g}$ and let θ be a Cartan involution commuting with τ. Then there exists an $h \in G_0^\tau$ such that $\mathrm{Ad}(h)\mathfrak{a}$ is θ-invariant.*

Proof. This follows from Lemma 3 and 4 . □

Lemma 6. [Wr72, Lemma 1.1.3.7] *Let K denote the analytic subgroup corresponding to $\mathfrak{k}$. Let $Y' = \mathrm{Ad}(k \exp X)Y$, where $k \in K$ and $X \in \mathfrak{p}$. If both Y and Y' are eigenvectors of θ, we have $[X, Y] = 0$.*

Lemma 7. *Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be two θ-invariant Cartan subspaces of $\mathfrak{g}$. Let G_1 be a θ-invariant subgroup of G. Denote the Lie algebra of G_1 by $\mathfrak{g}_1$. Assume that $G_1 = (G_1 \cap K)\exp(\mathfrak{g}_1 \cap \mathfrak{p})$. Then the Cartan subspaces $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate under G_1 if and only if they are conjugate under $G_1 \cap K$.*

Proof. One direction is obvious. Now assume that for some $g \in G_1$ we have $\mathrm{Ad}(g)\mathfrak{a}_1 = \mathfrak{a}_2$. Since the $\mathfrak{a}_i$ are θ-invariant, we have $\mathfrak{a}_i = (\mathfrak{a}_i \cap \mathfrak{k}) \oplus (\mathfrak{a}_i \cap \mathfrak{p})$. Further $\mathfrak{a}_i \cap \mathfrak{k}$ (respectively $\mathfrak{a}_i \cap \mathfrak{p}$) are exactly the $X \in \mathfrak{a}_i$ for which $\mathrm{ad}X$ has purely imaginary (respectively real) eigenvalues. This property is invariant under conjugation. Hence

$$\mathrm{Ad}(g)(\mathfrak{a}_1 \cap \mathfrak{k}) = \mathfrak{a}_2 \cap \mathfrak{k} \quad \text{and} \quad \mathrm{Ad}(g)(\mathfrak{a}_1 \cap \mathfrak{p}) = \mathfrak{a}_2 \cap \mathfrak{p}\,.$$

Write $g = k \exp Y$ with $k \in G_1 \cap K$ and $Y \in \mathfrak{g}_1 \cap \mathfrak{p}$. Let $X \in \mathfrak{a}_1 \cap \mathfrak{k}$. Then $\mathrm{Ad}(k \exp Y)X \in \mathfrak{a}_2 \cap \mathfrak{k}$. By Lemma 6 we get that $[Y, X] = 0$. So $\mathrm{Ad}(k \exp Y)X = \mathrm{Ad}(k)X$. It follows that

$$\mathrm{Ad}(g)|_{\mathfrak{a}_1 \cap \mathfrak{k}} = \mathrm{Ad}(k)|_{\mathfrak{a}_1 \cap \mathfrak{k}}\,.$$

In the same way we obtain $\mathrm{Ad}(g)|_{\mathfrak{a}_1\cap\mathfrak{p}} = \mathrm{Ad}(k)|_{\mathfrak{a}_1\cap\mathfrak{p}}$. Taken together this results in $\mathrm{Ad}(k)\mathfrak{a}_1 = \mathrm{Ad}(g)\mathfrak{a}_1 = \mathfrak{a}_2$. □

Note that the hypothesis on the Cartan decomposition $G_1 = (G_1 \cap K)\exp(\mathfrak{g}_1 \cap \mathfrak{p})$ in Lemma 7 is satisfied if G_1 has only finitely many connected components.

Lemma 8. *Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be two maximal abelian subalgebras of $\mathfrak{q}_\mathfrak{p}$. Then they are conjugate under $(H \cap K)_0$.*

Proof. In a reductive Lie algebra two maximal abelian subalgebras of $\mathfrak{p}$ are conjugate under a maximal compact subgroup. The algebra $\mathfrak{h}_\mathfrak{k} \oplus \mathfrak{q}_\mathfrak{p}$ is a reductive Lie algebra. Hence all maximal abelian subspaces in $\mathfrak{q}_\mathfrak{p}$ are conjugate under the analytic subgroup corresponding to $\mathfrak{h}_\mathfrak{k}$, which is $(K \cap H)_0$. □

Lemma 9. *Let $G_0^\tau \subseteq H \subseteq G^\tau$. Then two θ-invariant Cartan subspaces $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate under $H \cap K$ if and only if $\mathfrak{a}_1 \cap \mathfrak{k}$ and $\mathfrak{a}_2 \cap \mathfrak{k}$ are conjugate under $H \cap K$.*

Proof. The one direction is obvious. Now let us assume there exists a $k \in H \cap K$ such that $\mathrm{Ad}(k)(\mathfrak{a}_1 \cap \mathfrak{k}) = \mathfrak{a}_2 \cap \mathfrak{k}$. Then $\mathfrak{a}_3 := \mathrm{Ad}(k)\mathfrak{a}_1$ will also be a θ-stable Cartan subspace of $\mathfrak{g}$. Furthermore $\mathfrak{a}_2$ and $\mathfrak{a}_3$ both lie in $\mathfrak{z}_\mathfrak{g}(\mathfrak{a}_2 \cap \mathfrak{k})$, the centralizer of $\mathfrak{a}_2 \cap \mathfrak{k}$ in $\mathfrak{g}$. The subspaces $\mathfrak{a}_2 \cap \mathfrak{p}$ and $\mathfrak{a}_3 \cap \mathfrak{p}$ are both maximal abelian in $\mathfrak{z}_\mathfrak{g}(\mathfrak{a}_2 \cap \mathfrak{k}) \cap \mathfrak{q}_\mathfrak{p}$. Now the assertion follows if we apply Lemma 8 to $\mathfrak{z}_\mathfrak{g}(\mathfrak{a}_2 \cap \mathfrak{k})$. □

Theorem 10. *Suppose that (G, H, τ) is a semisimple symmetric space with G connected. Assume that H is connected. Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be two Cartan subspaces of $(\mathfrak{g}, \mathfrak{h})$ in $\mathfrak{q}$. Then $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate under G if and only if they are conjugate under H.*

Proof. One direction is obvious. To prove the other implication we first look at the case that G is simply connected. We assume that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate under G. With Lemma 5 we can assume that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are both θ-invariant. Then by applying Lemma 7 to G we can find a $k \in K$ such that $\mathrm{Ad}(k)\mathfrak{a}_1 = \mathfrak{a}_2$. In particular $\mathrm{Ad}(k)(\mathfrak{a}_1 \cap \mathfrak{k}) = \mathfrak{a}_2 \cap \mathfrak{k}$.

As $\mathfrak{k}$ is reductive we have $\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{k}'$, where $\mathfrak{z}(\mathfrak{k})$ is the center of $\mathfrak{k}$ and $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$. Both $\mathfrak{z}(\mathfrak{k})$ and $\mathfrak{k}'$ are τ-invariant. Since G is simply connected so is K. Therefore we have that $K = V \times K'$, where $V = \exp \mathfrak{z}(\mathfrak{k})$ is a vector group and $K' = \exp \mathfrak{k}'$ is semisimple and hence compact. Because V acts trivially on $\mathfrak{k}$ we can find a $k' \in K'$ such that $\mathrm{Ad}(k')(\mathfrak{a}_1 \cap \mathfrak{k}) = \mathfrak{a}_2 \cap \mathfrak{k}$. We denote by $\mathfrak{a}_i'$ the projection of $\mathfrak{a}_i \cap \mathfrak{k}$ to $\mathfrak{k}'$. The $\mathfrak{a}_i'$ are abelian and $\mathrm{Ad}(k')\mathfrak{a}_1' = \mathfrak{a}_2'$.

We have $K' = (H \cap K')_0 \exp(\mathfrak{q} \cap \mathfrak{k}')$, cf., [He78, Lemma 6.3 and Theorem 6.7]. Thus we can write $k' = m \exp Y$ with $m \in (H \cap K')_0$ and $Y \in \mathfrak{q} \cap \mathfrak{k}'$. Since $\mathfrak{a}_1'$ and $\mathfrak{a}_2'$ both lie in $\mathfrak{q} \cap \mathfrak{k}'$ we have for $U \in \mathfrak{a}_1'$ that $\mathrm{Ad}(k')U = \tau(-\mathrm{Ad}(k')U) = \mathrm{Ad}(\tau(k'))U$ and therefore $\exp(2Y) = \tau(k')^{-1}k' \in Z_K(\mathfrak{a}_1')$. We claim that there exists an abelian subspace $\mathfrak{b} \subseteq \mathfrak{q} \cap \mathfrak{k}'$ such that $\mathfrak{a}_1' \subseteq \mathfrak{b}$ and $\exp(2Y) \in \exp(\mathfrak{b})$.

Denote by S the abelian group generated by $\exp(2Y)$ and $\exp(\mathfrak{a}_1')$. We claim that $S \subseteq \exp(\mathfrak{q} \cap \mathfrak{k}')$. To prove this we recall from the polar decomposition for compact symmetric spaces that $\exp(\mathfrak{q} \cap \mathfrak{k}')$ is precisely the connected component of e in the set $Q = \{k \in K' \mid \tau(k)^{-1} = k\}$ (cf. [Lo69, Prop. IV.4.4]). Note that $\exp(n2Y)\exp(X) \in Q$ for each $n \in \mathbb{Z}$ and $X \in \mathfrak{a}_1'$ since $\exp(n2Y), \exp(X) \in \exp(\mathfrak{q} \cap \mathfrak{k}') \subseteq Q$ commute. Since $\exp(\mathfrak{a}_1')$ is connected and contains the identity we see that $\exp(n2Y)\exp(X)$ is in the same connected component as $\exp(n2Y)$, i.e. in $\exp(\mathfrak{q} \cap \mathfrak{k}')$. Thus we obtain $S \subseteq \exp(\mathfrak{q} \cap \mathfrak{k}')$.

Since $\exp(\mathfrak{q} \cap \mathfrak{k}')$ is closed in K' we obtain $\overline{S} \subseteq \exp(\mathfrak{q} \cap \mathfrak{k}')$. Thus $\overline{S}$ is compact and $\overline{S}/\overline{S}_0$ is finite. Since $\overline{S}/\overline{S}_0$ is generated by $\exp(2Y)\overline{S}_0$ it is also cyclic. Therefore we can find an element $x \in \overline{S}$ such that $\{x^n : n \in \mathbb{N}\}$ lies dense in $\overline{S}$. We write $x = \exp Z$ for $Z \in \mathfrak{q} \cap \mathfrak{k}'$. Then $\overline{\exp(\mathbb{R}Z)}$ is a connected abelian group lying in $\exp(\mathfrak{q} \cap \mathfrak{k}')$ and containing

$\exp(2Y)$ and $\exp(\mathfrak{a}_1')$. We denote it by B. Then $\mathfrak{b}$, the Lie algebra of B, is the desired subspace.

Let $X \in \mathfrak{b}$ be such that $\exp(2X) = \exp(2Y)$ and define $k_1 = k' \exp(-X)$. Since B is abelian we have

$$\begin{aligned} k_1^{-1}\tau(k_1) &= \exp(X)\exp(-Y)m^{-1}m\exp(-Y)\exp(X) \\ &= \exp(-2Y)\exp(2X) = e. \end{aligned}$$

So $k_1 \in G^\tau \cap K$. But G is simply connected, so by [Lo69, p. 337] we have that G^τ is connected. Therefore k_1 lies in $H \cap K$. Since $[X, \mathfrak{a}_1'] = 0$ we obtain $\mathrm{Ad}(k_1)|\mathfrak{a}_1' = \mathrm{Ad}(k')|\mathfrak{a}_1'$. Then obviously $\mathrm{Ad}(k_1)(\mathfrak{a}_1 \cap \mathfrak{k}) = \mathrm{Ad}(k')(\mathfrak{a}_1 \cap \mathfrak{k}) = \mathfrak{a}_2 \cap \mathfrak{k}$. Now Lemma 9 completes the proof.

If G is not simply connected we can look at the universal covering group $\tilde{G}$. Then $\mathrm{Ad}(G) = \mathrm{Ad}(\tilde{G})$ and $\mathrm{Ad}(G_0^\tau) = \mathrm{Ad}(\tilde{G}_0^\tau)$ which completes the proof in the general case. □

We would like to remark here that the classification of H_0-conjugacy classes of Cartan subspaces will be given in [Helm99]. Note the following simple consequence of our main theorem.

Corollary 11. *Let G be a connected semisimple group, G/H a symmetric space, and $\mathfrak{a}$ a θ-invariant Cartan subspace of $\mathfrak{g}$. Then*

$$H = N_{H\cap K}(\mathfrak{a})H_0.$$

Proof. Note first that $H = (H \cap K)\exp(\mathfrak{h} \cap \mathfrak{p})$. Pick an $h \in H$. With Theorem 10 we can find an $h' \in H_0$ such that $\mathrm{Ad}(h)\mathfrak{a} = \mathrm{Ad}(h')\mathfrak{a}$. Write $h(h')^{-1} = k\exp(X)$ with $k \in H \cap K$ and $X \in \mathfrak{h} \cap \mathfrak{p}$. Then Lemma 6 shows that $k \in N_{H\cap K}(\mathfrak{a})$ which implies $h \in N_{H\cap K}(\mathfrak{a})H_0$. □

Using Lemma 5 we see that without the assumption that $\mathfrak{a}$ is θ-invariant we can still show that $H = N_{H\cap K}(\mathfrak{a})H_0$ holds for *some* subgroup K with maximal compactly embedded subalgebra.

3. The Double Coset Decomposition

Theorem 12. *Let G be a connected semisimple group and G/H a symmetric space. Let $\mathfrak{a}_1, \dots \mathfrak{a}_n$ be representatives of the H-conjugacy classes of Cartan subspaces. Then the set*

$$\bigcup_{j=1}^{n} HN_G(\mathfrak{a}_j)H$$

contains an open and dense subset of G.

Proof. This proof is almost the same as in [HN93] or [NO97]. For $g \in G$ we set $g^* := \tau(g)^{-1}$. Define $A_i = Z_G(\mathfrak{a}_i)$ and $B_i = \{g \in G : gg^* \in A_i\}$. Denote by G^{reg} the set of regular elements in G. Then the set $G' = \{g \in G : gg^* \in G^{\mathrm{reg}}\}$ is open and dense in G and [OM80, Theorem 2.ii] tells us, that

$$G' \subseteq \bigcup_{i=1}^{n} HB_iH.$$

Let $g \in B_i$. Then $\mathrm{Ad}(gg^*)|_{\mathfrak{a}_i} = id$. We have $\mathrm{Ad}(g^*) = \tau\mathrm{Ad}(g)^{-1}\tau$. Therefore $\mathrm{Ad}(g)$ commutes pointwise with τ on $\mathfrak{a}_i$. So $\mathrm{Ad}(g)\mathfrak{a}_i$ is a Cartan subspace and therefore

$$\mathrm{Ad}(h)\mathrm{Ad}(g)\mathfrak{a}_i = \mathfrak{a}_j$$

for some $j \in \{1, \ldots, n\}$ and $h \in H$. By Theorem 10 we have that $i = j$ and thus $hg \in N_G(\mathfrak{a}_i)$. It follows that

$$HB_iH \subseteq HN_G(\mathfrak{a}_i)H$$

which proves the claim. □

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