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ON THE CLASSIFICATION OF k-INVOLUTIONS

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Abstract. Let G be a connected reductive algebraic group defined over a field k of characteristic not 2, θ an involution of G defined over k, H a k-open subgroup of the fixed point group of θ and G_k (resp. H_k) the set of k-rational points of G (resp. H). The variety G_k/H_k is called a symmetric k-variety. These varieties occur in many problems in representation theory, geometry and singularity theory. Over the last few decades the representation theory of these varieties has been extensively studied for $k = \mathbb{R}$ and $\mathbb{C}$. As most of the work in these two cases was completed, the study the representation theory over other fields, like local fields and finite fields began. The representations of a homogeneous space usually depend heavily on the fine structure of the homogeneous space, like the restricted root systems with Weyl groups, etc. Thus it is essential to study first this structure and the related geometry.

In this paper we give a characterization of the isomorphy classes of these symmetric k-varieties together with their fine structure of restricted root systems and also a classification of this fine structure for the real numbers, $\mathfrak{p}$-adic numbers, finite fields and number fields.

1. Introduction

Symmetric varieties are defined as the spherical homogeneous spaces G/H with G a reductive algebraic group and H the fixed point group of an involution θ. They occur in many problems in representation theory (see [BB81] and [Vog83]), geometry (see [PdC83] and [Abe88]) and singularity theory (see [LV83] and [Slo84]).

When G and θ are defined over a field k which is not necessarily algebraically closed, then G_k/H_k is called a *symmetric k-variety*. Here G_k and H_k denote the sets of k-rational points of G_k and H_k. The symmetric k-varieties also play an

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important role in several areas including representation theory and the cohomology of arithmetic subgroups (see [TW89]). Best known are the symmetric k-varieties over the real numbers (also called reductive symmetric spaces). For the representation theory in this case one mainly studies the decomposition into irreducible components of the regular representation of G_k on the Hilbert space $L^2(G_k/H_k)$ of square integrable functions on G_k/H_k (also called the harmonic analysis of the reductive symmetric space). This has been studied extensively in the last few decades. The first breakthrough was made in the early fifties when Harish-Chandra commenced his study of general semisimple Lie groups. Harish-Chandra's work [HC84] led to Plancherel formulas for Riemannian symmetric spaces (the reductive symmetric spaces $G_{\mathbb{R}}/H_{\mathbb{R}}$ with $H_{\mathbb{R}}$ compact) and the group itself. (Note that any group G is a symmetric k-variety. Namely consider $G_1 = G \times G$ and $\theta(x, y) = (y, x)$, then $H \simeq G$ embedded diagonally and $G_1/H \simeq G$ embedded anti-diagonally.) The case of general real symmetric k-varieties turned out to be much more complicated and over the last 30 years many people worked on this, including Brylinski, Carmona, Delorme, Faraut, Flensted-Jensen, Matsuki, Oshima, Sekiguchi, Schlichtkrul, and van der Ban (see [BD92, CD94, Del97, Far79, FJ80, OM84, OS80, Ban88, BS97]). The work on the Plancherel formula for real reductive symmetric spaces was recently completed by Delorme [Del97].

For other base fields the representations related to symmetric k-varieties have been studied for k a finite field (see for example [Lus90] and [Gro92]), for k a number field (see [JLR93]) and k a $\mathfrak{p}$-adic field. This latter case is in fact the natural next case to study now that the Plancherel formula for real reductive symmetric k-varieties has been completed. These symmetric k-varieties are also called $\mathfrak{p}$-adic symmetric spaces. For these the "groups case" has again been studied extensively. Much less is known for the general case, but recently a number of encouraging first results have been obtained (see for example [RR96, Bos92, HHb, HHa]). A major obstacle for studying the general symmetric k-varieties has been that there was no classification of these $\mathfrak{p}$-adic symmetric k-varieties together with their fine structure of restricted root systems etc. In this paper we remove this obstacle by giving a characterization of the isomorphy classes of these symmetric k-varieties and also classify the corresponding fine structure for a number of base fields, including the real numbers, $\mathfrak{p}$-adic numbers, finite fields and number fields. This classification of the fine structure of restricted root systems with Weyl groups etc. related to these symmetric k-varieties is possibly even more important than a classification of the symmetric k-varieties itself. In the real case this fine structure enabled one to analyze the representations involved in the Plancherel decomposition in much more detail and consequently it played a fundamental role in the study of the harmonic analysis of real reductive symmetric spaces. The first studies of the representations associated with these symmetric k-varieties over $\mathfrak{p}$-adic and finite fields indicate that in these cases the fine structure will play a similar important role. A classification of the symmetric k-varieties makes it also possible to study the representations of a number of explicit cases, so that one might get an idea what problems to expect in tacking the representations for the general symmetric k-varieties.

There are many ways in which one could characterize (and classify) the isomorphy classes of the k-involutions (and the related symmetric k-varieties). The method presented here is not necessarily the easiest one, but it describes and classifies at the same time the interplay of the fine structure of the symmetric k-variety and the group itself. This is in some way even more useful for the representation theory then a classification of the symmetric k-varieties itself as noted above.

To classify the isomorphy classes of k-involutions one needs to find a number of invariants which will characterize the isomorphy classes. One might expect that one can use (with some modifications) the invariants used in the characterization of involutions of groups defined over an algebraically closed field and the invariants used in the characterization of semisimple k-groups. Recall that semisimple algebraic groups defined over an algebraically closed field are completely characterized (modulo the center) by the corresponding Dynkin diagram. For isomorphy classes of involutions of groups defined over an algebraically closed field of characteristic not 2 it was shown in [Hel88] that they can be characterized by an extension of the Dynkin diagram for the group called the "θ-index". This θ-index completely determines the restricted root system of the symmetric variety G/G_θ, which is the root system of a maximal θ-split torus of G. (A θ-split torus S of G is a torus satisfying $\theta(a) = a^{-1}$ for all $a \in S$). For isomorphy classes of semisimple k-groups there also exists a natural extension of the Dynkin diagram which describes the fine k-structure of the group, including the restricted root system related to a maximal k-split torus. This diagram is often called a Γ-index, where Γ is the Galois group of a finite extension K of k, which splits a maximal k-torus containing a maximal k-split torus. However in this case the Γ-index does not suffice to characterize the isomorphy classes of semisimple k-groups and a second invariant is needed. A necessary and sufficient second condition is the isomorphy of the k-anisotropic kernels of the groups (i.e. the centralizer groups of the maximal k-split tori or equivalently the Levi factors of minimal parabolic k-subgroups). We note that for $k = \mathbb{R}$ this second invariant is not needed and the semisimple $\mathbb{R}$-groups are completely characterized by the Γ-index.

To classify the isomorphy classes of k-involutions it would be natural to try and combine the above classifications. Again one can define a natural extension of the Dynkin diagram for the group, which determines the restricted root system of the symmetric k-variety together with the multiplicities etc. This restricted root system is the root system of a maximal (θ, k)-split torus of G, (i.e. a torus which is both θ-split and k-split). In this case there are some complications, since this index is not necessarily uniquely determined by the isomorphy class of the k-involution. By requiring that the index is an extension of both the underlying Γ-index and θ-index we can solve the uniqueness problem. We can also combine this index now with the Γ-index and θ-index and get an index from which we can recover all three these indices. This index will be called a (Γ, θ)-index. From the characterization of the isomorphy classes of k-groups we know that this (Γ, θ)-index will not suffice to characterize the isomorphy classes and we will need to require at least the isomorphy of the k-involutions restricted to the k-anisotropic kernel. This is an invariant for the isomorphy classes of the k-involutions. Again this condition is not needed in the case that $k = \mathbb{R}$.

Unfortunately these 2 invariants do not suffice to characterize the isomorphy classes of k-involutions. There are several complications and also a third invariant is needed. The additional invariant essentially comes down to isomorphy classes of cosets of A/A^2 in a maximal (θ, k)-split torus A. A complicating factor here is that not all maximal (θ, k)-split tori are conjugate under H_k and consequently the isomorphy of the above cosets as well as the isomorphy of the k-involutions of the k-anisotropic kernel is not under the normalizer of the maximal (θ, k)-split torus, but under the set $(HZ_G(A))_k$, where A is a maximal (θ, k)-split torus. Again in the case that $k = \mathbb{R}$ these complications do not occur. In fact, for $k = \mathbb{R}$, the isomorphy of the above cosets can be reduced to isomorphy classes of elements of order 2 in the maximal (θ, k)-split torus A. Although the ideas behind this classification can be described relatively simply, the technical details are in fact quite complicated. Precise definitions and an outline follow.

Let G be a reductive algebraic group defined over an algebraically closed field of characteristic not 2, T a maximal torus of G, $X^*(T)$ the group of characters of T, $\Phi(T)$ the set of roots of T with respect to G and $W(T)$ the Weyl group of T with respect to G. Isomorphy classes of involutions of G were classified by reducing the problem to $W(T)$-conjugacy classes of certain involutions of $(X^*(T), \Phi(T))$ (see [Hel88]). This reduction can be obtained as follows. Let $\mathcal{C}$ be the set of isomorphy classes of involutions of G. An involution θ of G is called *normally related to* T if $T_\theta^- := \{t \in T \mid \theta(t) = t^{-1}\}^0$ is a maximal θ-split torus of G. Every involution of G is G-isomorphic to one normally related to T. So every isomorphy class in $\mathcal{C}$ has a representative which is normally related to T. In [Hel88, 3.7] it was shown that two involutions θ_1, θ_2 of G normally related to T are G-isomorphic if and only if the induced involutions of $(X^*(T), \Phi(T))$ are $W(T)$-conjugate. If we denote the set of $W(T)$-isomorphy classes of involutions of $(X^*(T), \Phi(T))$ by $\mathcal{T}$ then this result gives us a map $\rho : \mathcal{C} \to \mathcal{T}$, which is one to one. It follows that the classification of the G-isomorphy classes of involutions of G reduces to a classification of isomorphy classes of involutions of $(X^*(T), \Phi(T))$, which can be lifted to an involution of (G, T), normally related to T. Involutions of $(X^*(T), \Phi(T))$ can be described by an index, which describes at the same time the restricted root system of the symmetric variety (i.e. $\Phi(T_\theta^-)$). This index is called a θ-index. So this characterization also gives us all the fine structure of the symmetric varieties related to these involutions.

Next assume that all our groups and automorphisms are defined over an arbitrary field of characteristic not 2 and denote the set of k-rational point of a k-group G by G_k. Let $\mathcal{C}_k$ denote the set of G_k-isomorphy classes of k-involutions of G. To characterize $\mathcal{C}_k$ we can do something similar as for involutions of groups over algebraically closed fields. Only this time, since we have k-involutions and a k-structure, we do not only characterize the involutions on a maximal torus, but also on a maximal k-split torus A of G so that we obtain the fine structure of both the k-group and the symmetric k-variety G_k/H_k. Similarly as in the case of maximal tori, we call an involution θ of G *normally related to* A if $A_\theta^- := \{a \in A \mid \theta(a) = a^{-1}\}^0$ is a maximal (θ, k)-split torus of G. Let $T \supset A$ be a maximal k-torus of G, $W(A, T) = \{w \in W(T) \mid w(A) \subset A\}$ and let $\mathcal{T}(A)$ denote the set of $W(A, T)$-isomorphy classes of involutions of $(X^*(T), \Phi(T), \Phi(A))$. Again, one can show that every k-involution of G is G_k-isomorphic to one normally related to A. So if we denote the family of all k-involutions of G, which are normally related to A by $\mathcal{F}_k(A)$, then every class in $\mathcal{C}_k$ has a representative in $\mathcal{F}_k(A)$. As with the isomorphy classes of involutions of G, we get again a map $\rho_k : \mathcal{C}_k \to \mathcal{T}(A)$, using the condition that the k-involutions θ of G have to be normally related to A and T. The image of ρ_k consists of the $W(A, T)$-isomorphy classes of involutions of

$(X^*(T), \Phi(T), \Phi(A))$, which can be lifted to k-involutions of G, normally related to A and T. We call these involutions of $(X^*(T), \Phi(T), \Phi(A))$ admissible k-involutions.

Unfortunately the map ρ_k is in general not one-to-one, so in order to get a characterization of the G_k-isomorphy classes of k-involutions of G we will need to characterize the fibers of ρ_k as well. Let $N = N_G(A)$ be the normalizer of A in G and let $\mathcal{C}_k(A, G)$ denote the set of N-isomorphy classes of k-involutions, which are normally related to A. Similarly as for $\mathcal{C}_k$ we also have a natural map $\rho_N : \mathcal{C}_k(A, G) \longrightarrow \mathcal{T}(A)$. In Theorem 8.9 we show that ρ_N is one to one. Since there exist also a natural map ρ_1 of $\mathcal{C}_k$ into $\mathcal{C}_k(A, G)$ mapping a G_k-isomorphy class onto a N-isomorphy class, it suffices to characterize the fibers of ρ_1 instead of ρ_k.

The characterization of the fibers of ρ_1 can be split in 2 parts. The first part concerns the restrictions of the involutions to the k-anisotropic kernel G_0 of G, i.e. the semisimple part of $Z_G(A)$, where A is a maximal k-split torus of G. The k-anisotropic kernel of G is uniquely determined (up to k-isomorphism) by the k-isomorphism class of G and in fact the isomorphy classes of semisimple k-groups are characterized by a congruence of the Γ-indices and the isomorphy of the k-anisotropic kernels. Let $\mathcal{F}_k(A, Z_G(A)) = \{\theta | Z_G(A) \in \mathrm{Aut}(Z_G(A), G) \mid \theta \in \mathcal{F}_k(A)\}$ denote the set of the restrictions of the k-involutions in $\mathcal{F}_k(A)$ to $Z_G(A)$. Let $\mathcal{C}_k(Z_G(A), G)$ denote the isomorphy classes of the involutions in $\mathcal{F}_k(A, Z_G(A))$, which are isomorphic under G_k. Since any two involutions in $\mathcal{F}_k(A, Z_G(A))$ which are G_k-isomorphic are also N-isomorphic we get a natural map $\nu : \mathcal{C}_k(Z_G(A), G) \to \mathcal{C}_k(A, G)$. This map is clearly surjective and its fibers are essentially the G_k-isomorphy classes of k-involutions of $Z_G(A)$ (coming from involutions of G), which give the same N-isomorphy class. Finally by restricting the k-involutions in $\mathcal{F}_k(A)$ to $Z_G(A)$ we also get a natural map μ from $\mathcal{C}_k$ to $\mathcal{C}_k(Z_G(A), G)$. We note that essentially we have split the map ρ_k in 3 parts:

$$\mathcal{C}_k \xrightarrow{\mu} \mathcal{C}_k(Z_G(A), G) \xrightarrow{\nu} \mathcal{C}_k(A, G) \xrightarrow{\rho_N} \mathcal{T}(A).$$

The fibers of μ and ν as well as the isomorphy classes of the k-involutions can be characterized now as follows:

Corollary 8.14. *Let G be a connected semi-simple algebraic group defined over k, A a maximal k-split torus of G and θ_1, θ_2 k-involutions of G, normally related to A. Then θ_1 is G_k-isomorphic to $\theta_2 \, \mathrm{Int}(a)$ for some $a \in A^-_{\theta_2}$ if and only if $\theta_1 | Z_G(A)$ and $\theta_2 | Z_G(A)$ are isomorphic under G_k.*

Instead of isomorphy under G_k one can also consider isomorphy under $\mathrm{Int}_k(G)$ (the set of inner automorphisms of G, which are defined over k) or $\mathrm{Aut}_k(G)$ (the set of k-automorphisms of G). In these cases one gets similar characterizations by replacing G_k-isomorphy by $\mathrm{Int}_k(G)$ or $\mathrm{Aut}_k(G)$-isomorphy whenever appropriate.

To determine the fibers of μ we have to determine the G_k-isomorphy classes of the k-involutions $\theta\,\mathrm{Int}(a)$ with $a \in A_\theta^-$. Denote the set of $a \in A_\theta^-$ such that $\theta\,\mathrm{Int}(a)$ is a k-involution of G by $I_k(A_\theta^-)$. This is called the *set of k-inner elements of* A_θ^-. Two involutions $\theta\,\mathrm{Int}(a)$ and $\theta\,\mathrm{Int}(b)$ with $a, b \in I_k(A_\theta^-)$ are G_k-isomorphic if and only if $\theta(g)ag^{-1} = b \mod Z(G)$. So for the isomorphy of these involutions we will have to consider the θ-twisted action of G_k on $I_k(A_\theta^-)$. A characterization of these isomorphy classes is given in section 9. Using the action of the Weyl group of $\Phi(A_\theta^-)$ one can reduce to elements of $I_k(A_\theta^-)$ contained in a Weyl chamber. Unfortunately this does not reduce the classification of these involutions to the action of $Z_{G_k}(A_\theta^-)$. The major complicating factor here is that we have to consider isomorphy under G_k instead of $N_{G_k}(A)$, because not all maximal (θ, k)-split tori of G are $G_\theta(k)$-conjugate. The best one can do is to reduce to conjugacy under the set $(Z_G(A)G_\theta)_k$, which contains representatives for the Weyl group of $\Phi(A_\theta^-)$ as well. For the classification of the G_k-isomorphy classes of k-involutions of the k-anisotropic kernel of G one has in fact the same complication and one also needs to consider isomorphy under G_k instead of $N_{G_k}(A)$. Again it suffices to consider isomorphy under the set $(Z_G(A)G_\theta)_k$. In fact most of the results in the characterization of the isomorphy classes of k-involutions become much simpler if all maximal (θ, k)-split tori of G are $G_\theta(k)$-conjugate. We call pairs (G, θ) for which this is the case *special pairs* and throughout this paper we will prove a number of additional results for these pairs. It is well known that, for $k = \mathbb{R}$, all pairs are special and it is shown in [Hel99] that for $k = \mathbb{Q}_p$ most pairs are special as well.

In a number of cases, including $k = \mathbb{R}$, the G_k-isomorphy classes of the k-inner elements of A_θ^- can be reduced to isomorphy classes of elements of order 2 in A_θ^-. The Weyl group orbits of these elements were classified by Borel and de Siebenthal [BdS49].

To summarize, the classification of the k-involutions of G essentially reduces to the following 3 problems.

(1) classification of admissible k-involutions of $(X^*(T), \Phi(T), \Phi(A))$.
(2) classification of the G_k-isomorphy classes of k-involutions of the k-anisotropic kernel of G.
(3) classification of the G_k-isomorphy classes of k-inner elements $a \in I_k(A_\theta^-)$.

In the latter part of this paper we give a classification of the admissible k-involutions of $(X^*(T), \Phi(T), \Phi(A))$ for a number of base fields. For this we describe these k-involutions of $(X^*(T), \Phi(T), \Phi(A))$ by an index. The involution of $(X^*(T), \Phi(T))$ can again be described by a θ-index, but this index does not describe whether the involution is normally related to A, nor can one determine A and $\Phi(A)$ from this. On the other hand there exists also a natural index corresponding to the k-structure of G, which is called the Γ-index. Here Γ is the Galois group of a finite extension K/k for which T splits. This index describes both A and $\Phi(A)$. However, even if we combine the Γ-index and θ-index the resulting index does not determine the maximal (θ, k)-split torus contained in A nor its root system. For this we need to impose the following additional combinatorial condition. Let $\Phi_0(\theta) = \{\alpha \in \Phi(T) \mid \theta(\alpha) = \alpha\}$, $\Phi_0(\Gamma) = \{\alpha \in \Phi(T) \mid \sum_{\sigma\in\Gamma} \sigma(\alpha) = 0\}$ and $\Phi_0(\Gamma, \theta) = \{\alpha \in \Phi(T) \mid \sum_{\sigma\in\Gamma} \sigma(\theta(\alpha)) = \sum_{\sigma\in\Gamma} \sigma(\alpha)\}$. The action of Γ and θ on $(X^*(T), \Phi(T))$ is called a *basic action* if it satisfies the following condition:

(1)
If $\Phi_1 \subset \Phi_0(\Gamma, \theta)$ irreducible component, then $\Phi_1 \subset \Phi_0(\theta)$ or $\Phi_1 \subset \Phi_0(\Gamma)$.

Now call an index of $(X^*(T), \Phi(T))$ a (Γ, θ)-index if the action of Γ and θ is a basic action and it is both a θ-index and a Γ-index. We note that this index describes at the same time the restricted root system of the symmetric variety (i.e. $\Phi(T_\theta^-)$), the restricted root system of the k-structure (i.e. $\Phi(A)$) and the restricted root system of the symmetric k-variety (i.e. $\Phi(A_\theta^-)$). Moreover we have now a one to one correspondence between the isomorphy classes of the admissible k-involutions of $(X^*(T), \Phi(T))$ and the isomorphy classes of the corresponding (Γ, θ)-indices (see Proposition 10.36). We will call a (Γ, θ)-index admissible if the corresponding k-involution of $(X^*(T), \Phi(T), \Phi(A))$ is admissible. It suffices now to classify the admissible (Γ, θ)-indices. For this we first need a characterization of the isomorphy classes of the admissible (Γ, θ)-indices. From the above characterization of the isomorphy classes of k-involutions of G we already know that a necessary condition is that the underlying indices of the involution with respect to the maximal torus (i.e. the θ-index), the Γ-index of the k-structure and the index of the restriction of the involution to the k-anisotropic kernel are all admissible. Also condition (1) must be satisfied. In Theorem 10.45 we show that these do not suffice and show that it must satisfy an additional combinatorial condition to be an admissible (Γ, θ)-index. Finally in section 11 we use this result to classify the admissible (Γ, θ)-indices (i.e. the admissible k-involutions) for k the real numbers, $\mathfrak{p}$-adic numbers, a finite field or a number field. This includes a classification of the root systems for the corresponding symmetric k-varieties.

A brief outline of this paper follows. In section 2 we set the notation and review some basic facts about symmetric k-varieties. This includes a discussion of the natural root system of a symmetric k-variety. Section 3 is devoted to a characterization of the H_k-conjugacy classes of maximal (θ, k)-split tori by analyzing the $Z_{G_k}(A) \times G_\theta(k)$-orbits in the set $(Z_G(A)G_\theta)_k$ for a maximal k-split torus A of G. These conjugacy classes play a fundamental role in the classification. In section 4 we discuss the interplay of the fine structures of the symmetric variety (i.e. action of θ), the k-group (i.e. the action of the Galois group $\Gamma = \mathrm{Gal}(K/k)$ of a splitting extension of the maximal torus) and the symmetric k-variety (i.e. action of θ and Γ). In the next section we define the θ-index, Γ-index and (Γ, θ)-index corresponding to these actions and show that they are uniquely determined by their respective isomorphy classes. In section 6 we prove a number of results about k-automorphisms which will be needed for the classification and in section 7 we briefly review the characterization of isomorphy classes of involutions of semisimple groups defined over an algebraically closed field and the characterization of isomorphy classes of semisimple k-groups. Both these characterization are needed for the characterization of k-involutions, which is finally discussed in section 8. Section 9 discusses the G_k-isomorphy classes of the involutions $\theta\,\mathrm{Int}(a)$ for the k-inner elements $a \in I_k(A_\theta^-)$. The last 2 sections deal with the classification of the admissible k-involutions of $(X^*(T), \Phi(T))$. In section 10 we give a characterization of the corresponding (Γ, θ)-indices and in section 11 we give a classification of these for k the real numbers, $\mathfrak{p}$-adic fields, finite fields and number fields.

Some of the results in this paper were announced in [Hel94].

2. Preliminaries and Recollections

In this section we set the notations and recall a few results from [HW93] and [Hel88]. We will also discuss the relation between the orbits of minimal parabolic k-subgroup acting on a symmetric k-variety and the H_k-conjugacy classes of θ-stable maximal k-split tori. For this we will rephrase the characterization of these orbits in [HW93] by giving another characterization of the orbits, which is geared more toward the conjugacy classes of θ-stable maximal k-split tori. Our basic reference for reductive groups will be the papers of Borel and Tits [BT65, BT72] and also the books of Borel [Bor91], Humphreys [Hum75] and Springer [Spr81]. We shall follow their notations and terminology. All algebraic groups and algebraic varieties are taken over an arbitrary field k (of characteristic $\neq 2$) and all algebraic groups considered are linear algebraic groups.

2.1. **Notations.** Given an algebraic group G, the identity component is denoted by G^0. We use $L(G)$ (resp. $\mathfrak{g}$, the corresponding lower case German letter) for the Lie algebra of G. If S is a subset of G and H a closed subgroup of G, then we write $N_H(S)$ (resp. $Z_H(S)$) for the normalizer (resp. centralizer) of S in H. We write $Z(G)$ for the center of G. The commutator subgroup of G is denoted by $D(G)$ or $[G, G]$.

Let k be a field. An algebraic group defined over k shall also be called an algebraic k-group. For an extension K of k, the set of K-rational points of G is denoted by G_K or $G(K)$.

If G is a reductive k-group and A a torus of G then we denote by $X^*(A)$ (resp. $X_*(A)$) the group of characters of A (resp. one-parameter subgroups of A) and by $\Phi(A) = \Phi(G, A)$ the set of the roots of A in G. The group $X^*(A)$ can be put in duality with $X_*(A)$ by a pairing $\langle \cdot\,,\ \cdot \rangle$ defined as follows: if $\chi \in X^*(A),\ \lambda \in X_*(A)$, then $\chi(\lambda(t)) = t^{\langle \chi, \lambda \rangle}$ for all $t \in k^*$.

For a closed subgroup H of G we denote the Weyl group of H relative to A by $W_H(A) = N_H(A)/Z_H(A)$. If $H = G$, then we will also write $W(A) = W(G, A) = N_G(A)/Z_G(A)$. If $\alpha \in \Phi(G, A)$, then let U_α denote the unipotent subgroup of G corresponding to α. If A is a maximal torus, then U_α is one-dimensional. Given a quasi-closed subset ψ of $\Phi(G, A)$, the group G_ψ (resp. G^*_ψ) is defined in [BT65, 3.8]. If G^*_ψ is unipotent, ψ is said to be unipotent and often one writes U_ψ for G^*_ψ.

If T is a torus of G defined over k, then there are subtori T_a and T_d of T, where T_a is the largest anisotropic subtorus of T and T_d is the largest k-split subtorus of T defined over k. These tori satisfy: $T = T_a \cdot T_d$ and $T_a \cap T_d$ is finite (see [Bor91, 8.15]).

Throughout the paper G will denote a connected reductive algebraic k-group.

2.2. ***k*-automorphisms.** A mapping $\phi : G \to G$ is called a k-automorphism if ϕ is a bijective rational k-homomorphism whose inverse is a rational k-homomorphism as well. The group of k-automorphisms of G will be denoted by $\mathrm{Aut}_k(G)$. If k is algebraically closed we will also write $\mathrm{Aut}(G)$ instead. For $g \in G$ write $\mathrm{Int}(g)$ for the inner automorphism of G defined by $\mathrm{Int}(g)(x) := gxg^{-1}$. Denote the group of inner automorphisms of G by $\mathrm{Int}(G)$ and the group of inner k-automorphisms of G by $\mathrm{Int}_k(G)$. Then $\mathrm{Int}_k(G) = \mathrm{Int}(G) \cap \mathrm{Aut}_k(G)$. Note that $\mathrm{Int}_k(G) \supsetneq \mathrm{Int}(G_k) = \{\mathrm{Int}(g) \mid g \in G_k\}$. For a subset $R \subset G$ we will write $\mathrm{Int}(R)$ for $\{\mathrm{Int}(x) \mid x \in R\}$.

For a subgroup $S \subset G$ let

$$\mathrm{Aut}_k(G, S) = \{\phi \in \mathrm{Aut}_k(G) \mid \phi(S) \subset S\},$$
$$\mathrm{Int}_k(G, S) = \{\phi \in \mathrm{Int}_k(G) \mid \phi(S) = S\} = \mathrm{Int}_k(G) \cap \mathrm{Aut}_k(G, S) \text{ and}$$
$$\mathrm{Int}(G_k, S) = \{\phi \in \mathrm{Int}(G_k) \mid \phi(S) = S\} = \mathrm{Int}(G_k) \cap \mathrm{Aut}_k(G, S).$$

Note that $\{x \in G_k \mid \mathrm{Int}(x) \in \mathrm{Int}(G_k, S)\} = N_{G_k}(S)$ and $\{x \in G \mid \mathrm{Int}(x) \in \mathrm{Int}_k(G, S)\} \subsetneq N_G(S)$. If k is algebraically closed we will also write $\mathrm{Aut}(G, S)$ for $\mathrm{Aut}_k(G, S)$.

If T is a maximal torus of G defined over k, then by Chevalley's restriction Theorem (see [Che58]) we have $\mathrm{Int}_k(G, T) = \mathrm{Int}(G_k, T).(\mathrm{Int}(T) \cap \mathrm{Int}_k(G, T))$.

2.3. **Involutions of G.** Let k be a field of characteristic not two, G a connected algebraic k-group, θ an automorphism of G of order two and $G_\theta = \{g \in G \mid \theta(g) = g\}$ the set of fixed points of θ. This is a subgroup of G which is reductive if G is reductive. If G is semisimple and simply connected, then G_θ is connected, but in general G_θ is not necessarily connected. When G and θ are defined over k, the automorphism θ will also be called a k-involution of G.

If G is reductive and H a k-open subgroup of G_θ, then we call the variety G/H a *symmetric variety* and the variety G_k/H_k a *symmetric k-variety*. Symmetric varieties are spherical.

Given $g,\ x \in G$, the *twisted action* associated to θ is given by $(g, x) \mapsto g * x = gx\theta(g)^{-1}$. This action will also be called the *θ-twisted action*. Let $Q = \{g^{-1}\theta(g) \mid g \in G\}$ and $Q' = \{g \in G \mid \theta(g) = g^{-1}\}$. The set Q is contained in Q'. Both Q and Q' are invariant under the twisted action associated to θ. There are only a finite number of twisted G-orbits in Q' and each such orbit is closed (see [Ric82]). In particular, Q is a connected closed k-subvariety of G. Define a morphism $\tau_\theta : G \to G$ by

$$\tau_\theta(x) = \theta(x)x^{-1}, \quad (x \in G). \tag{2.3.1}$$

We will omit the subscript θ from this map if there is no ambiguity about the involution involved. The image $\tau(G) = Q$ is a closed k-subvariety of G and τ induces an isomorphism of the coset space G/G_θ onto $\tau(G)$. Note that $\tau(x) = \tau(y)$ if and only if $y^{-1}x \in G_\theta$ and $\theta(\tau(x)) = \tau(x)^{-1}$ for $x \in G$.

2.4. If $T \subset G$ is a torus and $\sigma \in \mathrm{Aut}(G, T)$ an involution, then we write $T_\sigma^+ = (T \cap G_\sigma)^0$ and $T_\sigma^- = \{x \in T \mid \sigma(x) = x^{-1}\}^0$. It is easy to verify that the product map

$$\mu : T_\sigma^+ \times T_\sigma^- \to T, \quad \mu(t_1, t_2) = t_1 t_2$$

is a separable isogeny. In particular $T = T_\sigma^+ T_\sigma^-$ and $T_\sigma^+ \cap T_\sigma^-$ is a finite group. (In fact it is an elementary abelian 2-group.) The automorphisms of $\Phi(G, T)$ and $W(G, T)$ induced by σ will also be denoted by σ. If $\sigma = \theta$ we reserve the notation T^+ and T^- for T_θ^+ and T_θ^- respectively. For other involutions of T, we shall keep the subscript.

Recall from [Hel88] that a torus A is called θ-*split* if $\theta(a) = a^{-1}$ for every $a \in A$. If A is a maximal θ-split torus of G, then $\Phi(G, A)$ is a root system with Weyl group $W(A) = N_G(A)/Z_G(A)$ (see [Ric82]). This is the root system associated with the symmetric variety G/H. To the symmetric k-variety G_k/H_k one can also associate a natural root system. To see this we consider the following tori:

Definition 2.5. A k-torus A of G is called (θ, k)-split if it is both θ-split and k-split.

Consider a maximal (θ, k)-split torus A in G. In [HW93, 5.9] it was shown that $\Phi(G, A)$ is a root system and $N_{G_k}(A)/Z_{G_k}(A)$ is the Weyl group of this root system. We can also obtain this root system by restricting the root system of G_k. Namely let $A_0 \supset A$ be a θ-stable maximal k-split torus of G. Then $A = (A_0)_\theta^-$ and $\Phi(G, A)$ can be identified with $\bar{\Phi}_\theta = \{\alpha|A \neq 0 \mid \alpha \in \Phi(G, A_0)\}$.

We will need several properties of the centralizer of a maximal (θ, k)-split torus. The key result in the study of these is the following result (see [HW93, 4.5]).

Proposition 2.6. *Let A be a maximal (θ, k)-split torus of G. Let C, L_1, L_2 denote the central, anisotropic and isotropic factors of $Z_G(A)$ over k respectively. Then we have the following conditions:*

(1) *A is the unique maximal (θ, k)-split torus of $Z_G(A)$.*
(2) *$L_2 \subset H$.*
(3) *If A_0 is any maximal k-split torus of $Z_G(A)$, then A_0 is θ-stable and moreover $CL_1 \subset Z_G(A_0)$.*

Corollary 2.7. *Let A be a maximal (θ, k)-split torus of G, $A_0 \supset A$ a maximal k-split torus and $S \supset A$ a maximal θ-split k-torus of G. Then A_0 and S commute. In particular there exists a maximal torus $T \subset Z_G(A)$ with $A_0 \subset T$ and $S \subset T$.*

Proof. Let C, L_1, L_2 denote the central, anisotropic and isotropic factors of $Z_G(A)$ over k respectively. Then $S \subset CL_1$. Since by Proposition 2.6(3) $CL_1 \subset Z_G(A_0)$ the result follows. □

Remark 2.8. If A is a maximal (θ, k)-split torus of G, then we will we will call a maximal k-torus $T \subset Z_G(A)$ a *θ-standard maximal k-torus* if T is θ-stable, contains a maximal k-split torus of G and T_θ^- is a maximal θ-split k-torus of G. These maximal k-tori will play an important role in the classification of k-involutions.

2.9. **P_k-orbits on G_k/H_k.** Let P be a minimal parabolic k-subgroup of G. The double cosets $P_k\backslash G_k/H_k$ play an important role in the classification of k-involutions. In this subsection we briefly review some results about these double cosets from [HW93]. There are several way's in which one can characterize the double cosets $P_k\backslash G_k/H_k$. One can characterize them as the P_k-orbits on the symmetric k-variety G_k/H_k (using the θ-twisted action), one can take the H_k-orbits on the flag variety G_k/P_k or one can consider the $P_k \times H_k$-orbits on G_k. All these characterizations are essentially the same. For more details see [HW93]. We will use the $P_k \times H_k$-orbits on G_k to characterize $P_k\backslash G_k/H_k$.

Let A be a θ-stable maximal k-split torus of P, $N = N_G(A)$, $Z = Z_G(A)$ and $W = W(A) = N_G(A)/Z_G(A)$ the corresponding Weyl group. As in [HW93, 6.7] set $\mathcal{V}_k = \{x \in G_k \mid \tau(x) \in N_k\}$. The group $Z_k \times H_k$ acts on $\mathcal{V}_k$ by $(x, z) \cdot y = xyz^{-1}$, $(x, z) \in Z_k \times H_k$, $y \in \mathcal{V}_k$. Let V_k be the set of $(Z_k \times H_k)$-orbits on $\mathcal{V}_k$. If $v \in V_k$, we let $x(v) \in \mathcal{V}_k$ be a representative of the orbit v in $\mathcal{V}_k$. The set V_k is essential in the study of orbits of minimal parabolic subgroups on the symmetric k-variety G_k/H_k. The inclusion map $\mathcal{V}_k \to G_k$ induces a bijection of the set V_k of $(Z_k \times H_k)$-orbits on $\mathcal{V}_k$ onto the set of $(P_k \times H_k)$-orbits on G_k (see [HW93]). The set V_k is in general infinite. In a number of cases one can show that there are only finitely many $(P_k \times H_k)$-orbits on G_k. If k is algebraically closed, the finiteness of V_k was proved by Springer [Spr84]. The finiteness of the orbit decomposition for $k = \mathbb{R}$ was discussed by Wolf [Wol74], Rossmann [Ros79] and Matsuki [Mat79]. For general local fields this result can be found in Helminck-Wang [HW93]. An example that in most cases the set V_k is infinite can be found in [HW93, 6.12].

2.10. **W-action on V_k.** The Weyl group W acts on V_k. This action is defined as follows. Let $v \in V_k$ and let $x = x(v)$. If $n \in N_k$, then $nx \in \mathcal{V}_k$ and its image in V_k depends only on the image of n in W. We thus obtain a (left) action of W on V_k, denoted by $(w, v) \to w \cdot v$ $(w \in W,\ v \in V_k)$.

Let $\mathcal{A}_k$ denote the set of maximal k-split tori of G and let $\mathcal{A}_k^\theta$ be the fixed point set of θ i.e., the set of θ-stable maximal k-split tori. The group H_k acts on $\mathcal{A}_k^\theta$ by conjugation.

If $x \in \mathcal{V}_k$, then $x^{-1}Ax$ is again a maximal k-split torus and conversely any θ-stable maximal k-split torus in $\mathcal{A}_k^\theta$ can be written as $x^{-1}Ax$ for some $x \in \mathcal{V}_k$.

If $v \in V_k$, then $x(v)^{-1}Ax(v) \in \mathcal{A}_k^\theta$. This determines a map ζ of V_k to the orbit set $\mathcal{A}_k^\theta/H_k$. It is easy to check that this map is independent of the choice of the representative $x(v)$ for v and is constant on W-orbits. So we also get a map of orbit sets: $\gamma_k : V_k/W \to \mathcal{A}_k^\theta/H_k$. In fact we have a bijection:

Proposition 2.11 ([Hel97, 1.9])**.** *Let G, $\mathcal{A}_k^\theta$ and γ_k be as above. Then $\gamma_k : V_k/W \to \mathcal{A}_k^\theta/H_k$ is bijective.*

Remark 2.12. The characterization of the isomorphy classes of k-involutions as given in this paper holds for any field k with only the restriction that the characteristic of k is not 2. For the classification of the irreducible indices corresponding to the isomorphy classes of the k-involutions we will restrict to the case that k is a perfect field. In fact we only give a classification of these indices for k the real numbers, $\mathfrak{p}$-adics fields, finite fields and number fields. The corresponding symmetric k-varieties are also mainly studied for these fields. So to avoid unnecessary technical difficulties we will assume for the remainder of this paper that k is a perfect field of characteristic not 2. We leave it to the reader to check that this restriction is not needed in section 8, where we give the characterization of the isomorphy classes of k-involutions.

3. H_k-conjugacy classes of maximal (θ, k)-split tori

The H_k-conjugacy classes of maximal (θ, k)-split tori will play an important role in the classification of the isomorphy classes of k-involutions. In this section we will prove a few facts about these conjugacy classes. Recall that a first characterization of the conjugacy classes of maximal (θ, k)-split tori was given in [HW93, 10.3]. This result is the following.

Proposition 3.1 ([HW93, 10.3])**.** *Let A_1 and A_2 be maximal (θ, k)-split tori of G and A a maximal k-split torus of G containing A_1. Then there exists $g \in (Z_G(A)H^0)_k$ such that $g^{-1}A_1g = A_2$.*

We note that one can replace $(Z_G(A)H^0)_k$ in the above result by $(H^0Z_G(A))_k$ and let $g \in (H^0Z_G(A))_k$ act on A via: gAg^{-1} (instead of $g^{-1}Ag$).

The maximal k-split tori containing the maximal (θ, k)-split tori are conjugate under $(Z_G(A)H^0)_k$ as well as follows from the following result.

Corollary 3.2. *Let A_1 and A_2 be maximal (θ, k)-split tori of G and $\tilde{A}_1 \supset A_1$ and $\tilde{A}_2 \supset A_2$ maximal k-split tori of G. Then there exists $g \in (Z_G(\tilde{A}_1)H^0)_k$ such that $g^{-1}A_1g = A_2$ and $g^{-1}\tilde{A}_1g = \tilde{A}_2$.*

Proof. By Proposition 3.1 there exists $g \in (Z_G(\tilde{A}_1)H^0)_k$ such that $g^{-1}A_1g = A_2$. Let $h \in H^0$ and $z \in Z_G(\tilde{A}_1)$ such that $g = zh$. Then $g^{-1}\tilde{A}_1g = h^{-1}\tilde{A}_1h$ and $\tilde{A}_2$ are θ-stable maximal k-split tori of $Z_G(A_2)$. Let $G_0 = [Z_G(A_2), Z_G(A_2)]$. By Proposition 2.6 $g^{-1}\tilde{A}_1g \cap G_0$ and $\tilde{A}_2 \cap G_0$ are maximal k-split tori of $G_0 \cap H$, hence there exists $h_1 \in (G_0 \cap H)_k$ such that $h_1(g^{-1}\tilde{A}_1g \cap G_0)h_1^{-1} = \tilde{A}_2 \cap G_0$. But then also $h_1g^{-1}\tilde{A}_1gh_1^{-1} = \tilde{A}_2$. Clearly $gh_1^{-1} \in (Z_G(\tilde{A}_1)H^0)_k$. □

It follows from the above results that to characterize the H_k-conjugacy classes of the maximal (θ, k)-split tori one needs to analyze the $H_k \times Z_k$-orbits in $(H^0Z_G(A))_k$ (or equivalently the $Z_k \times H_k$-orbits in $(Z_G(A)H^0)_k$). Before we characterize these orbits we first note the following:

Lemma 3.3. *Let $g = zh \in (Z_G(A)H^0)_k$, where $z \in Z_G(A)$ and $h \in H$. Then $\tau(z) \in \tau(Z_k)$ if and only if there is a $h_1 \in H_k$ and $z_1 \in Z_k$ such that $zh = z_1h_1$.*

Proof. If $\tau(z) \in \tau(Z_k)$ then there exists $z_1 \in Z_k$ such that $\tau(z) = \tau(z_1)$. But then $z_1^{-1}z \in H$. Take $h_1 = z_1^{-1}zh$. Then $zh = z_1h_1 \in (Z_G(A)H^0)_k$ and since $z_1 \in Z_k$ it follows that $h_1 \in H_k$. The opposite statement is immediate. □

3.4. Let V_1 be the set of representatives of the double cosets $Z_k\backslash(Z_G(A)H^0)_k/H_k$. This set basically consists out of a set of representatives for the H_k-conjugacy classes of maximal (θ, k)-split tori and the Weyl group coset $W(A, H)/W(A, H_k)$. In the following we make this all a bit more explicit.

Let $\mathcal{A}_1 \subset \mathcal{A}_k^\theta$ be the set of θ-stable maximal k-split tori containing a maximal (θ, k)-split torus. From Corollary 3.2 it follows that $\mathcal{A}_1 = \{g^{-1}Ag \mid g \in (Z_G(A)H^0)_k\}$. The group H_k acts on $\mathcal{A}_1$ by conjugation. Let $\mathcal{A}_1/H_k$ denote the orbit set. Let $\zeta : V_k \to \mathcal{A}_k^\theta/H_k$ be as in 2.10. Then $\zeta(V_1) = \mathcal{A}_1/H_k$. In the following we show that the fiber of ζ restricted to V_1 corresponds to $W(A, H)/W(A, H_k)$. First we need the following:

Proposition 3.5. *Let $g = zh \in (Z_G(A)H^0)_k$, where $z \in Z_G(A)$ and $h \in H$. If $zh \in (Z_G(A)H)_k \cap N_{G_k}(A)$ then $h \in N_H(A)$. Conversely if $h \in N_H(A)$, then there exists $z \in Z_G(A)$ such that $zh \in (Z_G(A)H)_k \cap N_{G_k}(A)$.*

Proof. Since $N_G(A) = N_{G_k}(A).Z_G(A)$ the first statement is clear.

Let $h \in N_H(A)$. Since $W(A)$ has representatives in $N_{G_k}(A)$, there exits $z \in Z_G(A)$ such that $zh \in N_{G_k}(A)$. It follows that $zh \in (Z_G(A)H)_k \cap N_{G_k}(A)$. □

Corollary 3.6. *$W(A, H)$ has representatives in $(Z_G(A)H)_k$.*

3.7. $W(A,H)$ act on V_k and V_1 is $W(A,H)$-stable. This can be seen as follows. Let $w \in W(A,H)$, $h \in N_H(A)$ a representative. By Proposition 3.5 there exists $z \in Z_G(A)$ such that $zh \in (Z_G(A)H)_k \cap N_{G_k}(A)$. Let $x(v) = z_1h_1$ be a representative of $v \in V_1$. Then $w \cdot x(v) = zhz_1h_1 = zhz_1h^{-1}hh_1 \in (Z_G(A)H)_k$. It is easy to verify that this is independent of the representatives z and h for w and also independent of the representative $x(v)$ of v.

Note that the set $Z_G(A)H \cap N_{G_k}(A)$ consists of representatives for the elements of $W(A,H)$. The $Z_k \times H_k$ orbits of these elements give a set of representatives isomorphic to $W(A,H)/W(A,H_k)$ as follows from the following result:

Proposition 3.8. *The map $\gamma_1 : V_1/W(A,H) \to \mathcal{A}_1/H_k$ is a bijection.*

Proof. Surjectivity follows from Proposition 3.1. As for injectivity let $g_1, g_2 \in (Z_G(A)H)_k$ and $A_1 = g_1^{-1}Ag_1$ and $A_2 = g_2^{-1}Ag_2$. Then $\zeta(g_1) = \zeta(g_2)$ if and only if A_1 and A_2 are H_k-conjugate. Say $h \in H_k$ such that $h^{-1}A_1h = A_2$. We may assume $A_1 = A_2$ Then $x := g_2g_1^{-1} \in N_{G_k}(A)$. Let $z_1, z_2 \in Z_G(A)$ and $h_1, h_2 \in H$ such that $g_1 = z_1h_1$ and $g_2 = z_2h_2$. Then $x = g_2g_1^{-1} = z_2h_2h_1^{-1}z_1^{-1}hh_2^{-1}h_2h_1^{-1} \in (Z_G(A)H)_k \cap N_{G_k}(A)$. If w is the Weyl group element corresponding to x, then $g_2 = w \cdot g_1$. From Proposition 3.5 it follows now that $w \in W(A,H)$, which proves the result. □

Corollary 3.9. *Let $\{x(v) \mid v \in V_1\}$ be a set of representatives of V_1 in $(Z_G(A)H)_k$. Write $x(v) = z_vh_v$, where $h_v \in H$ and $z_v \in Z_G(A)$. Then $\{x(v) \cdot A = x(v)^{-1}Ax(v) = h_v^{-1}Ah_v \mid v \in V_1\}$ is a set of representatives for the H_k-conjugacy classes of θ-stable maximal k-split tori containing a maximal (θ,k)-split torus.*

Remark 3.10. If all maximal (θ,k)-split tori of G are H_k-conjugate, then $V_1 \simeq W(A,H)/W(A,H_k)$. This happens for example for any pair (G,θ) in the case that $k = \mathbb{R}$ and for many pairs (G,θ) in the case that k is the $\mathfrak{p}$-adic numbers. The classification of the k-involutions of G is considerably simpler when this happens. Therefor we define these pairs as follows.

Definition 3.11. Let G be a reductive k-group as above and $\theta \in \mathrm{Aut}_k(G)$ a k-involution. The pair (G,θ) is called a *special pair* if all maximal k-split tori of G containing a maximal (θ,k)-split torus are H_k-conjugate. If all k-involutions of G are special, then we will also call G *special*.

Corollary 3.12. *Assume (G,θ) is special. Then $V_1 \simeq W(A,H)/W(A,H_k)$.*

Remark 3.13. For many special pairs (G,θ) it also happens that $W_G(A)$ has representatives in H_k. (Here A is a maximal k-split torus of G containing a maximal (θ, k)-split torus). This implies that $W(A, H) = W(A, H_k)$ and hence $V_1 \simeq \{\mathrm{id}\}$. This happens for example when H_k is k-anisotropic and for many pairs (G, θ) in the case that $k = \mathbb{R}$ (see [Hel88]) and for many pairs (G, θ) in the case that k is the $\mathfrak{p}$-adic numbers (see also 9.23). However even in these cases it is not true in general as can be seen from the following example.

Example 3.14. Let $k = \mathbb{R}$, $G = \mathrm{SL}_2(\mathbb{C})$, $\theta \in \mathrm{Aut}(G)$ defined by $\theta(g) = {}^t g^{-1}$, $g \in G$ and let $A = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \mid a \in \mathbb{C}\}$ be the set of diagonal matrices. Then A is a maximal k-split torus of G, which is a maximal torus as well. Moreover $A = A_\theta^-$. Both G and θ are defined over $\mathbb{R}$, $G_\mathbb{R} = \mathrm{SL}_2(\mathbb{R})$ and $H = G_\theta = \mathrm{SO}_2(k) = \{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \mid a, b \in k,\ a^2 + b^2 = 1\}$. Note that in fact $\theta = \mathrm{Int}(x)$ where $x = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $\sigma = \mathrm{Int}(y)$, where $y = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $\sigma = \theta\,\mathrm{Int}(b)$, where $b = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) \in A$. The involution σ is also defined over $\mathbb{R}$ and $G_\sigma = \{\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right) \mid a, b \in k,\ a^2 - b^2 = 1\}$. If $g = \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right) \in G_\sigma$ and $\left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1} \end{smallmatrix}\right) \in A$, then $g \left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1} \end{smallmatrix}\right) g^{-1} = \left(\begin{smallmatrix} a^2r - b^2r^{-1} & abr^{-1} - abr \\ abr - abr^{-1} & a^2r^{-1} - b^2r \end{smallmatrix}\right)$, so $g \in N_G(A)$ if and only if $ab = 0$. If $b = 0$, then $g \in A = Z_G(A)$ and if $a = 0$ then $b = \pm i$. It follows that $W(A, G_\sigma(\mathbb{R})) = \{\mathrm{id}\} \subsetneq W(A, G_\sigma) = W(A)$. A similar computation gives $W(A, G_\theta(\mathbb{R})) = W(A)$.

Remark 3.15. Note that in general we do not need to have that $W_G(A) = W_H(A)$. However in a few cases we can actually show that these groups are equal. Examples are the case that H is anisotropic over k (see [HW93, 10.6]) and the case that $k = \mathbb{R}$ (see [Hel88, 6.16]). In general we can show the following:

Lemma 3.16. *Let $A_1 \supset A$ be a maximal k-split torus and $T \supset A_1$ a maximal k-torus of G such that $A_0 = T_\theta^-$ is maximal θ-split. Any element of $W_G(A)$ has a representative in $N_{G_k}(A) \cap N_{G_k}(A_1)$ or $N_G(A) \cap N_G(A_1) \cap N_G(T)$*

Proof. Let $A_1 \supset A$ be a maximal k-split torus, $W_1(A_1) = \{w \in W(A_1) \mid w(A) \subset A\}$ and $W_0(A_1) = \{w \in W(A_1) \mid w(a) = a,$ for all $a \in A\}$. Then $W(A) \simeq W_1(A_1)/W_0(A_1)$. Namely if $w \in W_G(A)$, $n \in N_{G_k}(A)$ a representative, then $A_2 = nA_1n^{-1} \subset Z_G(A)$ a maximal k-split torus. So there exists $z \in Z_{G_k}(A)$ such that $znA_1n^{-1}z^{-1} = A_1$, hence $zn \in N_{G_k}(A) \cap N_{G_k}(A_1)$.

The second statement follows with a similar argument. ☐

4. Group actions on root data

In 2.5 we saw that there is a natural fine structure of a restricted root system associated with a symmetric k-variety (or equivalently a k-involution θ) coming from a maximal (θ, k)-split torus. The underlying symmetric variety and the semisimple k-group have a similar fine structure of restricted root systems. In those cases the restricted root systems are related respectively to maximal θ-split and maximal k-split tori. The restricted root system of a maximal (θ, k)-split torus can be obtained as restrictions from either of these. All three of these restricted root systems can be obtained by group actions on the underlying root data. In this section we study these group actions on these root data and the relation between all the restricted root systems involved.

4.1. **Root Data.** To deal with the notion of root system in reductive groups it is quite useful to work with the notion of root datum. First we review a few facts about root data. These results can be found in [Spr79, §1].

4.1.1. A *root datum* is a quadruple $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$, where X and $X^\vee$ are free abelian groups of finite rank, in duality by a pairing $X \times X^\vee \to \mathbb{Z}$, denoted by $\langle \cdot\,,\ \cdot \rangle$, Φ and $\Phi^\vee$ are finite subsets of X and $X^\vee$ with a bijection $\alpha \to \alpha^\vee$ of Φ onto $\Phi^\vee$. If $\alpha \in \Phi$ we define endomorphisms s_α and $s_{\alpha^\vee}$ of X and $X^\vee$, respectively, by

$$s_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha,\ s_{\alpha^\vee}(\lambda) = \lambda - \langle \alpha, \lambda \rangle \alpha^\vee. \tag{4.1.1}$$

The following two axioms are imposed:

(1) If $\alpha \in \Phi$, then $\langle \alpha, \alpha^\vee \rangle = 2$;
(2) if $\alpha \in \Phi$, then $s_\alpha(\Phi) \subset \Phi$, $s_{\alpha^\vee}(\Phi^\vee) \subset \Phi^\vee$.

It follows from (4.1.1), that $s_\alpha^2 = 1$, $s_\alpha(\alpha) = -\alpha$ and similarly for $s_\alpha^\vee$. Put $E = X \otimes_\mathbb{Z} \mathbb{R}$. For a subset Ω of X we denote the subgroup of X generated by Ω by $\Omega_\mathbb{Z}$ and write $\Omega_\mathbb{Q} := \Omega_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$ and $\Omega_\mathbb{R} := \Omega_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}$. We consider $\Omega_\mathbb{Q}$ and $\Omega_\mathbb{R}$ as linear subspaces of E. Let $Q := \Phi_\mathbb{Z}$ be the subgroup of X generated by Φ and put $V = \Phi_\mathbb{R} = Q \otimes_\mathbb{Z} \mathbb{R}$. We consider V as a linear subspace of E. Define similarly the subgroup $Q^\vee$ of $X^\vee$ and the vector space $V^\vee$. If $\Phi \neq \emptyset$, then Φ is a not necessarily reduced root system in V in the sense of Bourbaki [Bou81, Ch.VI, no. 1]. The rank of Φ is by definition the dimension of V. The root datum Ψ is called semisimple if $X \subset V$. We observe that $s_{\alpha^\vee} = {}^t s_\alpha$ and $s_\alpha(\beta)^\vee = s_{\alpha^\vee}(\beta^\vee)$ as follows by an easy computation (c.f. Springer [Spr79, 1.4]). Let $(\cdot\,,\ \cdot)$ be a positive definite symmetric bilinear form on E, which is $\mathrm{Aut}(\Phi)$ invariant. Now the $s_\alpha (\alpha \in \Phi)$ are Euclidean reflections, so we have

$$\langle \chi, \alpha^\vee \rangle = 2(\alpha, \alpha)^{-1}.(\chi, \alpha) \qquad (\chi \in E, \alpha \in \Phi).$$

Consequently, we can identify $\Phi^\vee$ with the set $\{2(\alpha, \alpha)^{-1}\alpha \mid \alpha \in \Phi\}$ and $\alpha^\vee$ with $2(\alpha, \alpha)^{-1}\alpha$. If $\phi \in \mathrm{Aut}(X, \Phi)$, then its transpose ${}^t\phi$ induces an automorphism of $\Phi^\vee$, so Φ induces a unique automorphism in $\mathrm{Aut}(\Psi)$, the set of automorphisms of the root datum Ψ. We shall frequently identify $\mathrm{Aut}(X, \Phi)$ and $\mathrm{Aut}(\Psi)$.

For any closed subsystem Φ_1 of Φ let $W(\Phi_1)$ denote the finite group generated by the s_α for $\alpha \in \Phi_1$.

Example 4.2. If T is a torus in a reductive group G, such that $\Phi(T)$ is a root system with Weyl group $W(T)$, then the root datum associated to the pair (G, T) is: $(X^*(T), \Phi(T), X_*(T), \Phi^\vee(T))$, where $X^*(T)$, $\Phi(T)$, $X_*(T)$ and $\Phi^\vee$(T) are as defined in 2.1. So in each of the cases that T is either a maximal torus of G, a maximal k-split torus of G, a maximal θ-split torus of G or a maximal (θ, k)-split torus of G, then the above root datum exists.

Remark 4.3. If T_1 and T_2 are tori and ϕ is a homomorphism of T_1 into T_2, then the mapping ${}^t\phi$ of $X^*(T_2)$ into $X^*(T_1)$, defined by

$$ {}^t\phi(\chi_2) = \chi_2 \circ \phi, \quad \chi_2 \in X^*(T_2) \tag{4.3.1}$$

is a module homomorphism. If ϕ is an isomorphism, then ${}^t\phi^{-1}$ is a module isomorphism from $(X^*(T_1), \Phi(T_1))$ onto $(X^*(T_2), \Phi(T_2))$.

4.4. **Actions on root data.** In the study of k-involutions one has to combine the k-structure of the group with the structure of the involution. For this one has to combine the actions on the related root data. This can be seen as follows. Let G be a reductive k-group, T a maximal k-torus of G, $X = X^*(T)$, $\Phi = \Phi(T)$, K a finite Galois extension of k which splits T and $\Gamma = \mathrm{Gal}(K/k)$ the Galois group of K/k. If $\phi \in \mathrm{Aut}(G, T)$ is defined over k, then $\phi^\star := {}^t(\phi|T)^{-1}$ satisfies $\phi^{\star\sigma} = \phi^\star$, i.e.

$$\sigma\phi^\star = \phi^\star\sigma \text{ for all } \sigma \in \Gamma. \tag{4.4.1}$$

If $\theta \in \mathrm{Aut}(G, T)$ is a k-involution, then we will also write θ for $\theta^\star := {}^t(\theta|T)^{-1} \in \mathrm{Aut}(X, \Phi)$. Both Γ and θ act on (X, Φ). Let $\mathcal{E}_\theta = \{1, -\theta\} \subset \mathrm{Aut}(X, \Phi)$ be the subgroup spanned by $-\theta|T$. Let $\mathcal{E}_\Gamma \subset \mathrm{Aut}(X, \Phi)$ be the subgroup corresponding to the action of Γ on (X, Φ) and let $\Gamma_\theta = \mathcal{E}_\Gamma.\mathcal{E}_\theta$ be the subgroup of $\mathrm{Aut}(X, \Phi)$ generated by $\mathcal{E}_\Gamma$ and $\mathcal{E}_\theta$. By (4.4.1) Γ_θ is a finite subgroup of $\mathrm{Aut}(X, \Phi)$. The actions of Γ, θ, resp. Γ_θ on (X, Φ) all lead to natural restricted root systems and as it turns out these are precisely the restricted root systems related to a maximal k-split, θ-split resp. (θ, k)-split torus. Since all three these actions on the root datum can be described in a similar manner we will consider in the remainder of this section the action of a finite group $\mathcal{E}$ on (X, Φ).

4.5. Let Ψ be a root datum with $\Phi \neq \emptyset$, as in 4.1.1 and let $\mathcal{E}$ be a finite group acting on Ψ. For $\sigma \in \mathcal{E}$ and $\chi \in X$ we will also write χ^σ or $\sigma(\chi)$ for the element $\sigma.\chi \in X$. Write $W = W(\Phi)$ for the Weyl group of Φ. Now define the following:

$$X_0 = X_0(\mathcal{E}) = \left\{ \chi \in X \mid \sum_{\sigma \in \mathcal{E}} \chi^\sigma = 0 \right\} \tag{4.5.1}$$

Then X_0 is a co-torsion free submodule of X, invariant under the action of $\mathcal{E}$. Let $\Phi_0 = \Phi_0(\mathcal{E}) = \Phi \cap X_0$. This is a closed subsystem of Φ invariant under the action of $\mathcal{E}$. Denote the Weyl group of Φ_0 by W_0 and identify it with the subgroup of $W(\Phi)$ generated by the reflections $s_\alpha, \alpha \in \Phi_0$. Put $W^{\mathcal{E}} = \{w \in W \mid w(X_0) = X_0\}$, $\bar{X}_{\mathcal{E}} = X/X_0(\mathcal{E})$ and let π be the natural projection from X to $\bar{X}_{\mathcal{E}}$. If we take $A = \{t \in T \mid \chi(t) = e \text{ for all } \chi \in X_0\}$ to be the annihilator of X_0 and $Y = X^*(A)$, then Y may be identified with $\bar{X}_{\mathcal{E}} = X/X_0$. Let $\bar{\Phi}_{\mathcal{E}} = \pi(\Phi - \Phi_0(\mathcal{E}))$ denote the set of *restricted roots of* Φ *relative to* $\mathcal{E}$.

Remark 4.6. In the case that $\mathcal{E} = \Gamma$, then X_0 is the annihilator of a maximal k-split torus A of T. Similarly in the case that $\mathcal{E} = \mathcal{E}_\theta$, then X_0 is the annihilator of a maximal θ-split torus A of G. In both these cases if A is maximal k-split resp. θ-split in G then $\bar{\Phi}_{\mathcal{E}}$ is the root system of $\Phi(A)$ with Weyl group $\bar{W}_{\mathcal{E}}$.

We define now an order on (X, Φ) related to the action of $\mathcal{E}$ as follows.

Definition 4.7. A linear order on X which satisfies

$$\text{if } \chi \succ 0 \text{ and } \chi \notin X_0, \text{ then } \chi^\sigma \succ 0 \text{ for all } \sigma \in \mathcal{E} \tag{4.7.1}$$

is called a $\mathcal{E}$*-linear order*. A fundamental system of Φ with respect to a $\mathcal{E}$-linear order is called a $\mathcal{E}$*-fundamental system of* Φ or a $\mathcal{E}$*-basis of* Φ.

A $\mathcal{E}$-linear order on X induces linear orders on $Y = X/X_0$ and X_0, and conversely, given linear orders on X_0 and on Y, these uniquely determine a $\mathcal{E}$-linear order on X, which induces the given linear orders (i.e., if $\chi \notin X_0$, then define $\chi \succ 0$ if and only if $\pi(\chi) \succ 0$). Instead of the above $\mathcal{E}$-linear order one can give a more general definition of a linear order on X, using only the fact that X_0 is a co-torsion free submodule of X (see [Sat71, §2.1]).

In the following we give a number of properties of an $\mathcal{E}$-linear order on X.

4.8. **Restricted fundamental system.** Fix a $\mathcal{E}$-linear order $\succ$ on X, let Δ be a $\mathcal{E}$-fundamental system of Φ and let Δ_0 be a fundamental system of Φ_0 with respect to the induced order on X_0. Let $A = \{t \in T \mid \chi(t) = e \text{ for all } \chi \in X_0\}$ be the annihilator of X_0 and define $\bar{\Delta}_{\mathcal{E}} = \pi(\Delta - \Delta_0)$. This is called a *restricted fundamental system* of Φ relative to A or also a *restricted fundamental system* of $\bar{\Phi}_{\mathcal{E}}$. The following proposition lists some properties of these fundamental systems.

Proposition 4.9. *Let X, X_0, Φ, Φ_0, $\bar{\Phi}_{\mathcal{E}}$, etc. be defined as above and let Δ, Δ' be $\mathcal{E}$-fundamental systems of Φ. Then we have the following*

(1) $\Delta_0 = \Delta \cap \Phi_0$.
(2) $\Delta = \Delta'$ *if and only if* $\Delta_0 = \Delta'_0$ *and* $\bar{\Delta}_{\mathcal{E}} = \bar{\Delta}'_{\mathcal{E}}$.
(3) *If* $\bar{\Delta}_{\mathcal{E}} = \bar{\Delta}'_{\mathcal{E}}$, *then there exists a unique* $w_0 \in W_0$ *such that* $\Delta' = w_0\Delta$.

Proof. (1). Assume rank $\Phi = n$, $\Delta = \{\alpha_1, \dots, \alpha_n\}$ and $\Delta_0 = \{\alpha_1, \dots, \alpha_m\}$, $m \le n$. It suffices to show that each $\alpha \in \Phi_0$ is a linear combination of the α_i's in Δ_0. Write $\alpha = \sum_{i=1}^{n} r_i\alpha_i$, $r_i \in \mathbb{Z}$. We may assume $\alpha \succ 0$, i.e. $r_i \ge 0$. Since $\alpha \in \Phi_0$ we have $\sum_{\sigma\in\mathcal{E}} \alpha^\sigma = 0$. Since $\alpha_1, \dots, \alpha_m \in \Delta_0$ we get: $\sum_{\sigma\in\mathcal{E}} \alpha^\sigma = \sum_{\sigma\in\mathcal{E}} (r_{m+1}\alpha_{m+1} + \dots r_n\alpha_n)^\sigma$. By the definition of $\mathcal{E}$-linear order $\alpha_j^\sigma \succ 0$ for $m+1 \le j \le n$ and $\sigma \in \mathcal{E}$. So if any of the $r_j \ne 0$, $m+1 \le j \le n$, then $\sum_{\sigma\in\mathcal{E}} \alpha^\sigma \succ 0$, what contradicts the fact that $\alpha \in \Phi_0$.

(2). It suffices to show $\Longleftarrow$. Let $\succ$ be the $\mathcal{E}$-linear order defining Δ and $\succ'$ the $\mathcal{E}$-linear order defining Δ'. Let $\Phi^+ = \{\alpha \in \Phi \mid \alpha \succ 0\}$ and $\Phi^+_{\succ'} = \{\alpha \in \Phi \mid \alpha \succ' 0\}$. We will show that $\Phi^+ = \Phi^+_{\succ'}$, what implies the result. Let $\alpha \in \Delta$. If $\alpha \in \Delta_0 = \Delta'_0$, then $\alpha \succ' 0$. If $\alpha \notin \Delta_0$, then $\pi(\alpha) \in \bar{\Delta} = \bar{\Delta}'$, hence also $\alpha \succ' 0$. Since Δ determines Φ^+, it follows that $\Phi^+ \subset \Phi^+_{\succ'}$. The same argument shows $\Phi^+_{\succ'} \subset \Phi^+$, hence $\Phi^+ = \Phi^+_{\succ'}$.

(3). Since Δ_0 and Δ'_0 are fundamental systems of Φ_0, there exists a unique $w_0 \in W_0$ such that $w_0\Delta_0 = \Delta'_0(\Gamma)$. But then $w_0\Delta \cap \Phi_0 = \Delta'_0(\Gamma)$ and $\pi(w_0\Delta) = \bar{\Delta}_{\mathcal{E}} = \bar{\Delta}'_{\mathcal{E}}$. So by (2) $\Delta' = w_0\Delta$. □

4.10. **Restricted Weyl group.** There is a natural (Weyl) group associated with the set of restricted roots, which is related to $W^{\mathcal{E}}/W_0$. Since W_0 is a normal subgroup of $W^{\mathcal{E}}$, every $w \in W^{\mathcal{E}}$ induces an automorphism of $\bar{X}_{\mathcal{E}} = X/X_0 = Y$. Denote the induced automorphism by $\pi(w)$. Then $\pi(w\chi) = \pi(w)\pi(\chi)$ ($\chi \in X$). Define $\bar{W}_{\mathcal{E}} = \{\pi(w) \mid w \in W^{\mathcal{E}}\}$. We call this the *restricted Weyl group*, with respect to the action of $\mathcal{E}$ on X. It is not necessarily a Weyl group in the sense of Bourbaki [Bou81, Ch.VI,no.1]. However we can show the following.

Proposition 4.11. *Let X, X_0, Φ, Φ_0, $\bar{\Phi}_{\mathcal{E}}$, Δ, Δ_0, $\bar{\Delta}_{\mathcal{E}}$, W_0, $W^{\mathcal{E}}$, $\bar{W}_{\mathcal{E}}$ be defined as above and let A be the annihilator of X_0. Then we have the following:*

(1) *If* $w \in W^{\mathcal{E}}$, *then* $w(\Delta)$ *is an* $\mathcal{E}$*-fundamental system.*
(2) *Let* $w \in W^{\mathcal{E}}$. *Then* $w \in W_0$ *iff* $\pi(w) = 1$ *iff* $\pi(w)\bar{\Delta}_{\mathcal{E}} = \bar{\Delta}_{\mathcal{E}}$.
(3) $\bar{W}_{\mathcal{E}} \cong W^{\mathcal{E}}/W_0$.

(4) *$W^{\mathcal{E}}/W_0 \cong N_G(A)/Z_G(A)$, where $N_G(A)$ and $Z_G(A)$ are, respectively, the normalizer and centralizer of A in G.*

Proof. (1). For $w \in W^{\mathcal{E}}$ define an order $\succ_w$ on X as follows:

$$\text{if } \chi \in X \text{ and } \chi \notin X_0, \text{ then } \chi \succ_w 0 \text{ if and only if } w(\chi) \succ 0.$$

Since $w(X_0) = X_0$ the order $\succ_w$ is an $\mathcal{E}$-linear order on X and $w(\Delta)$ is an $\mathcal{E}$-fundamental system of Φ with respect to this order.

(2). If $w \in W_0$, then from the definition of $\pi(w)$ it follows that $\pi(w) = 1$, which implies that $\pi(w)\bar{\Delta}_{\mathcal{E}} = \bar{\Delta}_{\mathcal{E}}$. So it suffices to show that the latter condition implies that $w \in W_0$. Since $w(\Delta)$ and Δ are both $\mathcal{E}$-fundamental systems it follows from Proposition 4.9(3) that there exists $w_0 \in W_0$ such that $w_0 w(\Delta) = \Delta$, what implies that $w = w_0^{-1} \in W_0$.

(3) is immediate from (1) and (2).

(4). Let $n \in N_G(T)$ and $w \in W(T)$ the corresponding Weyl group element. Then $w(X_0) = X_0$ if and only if $n \in N_G(A)$. It follows that $w \in W^{\mathcal{E}}$ if and only if $n \in N_G(A)$. By (2) $w \in W_0$ if and only if $\pi(w) = 1$. This is true if and only if $n \in Z_G(A)$. Since $N_G(A) = (N_G(A) \cap N_G(T)) \cdot Z_G(A)$ the result follows. □

Remarks 4.12. (1) In the case that A is a maximal k-split, θ-split or (θ, k)-split torus, then $\bar{\Phi}_{\mathcal{E}}$ is actually a root system with Weyl group $\bar{W}_{\mathcal{E}}$. The general question when $\bar{\Phi}_{\mathcal{E}}$ is a root system in $Y = X/X_0$ was studied in [Sch69].

(2) In the remainder of this section we will also write $\bar{\Phi}$, $\bar{\Delta}$, $\bar{W}$ instead of $\bar{\Phi}_{\mathcal{E}}$, $\bar{\Delta}_{\mathcal{E}}$, $\bar{W}_{\mathcal{E}}$ whenever it causes no confusion.

4.13. **Action of $\mathcal{E}$ on Δ.** From Proposition 4.11 it follows that $W^{\mathcal{E}}$ acts on the set of $\mathcal{E}$-fundamental systems of Φ. There is also a natural action of $\mathcal{E}$ on this set. If Δ is a $\mathcal{E}$-fundamental system of Φ, and $\sigma \in \mathcal{E}$, then the $\mathcal{E}$-fundamental system $\Delta^\sigma = \{\alpha^\sigma \mid \alpha \in \Delta\}$ gives the same restricted basis as Δ, i.e. $\bar{\Delta}^\sigma = \bar{\Delta}$. This follows from the fact that $\alpha_i \equiv \alpha_i^\sigma \mod X_0)$ for all $\alpha_i \in \Delta$, $\sigma \in \mathcal{E}$. From Proposition 4.9 it follows that there is a unique element $w_\sigma \in W_0$ such that $\Delta^\sigma = w_\sigma \Delta$. This means we can define a new operation of $\mathcal{E}$ on X as follows:

$$\chi^{[\sigma]} = w_\sigma^{-1} \chi^\sigma, \quad \chi \in X, \quad \sigma \in \mathcal{E}. \tag{4.13.1}$$

It is easily verified that $\chi \to \chi^{[\sigma]}$ is an automorphism of the triple (X, Φ, Δ) and that $\chi^{[\sigma][\gamma]} = \chi^{[\sigma\gamma]}$ for all $\sigma, \gamma \in \mathcal{E}$, $\chi \in X$.

In the following we prove some properties of the action of $\mathcal{E}$ on Δ which will be needed lateron. We will assume X_0 is as defined in (4.5.1) and $\succ$ is an $\mathcal{E}$ order on X.

Lemma 4.14. *Let $\lambda_j \in \bar{\Delta}$ and $\alpha_i \in \Delta$ such that $\pi(\alpha_i) = \lambda_j$. If $\sigma \in \mathcal{E}$, then we have the following:*

(1) $\alpha_i^\sigma = \alpha_p + \sum_{\alpha_r \in \Delta_0} c_{i,r}(\sigma)\alpha_r$ *for some* $\alpha_p \in \pi^{-1}(\lambda_j),\ c_{i,r}(\sigma) \in \mathbb{Z}$.

(2) $\alpha_i^{[\sigma]} = \alpha_p + \sum_{\alpha_r \in \Delta_0} b_{i,r}(\sigma)\alpha_r$ *for some* $\alpha_p \in \pi^{-1}(\lambda_j),\ b_{i,r}(\sigma) \in \mathbb{Z}$.

Proof. Let $\operatorname{rank}(\Phi) = n$. Write $\alpha_i^\sigma = \sum_{r=1}^n c_{i,r}(\sigma)\alpha_r$, where $c_{i,r}(\sigma) \in \mathbb{Z}$. Since $\alpha_i \in \Delta$ and Δ is a $\mathcal{E}$-fundamental system of Φ we may assume that $c_{i,r}(\sigma) \geq 0$ if $\alpha_i \notin \Delta_0$, and $c_{i,r}(\sigma) = 0$ if $\alpha_i \in \Delta_0$ and $\alpha_r \notin \Delta_0$. Reorder the fundamental roots, if necessary, so that $\Delta - \Delta_0 = \{\alpha_1, \dots, \alpha_m\}$ and $\Delta_0 = \{\alpha_{m+1}, \dots, \alpha_n\}$. Then the matrices $(c_{ij}(\sigma))_{1 \leq i,j \leq n}$ are integral, and of the form $\begin{pmatrix} A_\sigma & B_\sigma \\ 0 & D_\sigma \end{pmatrix}$, where all entries of A_σ and B_σ are ≥ 0. Since the product of the matrices $(c_{ij}(\sigma))$ and $(c_{ij}(\sigma^{-1}))$ is the identity matrix, it follows that A_σ is necessarily a permutation matrix, hence if $\alpha_i \notin \Delta_0,\ \alpha_i^\sigma = \alpha_p + \sum_{\alpha_r \in \Delta_0} c_{i,r}(\sigma)\alpha_r$. Since $\pi(\alpha_i) = \pi(\alpha_i^\sigma) = \lambda_j$ it follows that $\alpha_p \in \pi^{-1}(\lambda_j)$.

(2). For $\sigma \in \mathcal{E}$ let $w_\sigma \in W_0$ such that $\alpha_i^{[\sigma]} = w_\sigma^{-1}\alpha_i^\sigma$. Let $c_{i,r}(\sigma) \in \mathbb{Z}$ and $\alpha_p \in \pi^{-1}(\lambda_j)$ such that $\alpha_i^\sigma = \alpha_p + \sum_{\alpha_r \in \Delta_0} c_{i,r}(\sigma)\alpha_r$. Then

$$\alpha_i^{[\sigma]} = w_\sigma^{-1}(\alpha_p + \sum_{\alpha_r \in \Delta_0} c_{i,r}(\sigma)\alpha_r) = w_\sigma^{-1}(\alpha_p) + w_\sigma^{-1}(\sum_{\alpha_r \in \Delta_0} c_{i,r}(\sigma)\alpha_r).$$

Since $w_\sigma^{-1} \in W_0$ it follows that $w_\sigma^{-1}(\sum_{\alpha_r \in \Delta_0} c_{i,r}(\sigma)\alpha_r) = \sum_{\alpha_r \in \Delta_0} d_{i,r}(\sigma)\alpha_r$ for some $d_{i,r}(\sigma) \in \mathbb{Z}$. Similarly $w_\sigma^{-1}(\alpha_p) = \alpha_p + \sum_{\alpha_r \in \Delta_0} e_{i,r}(\sigma)\alpha_r$ for some $e_{i,r}(\sigma) \in \mathbb{Z}$. Let $b_{i,r}(\sigma) = d_{i,r}(\sigma) + e_{i,r}(\sigma)$. Then $\alpha_i^{[\sigma]} = \alpha_p + \sum_{\alpha_r \in \Delta_0} b_{i,r}(\sigma)\alpha_r$. □

Lemma 4.15. *Let $\Omega = \Delta_0(\mathcal{E}) \cup \{\alpha^{[\sigma]} - \alpha \mid \alpha \in \Delta - \Delta_0(\mathcal{E})$ and $\alpha^{[\sigma]} \neq \alpha\}$. Then $X_0(\mathcal{E})_\mathbb{Q} = \Omega_\mathbb{Q}$ and $\operatorname{card}\Omega = \operatorname{rank} X_0(\mathcal{E})$.*

Proof. Clearly Ω is a linear independent set and $\operatorname{rank} X_0(\mathcal{E}) \geq \operatorname{card}\Omega$. So it suffices to show that Ω generates $X_0(\mathcal{E})$. From the definition of $X_0(\mathcal{E})$ and $X^{\mathcal{E}}(\mathcal{E})$ it is clear that $X_0(\mathcal{E})_\mathbb{Q}$ is generated over $\mathbb{Q}$ by the set $\{\alpha^\sigma - \alpha \mid \sigma \in \Gamma, \alpha \in \Delta\}$. If $\alpha \in \Delta_0(\mathcal{E})$, then $\alpha^\sigma \in \Phi \cap X_0(\mathcal{E}) = \Phi_0(\mathcal{E})$. Since $\Delta_0(\mathcal{E})$ is a fundamental system of $\Phi_0(\mathcal{E})$ it follows that $\alpha^\sigma - \alpha \in \Delta_0(\mathcal{E})_\mathbb{Z} \subset \Omega_\mathbb{Z}$. If $\alpha \in \Delta - \Delta_0(\mathcal{E})$, then for all $\sigma \in \Gamma$ we have $\pi(\alpha) = \pi(\alpha^\sigma) = \lambda$ for some $\lambda \in \bar{\Delta}_\mathcal{E}$. By Lemma 4.14 we get $\alpha^{[\sigma]} \in \gamma^{-1}(\lambda)$ and $\alpha^\sigma = \alpha^{[\sigma]} + \gamma$ for some $\gamma \in \Delta_0(\mathcal{E})_\mathbb{Z}$. But then $\alpha^\sigma - \alpha = \alpha^{[\sigma]} - \alpha + \gamma \in \Omega_\mathbb{Z}$. □

Corollary 4.16. *Let $X,\ X_0(\mathcal{E}),\ \Phi,\ \Phi_0(\mathcal{E}),\ \bar{\Phi}_\mathcal{E},\ \Delta,\ \Delta_0$ be defined as above and let $\bar{\Delta}_\mathcal{E} = \{\lambda_1, \dots, \lambda_r\}$ be a restricted fundamental system of $\bar{\Phi}_\mathcal{E}$, with the (λ_i mutually distinct). Then $\lambda_1, \dots, \lambda_r$ are linearly independent.*

Proof. Since Δ spans X it follows that $\bar{\Delta}_{\mathcal{E}}$ spans $\bar{X}_{\mathcal{E}}$, so rank $\bar{X}_{\mathcal{E}} \le r$. But since rank $X =$ rank $X_0(\mathcal{E}) +$ rank $\bar{X}_{\mathcal{E}}$ it follows from Lemma 4.15 that rank $\bar{X}_{\mathcal{E}} = r$, hence $\lambda_1, \dots, \lambda_r$ are linearly independent. □

The diagram automorphism $[\sigma]$ relates the simple roots in Δ, which are lying above a restricted root in $\bar{\Delta}_{\mathcal{E}}$:

Lemma 4.17. *Let Δ be a (Γ, θ)-basis of Φ and $\alpha, \beta \in \Delta$, $\alpha \neq \beta$ such that $\pi(\alpha) = \pi(\beta) \neq 0$. Then there is a $\sigma \in \mathcal{E}$ such that $\beta = \alpha^{[\sigma]}$.*

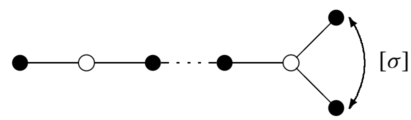
Proof. For each $\sigma \in \mathcal{E}$ let $w_\sigma \in W_0$ such that $[\sigma] = w_\sigma^{-1}\sigma$. Since $\pi(\alpha) = \pi(\beta) \neq 0$ we have $\alpha \equiv \beta \mod X_0(\mathcal{E})$. But then $\sum_{\sigma\in\mathcal{E}} \alpha^\sigma = \sum_{\sigma\in\mathcal{E}} \beta^\sigma$. On the other hand $\sum_{\sigma\in\mathcal{E}} \alpha^\sigma = \sum_{\sigma\in\mathcal{E}} w_\sigma \alpha^{[\sigma]} = \sum_{\sigma\in\mathcal{E}} \alpha^{[\sigma]} + \delta_1$ with $\delta_1 \in \mathrm{Span}(\Delta_0(\mathcal{E}))$. Similarly $\sum_{\sigma\in\mathcal{E}} \beta^\sigma = \sum_{\sigma\in\mathcal{E}} \beta^{[\sigma]} + \delta_2$ with $\delta_2 \in \mathrm{Span}(\Delta_0(\mathcal{E}))$. But then we have $\sum_{\sigma\in\mathcal{E}} (\alpha^{[\sigma]} - \beta^{[\sigma]}) = \delta_1 - \delta_2$. It follows that $\delta_1 = \delta_2$ and $\beta = \alpha^{[\sigma]}$ for some $\sigma \in \mathcal{E}$. □

5. (Γ, θ)-indices

The actions of Γ and θ on the root datum can be described by an index. These indices not only determine the fine structure of restricted root systems with multiplicities etc. of the corresponding k-group and symmetric variety, but also play an important role in the classifications of k-groups and symmetric varieties (or equivalently involutions of reductive groups). In this section we extend these indices to get an index which describes the action of a k-involution. Similar as for k-groups and symmetric varieties this index describes the fine structure of restricted root systems with multiplicities etc. of the corresponding symmetric k-variety, but also plays again an important role in the classification of k-involutions in section 8.

5.1. **The index of $\mathcal{E}$.** Throughout this section let Ψ be a semisimple root datum with $\Phi \neq \emptyset$, as in (4.1.1), $\mathcal{E}$ a (finite) group acting on Ψ, like in 4.5, Δ a $\mathcal{E}$-basis of Φ and $\Delta_0 = \Delta_0(\mathcal{E}) = \Delta \cap X_0(\mathcal{E})$. In equation (4.13.1) we defined an action of $\mathcal{E}$ on Δ, which we denote by $[\sigma]$. The action of $\mathcal{E}$ on Ψ is essentially determined by Δ, Δ_0 and $[\sigma]$. Following Tits [Tit66] we will call the quadruple $(X, \Delta, \Delta_0, [\sigma])$ an *index of* $\mathcal{E}$ or an $\mathcal{E}$-index. We will also use the name $\mathcal{E}$-diagram, following the notation in Satake [Sat71, 2.4].

5.2. As in [Tit66] we make a diagrammatic representation of the index of $\mathcal{E}$ by coloring black those vertices of the ordinary Dynkin diagram of Φ, which represent roots in $\Delta_0(\mathcal{E})$ and indicating the action of $[\sigma]$ on Δ by arrows. An example in type D_l is:



To use these $\mathcal{E}$-indices in the characterization of isomorphy classes of reductive k-groups or involutions, we need a notion of isomorphism between these indices.

Definition 5.3. Let Ψ and Ψ' be semisimple root data and $\mathcal{E}$ a group acting on them. A *congruence* φ of the $\mathcal{E}$-index $(X, \Delta, \Delta_0, [\sigma])$ of Ψ onto the $\mathcal{E}$-index $(X', \Delta', \Delta_0', [\sigma]')$ of Ψ' is an isomorphism which maps $(X, \Delta, \Delta_0) \to (X', \Delta', \Delta_0')$, and satisfies $[\sigma]' = \varphi[\sigma]\varphi^{-1}$.

For k-involutions it suffices to consider two actions of $\mathcal{E}$ on the same root datum. In that case we will also use the term *isomorphic* $\mathcal{E}$-indices instead of congruent $\mathcal{E}$-indices. In this case one can differentiate between inner and outer automorphisms.

Definition 5.4. Let Ψ be a root datum and $\mathcal{E}_1,\ \mathcal{E}_2 \subset \mathrm{Aut}(\Psi)$ the subgroups of $\mathrm{Aut}(\Psi)$ corresponding to actions of $\mathcal{E}$ on Ψ. Two indices $(X, \Delta, \Delta_0(\mathcal{E}_1), [\sigma]_1)$ and $(X, \Delta', \Delta_0'(\mathcal{E}_2), [\sigma]_2)$ are said to be $W(\Phi)$- (resp. $\mathrm{Aut}(\Phi)$)-*isomorphic* if there is a $w \in W(\Phi)$ (resp. $w \in \mathrm{Aut}(\Phi)$), which maps $(\Delta, \Delta_0(\mathcal{E}_1))$ onto $(\Delta', \Delta_0'(\mathcal{E}_2))$ and satisfies $w[\sigma]_1 w^{-1} = [\sigma]_2$. Instead of $W(\Phi)$-isomorphic we will also use the term *isomorphic*.

Remark 5.5. An index of $\mathcal{E}$ may depend on the choice of the $\mathcal{E}$-basis of Φ, i.e. for two $\mathcal{E}$-bases Δ, Δ', the corresponding indices $(X, \Delta, \Delta_0(\mathcal{E}), [\sigma])$ and $(X, \Delta', \Delta_0'(\mathcal{E}), [\sigma]')$ need not be isomorphic. However this cannot happen if $\bar{\Phi}_{\mathcal{E}}$ is a root system with Weyl group $\bar{W}_{\mathcal{E}}$:

Proposition 5.6. *Let Ψ be a semisimple root datum and $\mathcal{E} \subset \mathrm{Aut}(\Psi)$ a group acting on Ψ such that $\bar{\Phi}_{\mathcal{E}}$ is a root system with Weyl group $\bar{W}_{\mathcal{E}}$. If Δ, Δ' are $\mathcal{E}$-bases of Φ, then $(X, \Delta, \Delta_0(\mathcal{E}), [\sigma])$ and $(X, \Delta', \Delta_0'(\mathcal{E}), [\sigma]')$ are isomorphic.*

Proof. Let $\bar{\Delta}_{\mathcal{E}}$ and $\bar{\Delta}'_{\mathcal{E}}$ be restricted fundamental systems of $\bar{\Phi}_{\mathcal{E}}$ induced by Δ and Δ' and let $\bar{w} \in \bar{W}_{\mathcal{E}}$ such that $\bar{w}(\bar{\Delta}'_{\mathcal{E}}) = \bar{\Delta}_{\mathcal{E}}$. Since by Proposition 4.11(3) $\bar{W}_{\mathcal{E}} = W^{\mathcal{E}}/W_0$ there exists $w_1 \in W^{\mathcal{E}}$ such that $\pi(w_1) = \bar{w}$. By Proposition 4.11(1) $w_1(\Delta') \cap \Phi_0$ is a basis of Φ_0, hence there exists $w_0 \in W_0$ such that

$w_0w_1(\Delta') \cap \Phi_0 = \Delta_0(\mathcal{E})$. Let $w = w_0w_1$. Then from Proposition 4.11(2) it follows that $w(\Delta') = \Delta$ and $w(\Delta'_0(\mathcal{E})) = \Delta_0(\mathcal{E})$.

It remains to show that w satisfies $[\sigma] = w[\sigma]'w^{-1}$. Let $\sigma \in \mathcal{E}$ and w_σ, $w'_\sigma \in W_0$ such that $\sigma(\Delta) = w_\sigma(\Delta)$ and $\sigma(\Delta') = w'_\sigma(\Delta')$. Then $[\sigma] = w_\sigma^{-1}\sigma$ and $[\sigma]' = (w'_\sigma)^{-1}\sigma$. Now

$$\begin{aligned} w_\sigma(\Delta) &= w_\sigma w(\Delta') = \sigma(\Delta) = \sigma w(\Delta') \\ &= \sigma w \sigma^{-1}\sigma(\Delta') = \sigma w\sigma^{-1}w'_\sigma(\Delta'). \end{aligned} \tag{5.6.1}$$

It follows that $w_\sigma w(\Delta') = \sigma w\sigma^{-1}w'_\sigma(\Delta')$, hence $\sigma w^{-1}\sigma^{-1}w_\sigma w(\Delta') = w'_\sigma(\Delta')$. Since both $\sigma w^{-1}\sigma^{-1}w_\sigma w$ and $w'_\sigma \in W$ it follows from (5.6.1) that

$$\sigma w^{-1}\sigma^{-1}w_\sigma w = w'_\sigma. \tag{5.6.2}$$

Now if $\chi \in X$, then

$$\begin{aligned} w[\sigma]'w^{-1}(\chi) &= w(w'_\sigma)^{-1}\sigma w^{-1}(\chi) = ww^{-1}w_\sigma^{-1}\sigma w\sigma^{-1}\sigma w^{-1}(\chi) \\ &= w_\sigma^{-1}\sigma(\chi) = [\sigma](\chi), \end{aligned} \tag{5.6.3}$$

what proves the result. □

Remark 5.7. In the case that $\bar{\Phi}_{\mathcal{E}}$ is a root system with Weyl group $\bar{W}_{\mathcal{E}}$, then the restricted root system together with the multiplicities of the roots can be easily determined from the $\mathcal{E}$-index. See for example [Hel88].

For the general congruence of the $\mathcal{E}$-indices we will use the following result:

Theorem 5.8. *Let G_1, G_2 be connected semisimple groups defined over k. For $i = 1, 2$ let T_i be a maximal k-torus of G_i, $\Psi_i = (X^*(T_i), \Phi(T_i), X_*(T_i), \Phi^\vee(T_i))$ the root datum corresponding to (G_i, T_i), $\mathcal{E}$ a (finite) group acting on Ψ_i, $X_0(\mathcal{E}, T_i) = \{\chi \in X^*(T_i) \mid \sum_{\sigma\in\mathcal{E}} \chi^\sigma = 0\}$, $A_i = \{t \in T_i \mid \chi(t) = e$ for all $\chi \in X_0(\mathcal{E}, T_i)\}$ the annihilator of $X_0(\mathcal{E}, T_i)$, $\Delta(T_i)$ a $\mathcal{E}$-basis of $\Phi(T_i)$, $\Delta_0(T_i) = \Delta(T_i) \cap X_0(\mathcal{E})$ and $[\sigma]_i$ the action of $\mathcal{E}$ on $\Delta(T_i)$. If $\varphi : (G_1, T_1, A_1) \to (G_2, T_2, A_2)$ is a k-isomorphism and $\varphi^\star = {}^t(\varphi|T_1)^{-1}$ is as in* (4.3.1), *then there exists a unique $w \in W^{\mathcal{E}}(T_2)$ such that $w(\varphi^\star(\Delta(T_1))) = \Delta(T_2)$ and $\varphi^{[\star]} := w\varphi^\star$ is a congruence from $(X^*(T_1), \Delta(T_1), \Delta_0(T_1), [\sigma]_1)$ to $(X^*(T_2), \Delta(T_2), \Delta_0(T_2), [\sigma]_2)$.*

Proof. Since $\phi : (G_1, T_1, A_1) \to (G_2, T_2, A_2)$ is a k-isomorphism it follows that the induced map $\varphi^\star : (X^*(T_1), \Phi(T_1), X_0(T_1)) \to (X^*(T_2), \Phi(T_2), X_0(T_2))$ is an isomorphism as well. Since $\varphi^\star(\Phi^+(T_1))$ is a set of positive roots with respect to a $\mathcal{E}$-linear order on $\Phi(T_2)$ it follows that $\varphi^\star(\Delta(T_1))$ is a $\mathcal{E}$-basis of $\Phi(T_2)$. Since $\Phi(A_2)$ is a root system with Weyl group $W(A_2)$ it follows from Proposition 4.11 that there exists a unique $w \in W^{\mathcal{E}}(T_2)$ such that $w(\varphi^{\star}(\Delta(T_1))) = \Delta(T_2)$. From Proposition 5.6 it follows now that the $\mathcal{E}$-indices $(X^*(T_2), \Delta(T_2), \Delta_0(T_2), \phi^{\star}[\sigma]_1(\phi^{\star})^{-1})$ and $(X^*(T_2), \Delta(T_2), \Delta_0(T_2), [\sigma]_2)$ are congruent. Let $\varphi^{[\star]} := w\varphi^{\star}$. With a similar argument as in (5.6.1) and (5.6.3) it follows now that $\varphi^{[\star]}$ is a congruence of the $\mathcal{E}$-indices $(X^*(T_1), \Delta(T_1), \Delta_0(T_1), [\sigma]_1)$ and $(X^*(T_2), \Delta(T_2), \Delta_0(T_2), [\sigma]_2)$. □

Definition 5.9. If $\phi : (G_1, T_1, A_1) \to (G_2, T_2, A_2)$ is a k-isomorphism as in Theorem 5.8, then we will call the congruence $\varphi^{[\star]} := w\varphi^{\star}$ of the $\mathcal{E}$-indices $(X^*(T_1), \Delta(T_1), \Delta_0(T_1), [\sigma]_1)$ and $(X^*(T_2), \Delta(T_2), \Delta_0(T_2), [\sigma]_2)$ the *congruence associated with* φ.

In the cases of $\mathcal{E} = \mathcal{E}_\theta$ and $\mathcal{E} = \Gamma$ we get the well known θ-index and Γ-index, which are essential in the respective classifications. Since the classification of k-involutions depends on a classification of these, we will briefly review these in the next subsections. First we need still a notion of irreducibility for $\mathcal{E}$-indices.

Definition 5.10. Let $\mathcal{E} \subset \mathrm{Aut}(X, \Phi)$ be a subgroup and Δ a $\mathcal{E}$-basis of Φ. An index $\mathcal{D} = (X, \Delta, \Delta_0, [\sigma])$ is $\mathcal{E}$-*irreducible* if Δ is not the union of two mutually orthogonal $[\sigma]$-invariant (non-empty) subsystems Δ', Δ''. The system $\mathcal{D}$ is *absolutely irreducible* if Δ is connected. In the case $\mathcal{E} = \mathcal{E}_\Gamma$ (resp. $\mathcal{E}_\theta$) we will also call an $\mathcal{E}$-irreducible index an k-irreducible index (resp. θ-irreducible index).

5.11. ***θ*-index.** In this subsection we discuss the index associated with an involutorial automorphism of a reductive algebraic group. Let G be a reductive algebraic group, $\theta \in \mathrm{Aut}(G)$ an involution and T a θ-stable maximal torus of G. Write $X = X^*(T)$, $\Phi = \Phi(T)$ and let $\mathcal{E}_\theta = \{1, -\theta\} \subset \mathrm{Aut}(X, \Phi)$ be the subgroup spanned by $-\theta|T$. In this case we will also write $X_0(\theta)$, $\bar{X}_\theta$, $\Phi_0(\theta)$, $\bar{\Phi}_\theta$, $W_1(\theta)$, $\bar{W}_\theta$, $\Delta_0(\theta)$, $\bar{\Delta}_\theta$ instead of, respectively, $X_0(\mathcal{E}_\theta)$, $\bar{X}_{\mathcal{E}_\theta}$, $\Phi_0(\mathcal{E}_\theta)$, $\bar{\Phi}_{\mathcal{E}_\theta}$, $W_0(\mathcal{E}_\theta)$, $W_1(\mathcal{E}_\theta)$, $\bar{W}_{\mathcal{E}_\theta}$, $\Delta_0(\mathcal{E}_\theta)$, $\bar{\Delta}_{\mathcal{E}_\theta}$. A $\mathcal{E}_\theta$-order on X will also be called a θ-*order* on X, a $\mathcal{E}_\theta$-basis of Φ a θ-*basis* of Φ and a $\mathcal{E}_\theta$-index a θ-index.

Let Δ be a θ-basis of Φ. To find the θ-index we need to find the action of $[-\theta]$ on (X, Φ, Δ). Since $\theta(-\Delta)$ is also a θ-basis of Φ with the same restricted basis, it follows from Proposition 4.9 that there is $w_0(\theta) \in W_0(\theta)$ such that $w_0(\theta)\theta(\Delta) = -\Delta$. Put $\theta^* = \theta^*(\Delta) = -w_0(\theta)\theta$. Then $\theta^* = [-\theta]$. Note that $\theta^*(\Delta) \in \mathrm{Aut}(X, \Phi, \Delta) = \{\phi \in \mathrm{Aut}(X, \Phi) \mid \phi(\Delta) = \Delta\}$, $\theta^*(\Delta)^2 = \mathrm{id}$ and $\theta^*(\Delta_0(\theta)) = \Delta_0(\theta)$.

Remarks 5.12. (1) $\theta^* = [-\theta]$ can be described by its action on the Dynkin diagram of Δ. Notice that

(a) if Φ is irreducible, then θ^* is either the identity or a diagram automorphism of order 2. The latter happens only if Φ is either of type $A_l(l(\geq 2)$, $D_{2l+1}(l(\geq 2)$ or E_6.

(b) if $\Phi = \Phi_1 \cup \Phi_2$ with Φ_1, Φ_2 irreducible and $\theta(\Phi_1) = \Phi_2$, then θ^* exchanges the Dynkin diagrams of Φ_1 and Φ_2. In particular $\Phi_0(\theta) = \emptyset$, so $w_0(\theta) = \mathrm{id}$ and $\theta = -\theta^*$.

(2) If $\theta = \mathrm{id}$ and Δ is a basis of Φ, then $\theta^*(\Delta) = -w_0(\mathrm{id})$ is called the *opposition involution* of Δ. In this case we shall also write $\mathrm{id}^*(\Delta)$ for $\theta^*(\Delta)$. For Φ irreducible the opposition involution is non-trivial if and only if Φ is either of type $A_l(l(\geq 2)$, $D_{2l+1}(l(\geq 2)$ or E_6.

(3) The action of θ^* on $\Delta_0(\theta)$ is determined by $\Delta_0(\theta)$, because $\theta^*|\Delta_0(\theta) = -w_0(\theta)$ is the opposition involution of $\Delta_0(\theta)$, which is uniquely determined on each irreducible component of $\Phi_0(\theta)$ by the type of the root system $\Phi_0(\theta)$. So for the θ-index we can omit the action of θ^* on $\Delta_0(\theta)$.

(4) For Φ irreducible, the action of θ^* can only be non-trivial if Φ is of type A_l $(l \geq 2)$, D_l $(l \geq 4)$ or E_6.

(5) The involution θ is determined by its θ-index, since $\theta = -\theta^* w_0(\theta)$ and $w_0(\theta)$ is completely determined by the type of $W_0(\theta)$.

The indices of involutions of (X, Φ) can be easily determined using the following result from [Hel88]:

Lemma 5.13 ([Hel88, Lemma 2.14])**.** *Let Δ be a basis of Φ, $\Delta_0 \subset \Delta$ a subset and $\theta^* \in \mathrm{Aut}(X, \Phi, \Delta)$ such that $\theta^*(\Delta_0) = \Delta_0$, $(\theta^*)^2 = \mathrm{id}$. Let X_0 be the $\mathbb{Z}$-span of Δ_0 in X and $\Phi(\Delta_0) = \Phi \cap X_0$. Then there is an involution $\theta \in \mathrm{Aut}(X, \Phi)$ with index $(X, \Delta, \Delta_0, \theta^*)$ if and only if $\theta^*|\Delta_0 = \mathrm{id}^*(\Delta_0)$ (the opposition involution of Δ_0 with respect to $\Phi(\Delta_0)$).*

Remark 5.14. The above θ-index may depend on the choice of the θ-basis. However if T_θ^- is a maximal θ-split torus, then by [Ric82, 4.7] $\bar{\Phi}_\theta = \Phi(T_\theta^-)$ is a root system and by Proposition 5.6 the θ-index does not depend on the θ-basis. Combined with the conjugacy of the maximal θ-split tori under G_θ^0 it follows now that the θ-index is uniquely determined by the G-isomorphism class of θ:

Proposition 5.15. *Let A be a maximal θ-split torus of G, $T \supset A$ a maximal torus and Δ a θ-basis of $\Phi(T)$. The θ-index $(X, \Delta, \Delta_0, \theta^*)$ is uniquely determined (up to congruence) by the isomorphy class of θ.*

Proof. Let $\theta_1, \theta_2 \in \mathrm{Aut}(G)$ be involutions and assume $\phi \in \mathrm{Aut}(G)$ such that $\phi\theta_2\phi^{-1} = \theta_1$. For $i = 1, 2$ let T_i be a θ_i-stable maximal torus with $(T_i)_{\theta_i}^-$ a maximal θ_i-split torus of G and let $\Delta(T_i)$ be a θ_i-basis of $(X^*(T_i), \Phi(T_i))$

with respect to θ_i. Now $T_3 = \phi(T_2)$ is a θ_1-stable maximal torus with $(T_3)^-_{\theta_1}$ a maximal θ_1-split torus of G. By Richardson [Vus74, §1] there exists $h \in G^0_{\theta_1}$ such that $hT_3h^{-1} = T_1$. Replacing ϕ by $\mathrm{Int}(h)\phi$ we may assume that $\phi(T_2) = T_1$ and $\phi((T_2)^-_{\theta_2}) = (T_1)^-_{\theta_1}$. Now the result follows from Theorem 5.8. □

Remark 5.16. We will see in 7.1 that for involutions there is in fact a bijective correspondence between conjugacy classes of θ-indices and isomorphy classes of involutions. For isomorphy classes of reductive k-groups or k-involutions, the respective indices do not characterize the isomorphism classes.

5.17. **Γ-index.** In this subsection we introduce the index related to the isomorphy classes of semisimple k-groups. For the remainder of this section let G be a reductive k-group, A a k-split torus of G, $T \supset A$ a maximal k-torus, K the smallest Galois extension of k which splits T, $\Gamma = \mathrm{Gal}(K/k)$ the Galois group of K/k, $X = X^*(T)$, $\Phi = \Phi(T)$, $X_0 = X_0(\Gamma)$, $\Phi_0 = \Phi_0(\Gamma)$, etc. Let $G_0 = G(\Phi_0)$ denote the connected semisimple subgroup of G generated by $\{U_\alpha \mid \alpha \in \Phi_0\}$. The group G_0 is the semisimple part of $Z_G(A)$. If A is a maximal k-split torus, then G_0 is anisotropic over k and is uniquely determined (up to k-isomorphy) by the k-isomorphism class of G. In that case G_0 is also called the k-anisotropic kernel of G.

5.18. Let Δ be a Γ-basis of Φ, and let $\Delta_0 = \Delta \cap X_0$. As in (4.13.1) we have an action of Γ on Δ, which we denote by $[\sigma]$. The 4-tuple $(X, \Delta, \Delta_0, [\sigma])$ is called the Γ-*index of* (G, T, A). If A is a maximal k-split torus of G, then we will also call this the Γ-index of G. It was shown by Tits [Tit66] that the k-isomorphism class of G uniquely determines, up to congruence, the Γ-index of G. Using Proposition 5.6 this can also be seen easily as follows.

Let G_1, G_2 be connected semisimple groups defined over k and $\phi : G_1 \to G_2$ a k-isomorphism. For $i = 1, 2$ let $A_i \subset G_i$ be a maximal k-split torus, $T_i \supset A_i$ a maximal k-torus of G_i and $\Delta(T_i)$ a Γ-basis of $\Phi(T_i)$. Now $\phi(A_1)$ is a maximal k-split torus of G_2, hence there exists a $g \in G_k$ such that $\mathrm{Int}(g)\phi(A_1) = A_2$. Then $\mathrm{Int}(g)\phi(T_1) \supset A_2$ is a maximal k-torus. Let K be the smallest Galois extension of k which splits T_1 and T_2. Then there exists $x \in G_K$ such that $\mathrm{Int}(x)\,\mathrm{Int}(g)\phi(T_1) = T_2$. Let $\phi_1 = \mathrm{Int}(x)\,\mathrm{Int}(g)\phi$. Then $\phi_1 : (G_1, T_1, A_1) \to (G_2, T_2, A_2)$ is a K-isomorphism and by Theorem 5.8 $\varphi_1^\star = {}^t(\varphi_1|T_1)^{-1}$ as in (4.3.1) (modulo a Weyl group element of $W(T_2)$) is a congruence from the Γ-index of (G_1, T_1, A_1) onto the Γ-index of (G_2, T_2, A_2). Summarized we have now the following result:

Proposition 5.19 ([Tit66])**.** *The k-isomorphism class of G uniquely determines (up to congruence) the* Γ*-index* $(X, \Delta, \Delta_0(\Gamma), [\sigma])$ *of G.*

Remark 5.20. In the special case that G is k-anisotropic ($G = G_0$), one has $\Delta = \Delta_0(\Gamma)$, so the Γ-index of G may be abbreviated by $(X, \Delta_0(\Gamma), [\sigma])$. Applying this to the k-anisotropic kernels G_0, G_0' of G, G' it is easily seen that a congruence $\phi : (X, \Delta, \Delta_0(\Gamma), [\sigma]) \to (X', \Delta', \Delta_0'(\Gamma), [\sigma]')$ induces a congruence $\phi_0 : (X_0, \Delta_0(\Gamma), [\sigma]|X_0) \to ((X_0', \Delta', \Delta_0'(\Gamma), [\sigma]'|X_0')$ of the Γ-index of G_0 onto the Γ-index of G_0'. The map ϕ_0 is called the *restriction* of ϕ to $(X_0, \Delta_0(\Gamma), [\sigma]|X_0)$.

5.21. **Γ_θ-index.** In this subsection we discuss indices related to the isomorphy classes of k-involutions.

Let G be a connected semisimple k-group, $\theta \in \mathrm{Aut}(G)$ an k-involution, A a (θ, k)-split torus of G, $T \supset A$ a θ-stable maximal k-torus of G and $X = X^*(T)$, $\Phi = \Phi(T)$. Let K be a finite Galois extension of k which splits T, $\Gamma = \mathrm{Gal}(K/k)$ the Galois group of K/k as in subsection 5.17 and $\mathcal{E}_\theta = \{1, -\theta\} \subset \mathrm{Aut}(X, \Phi)$ be the subgroup spanned by $-\theta|T$ as in 5.11. Let $\mathcal{E}_\Gamma \subset \mathrm{Aut}(X, \Phi)$ be the subgroup corresponding to the action of Γ on (X, Φ) and let $\Gamma_\theta = \mathcal{E}_\Gamma.\mathcal{E}_\theta$ the subgroup of $\mathrm{Aut}(X, \Phi)$ generated by $\mathcal{E}_\Gamma$ and $\mathcal{E}_\theta$. As in 4.5.1 let $X_0 = X_0(\Gamma_\theta)$, $\Phi_0 = \Phi_0(\Gamma_\theta)$, etc. We will also use the notation $\Phi_0(\Gamma, \theta)$ (resp. $\Delta_0(\Gamma, \theta)$) for $\Phi_0(\Gamma_\theta)$ (resp. $\Delta_0(\Gamma_\theta)$). In addition, let $G_0 = G(\Phi_0)$ denote the connected semisimple subgroup of G generated by $\{U_\alpha \mid \alpha \in \Phi_0\}$. The group G_0 is the semisimple part of $Z_G(A)$. Moreover $\bar{\Phi}_{\Gamma_\theta} = \Phi(A)$ is the set of restricted roots of A, which, by [HW93, 5.9] is a root system if A is a maximal (θ, k)-split torus of G. Let Δ be a Γ_θ-bases of Φ, and let $\Delta_0 = \Delta \cap X_0$. Similar as in (4.13.1) we have an action of Γ_θ on Δ, which we denote by $[\sigma]$. The 4-tuple $(X, \Delta, \Delta_0, [\sigma])$ is called *the* Γ_θ*-index of* (G, T, A, θ). If A is a maximal (θ, k)-split torus of G, then we will also call this *the* Γ_θ*-index of* (G, T, θ).

In the case of θ-indices or Γ-indices the indices did not depend on the choice of the maximal torus, when one choose the torus A involved to be maximal. The above Γ_θ-index of (G, T, θ) depends on the choice of $T \supset A$. For example one can choose T such that T_θ^- is maximal θ-split or one can choose T such that T_θ^+ is a maximal torus of $Z_G(A) \cap H$. In most cases this leads to non congruent Γ_θ-indices. We can obtain a Γ_θ-index uniquely determined by the isomorphy class of the k-involution by taking A maximal (θ, k)-split and $T \supset A$ a θ-standard maximal k-torus of $Z_G(A)$ as in 2.8, i.e. T contains a maximal k-split torus and T_θ^- is a maximal θ-split k-torus of G. We will call a Γ_θ-index of (G, T, A, θ) a Γ_θ*-index of* (G, θ) if A is a maximal (θ, k)-split and $T \supset A$ a θ-standard maximal k-torus of G. This index is uniquely determined by the isomorphy class of the k-involution θ:

Proposition 5.22. *Let θ_1 be a k-involution of G. The k-isomorphism class of θ_1 uniquely determines (up to congruence) the Γ_θ-index* $(X, \Delta, \Delta_0(\Gamma_\theta), [\sigma])$ *of* (G, θ_1).

Proof. Assume $\theta_2 \in \mathrm{Aut}_k(G)$ a k-involution and $\phi \in \mathrm{Aut}_k(G)$ a k-automorphism such that $\phi\theta_2\phi^{-1} = \theta_1$. Let A_1 be a maximal (θ_1, k)-split torus of G, $\tilde{A}_1 \supset A_1$ a maximal k-split torus of G and $T_1 \supset \tilde{A}_1$ a θ_1-standard maximal k-torus of G. Similarly let A_2 be a maximal (θ_2, k)-split torus of G, $\tilde{A}_2 \supset A_2$ a maximal k-split torus of G and $T_2 \supset \tilde{A}_2$ a θ_2-standard maximal k-torus of G. Let Δ_1 be a Γ_θ-basis of $\Phi(T_1)$ and Δ_2 a Γ_θ-basis of $\Phi(T_2)$. By Corollary 3.2 there exists a $h \in G_\theta$ such that $\mathrm{Int}(h)\phi(A_2) = A_1$ and $\mathrm{Int}(h)\phi(\tilde{A}_2) = \tilde{A}_1$. Let K be the smallest Galois extension of K_1 which splits T_1 and T_2 and let $T = \mathrm{Int}(h)\phi(T_2)$. Now $T^-_{\theta_1}$ and $(T_1)^-_{\theta_1}$ are maximal (θ_1, K)-split tori of $Z_G(\tilde{A}_1)$. Again by Corollary 3.2 there exists a $h_1 \in G_\theta$ such that $\mathrm{Int}(h_1)(T^-_{\theta_1}) = (T_1)^-_{\theta_1}$ and $\mathrm{Int}(h_1)(T) = T_1$. Let $\phi_1 = \mathrm{Int}(h_1)\,\mathrm{Int}(h)\phi$. Then ϕ_1 maps (G, T_2, A_2, θ_2) onto (G, T_1, A_1, θ_1) and preserves the Γ_θ-action. Now the result follows from Theorem 5.8. □

Remark 5.23. Similar as for the θ-index and Γ-index one easily determines the restricted root system of a maximal (θ, k)-split torus of G from the Γ_θ-index $(X,\ \Delta,\ \Delta_0,\ [\sigma])$ of (G, θ).

5.24. **(Γ, θ)-order.** The Γ_θ-index of (G, θ), as defined above, corresponds to a Γ_θ-order on (X, Φ). However there is a lot of additional structure present, which is not represented in the Γ_θ-index. We also have a θ-index and a Γ-index. This can be seen as follows. Assume A is a maximal (θ, k)-split torus of G, $\tilde{A} \supset A$ a maximal k-split torus of G and $T \supset \tilde{A}$ a θ-standard maximal k-torus. Let $X = X^*(T)$ and $\Phi = \Phi(T)$. Then we have the usual Γ-order on (X, Φ). On the other hand since T^-_θ is a maximal θ-split torus of G, we also have a θ-order on (X, Φ). Finally since A is maximal (θ, k)-split we also have a Γ_θ-order. All these can be defined simultaneously on (X, Φ) as follows.

Definition 5.25. Let Ψ be a semisimple root datum and let Γ, θ act on (X, Φ) as in 5.21. A linear order on X which is simultaneously a Γ-, θ- and Γ_θ-order is called a (Γ, θ)-order. A fundamental system of Φ with respect to a (Γ, θ)-order is called a *(Γ, θ)-fundamental system of* Φ.

From the above remarks it follows that if $A,\ A_1,\ S,\ T$ are as above, then a (Γ, θ)-order on (X, Φ) exists. However not every Γ_θ-order is a (Γ, θ)-order. Another characterization of a (Γ, θ)-order is given in the following result.

Proposition 5.26. *Let Ψ be a semisimple root datum and assume Γ, θ act on (X, Φ) as in 5.21. The following are equivalent:*

(1) *(X, Φ) has a (Γ, θ)-order.*
(2) *$\Phi_0(\Gamma, \theta) = \Phi_0(\Gamma) \cup \Phi_0(\theta)$.*
(3) *If $\Phi_1 \subset \Phi_0(\Gamma, \theta)$ irreducible component then $\Phi_1 \subset \Phi_0(\theta)$ or $\Phi_1 \subset \Phi_0(\Gamma)$.*

Proof. (1) $\Longrightarrow$ (2). Assume $\succ$ is a (Γ, θ)-order on (X, Φ) and let Φ^+ be the set of positive roots with respect to this order. Then $\succ$ induces an order on $\Phi_0(\Gamma, \theta)$ which is both a Γ-order and a θ-order of $\Phi_0(\Gamma, \theta)$. Namely suppose that there is an $\alpha \in \Phi^+ \cap \Phi_0(\Gamma, \theta)$ such that $\theta(\alpha) \neq \alpha$ and $\sum_{\sigma\in\Gamma} \alpha^\sigma \neq 0$. Then $\alpha \succ 0$, $-\theta(\alpha) \succ 0$ and for all $\sigma \in \Gamma$: $\sigma(\alpha) \succ 0$ and $-\sigma\theta(\alpha) \succ 0$. It follows that for each $\sigma \in \Gamma_\theta$ we have $\alpha^\sigma \succ 0$, hence $\sum_{\sigma\in\Gamma_\theta} \alpha^\sigma \succ 0$. But since $\alpha \in \Phi_0(\Gamma, \theta)$ we have $\sum_{\sigma\in\Gamma_\theta} \alpha^\sigma = 0$, what contradicts the assumption.

(2) $\Longrightarrow$ (3). Assume $\Phi_0(\Gamma, \theta) = \Phi_0(\Gamma) \cup \Phi_0(\theta)$. Let $\Phi_1 \subset \Phi(\Gamma, \theta)$ be an irreducible component and let Δ be a basis of Φ_1. Assume $\Phi_1 \not\subset \Phi(\Gamma)$ and $\Phi_1 \not\subset \Phi(\theta)$. Then there exists $\alpha \in \Delta$ such that $\alpha \notin \Phi(\Gamma)$, i.e. $\sum_{\sigma\in\Gamma} \alpha^\sigma \neq 0$ and there exists $\beta \in \Delta$ such that $\beta \notin \Phi(\theta)$, i.e. $\theta(\beta) \neq \beta$. Since $\Phi_0(\Gamma, \theta) = \Phi_0(\Gamma) \cup \Phi_0(\theta)$ it follows that $\alpha \in \Phi(\theta)$ and $\beta \in \Phi_0(\Gamma)$. Since Φ_1 is irreducible, there exists a string of simple roots $\lambda_1 = \alpha, \lambda_2, \dots, \lambda_r = \beta$ connecting α and β. Moreover we can choose $\alpha, \beta \in \Delta$ such that for $i = 2, \dots, r-1$ we have $\lambda_i \in \Phi_0(\Gamma) \cap \Phi_0(\theta)$. Let $\lambda = \lambda_1 + \cdots + \lambda_r \in \Phi_1$. Then

$$\sum_{\sigma\in\Gamma} \lambda^\sigma = \sum_{\sigma\in\Gamma} \lambda_1^\sigma = \sum_{\sigma\in\Gamma} \alpha^\sigma \neq 0,$$

so $\lambda \notin \Phi_0(\Gamma)$. Similar $\theta(\lambda) = \lambda_1 + \cdots + \lambda_{r-1} + \theta(\beta) \neq \lambda$, so $\lambda \notin \Phi_0(\theta)$. But then $\lambda \notin \Phi_0(\theta) \cup \Phi_0(\Gamma) = \Phi_0(\Gamma, \theta)$, what contradicts the assumption.

(3) $\Longrightarrow$ (1). The condition implies that $\Phi_0(\Gamma, \theta)$ has an order $\succ_1$, which is both a Γ-order and a θ-order. By choosing any order on $\bar{\Phi}_{\Gamma_\theta}$ this order extends to a Γ_θ-order on Φ, which is then also a Γ-order and a θ-order. ☐

Using similar arguments as in the result above we can also prove the following result which is useful in the study of the (Γ, θ)-indices.

Lemma 5.27. *Let Ψ be a semisimple root datum and assume Γ, θ act on (X, Φ) as in 5.21. Assume $X = X_0(\Gamma, \theta)$ and $\succ$ is an order on (X, Φ), which is both a Γ-order and a θ-order. Then we have the following:*

(1) *If $\chi \in X$ with $\sigma(\chi) = \chi$ for all $\sigma \in \Gamma$ and $\theta(\chi) = -\chi$, then $\chi = 0$.*
(2) *If $\alpha \in \Phi = \Phi_0(\Gamma, \theta)$, then $\sum_{\sigma\in\Gamma} \alpha^\sigma = 0$ or $\theta(\alpha) = \alpha$.*

Proof. (1). Let $\chi \in X$ with $\sigma(\chi) = \chi$ for all $\sigma \in \Gamma$ and $\theta(\chi) = -\chi$, then for all $\sigma \in \Gamma_\theta$ we have $\sigma(\chi) = \chi$. But since $X = X_0(\Gamma, \theta)$ we have $0 = \sum_{\sigma\in\Gamma_\theta} \chi^\sigma = \sum_{\sigma\in\Gamma_\theta} \chi$, hence $\chi = 0$.

(2). Let $\alpha \in \Phi$. Assume $\sum_{\sigma\in\Gamma} \alpha^\sigma \neq 0$ and $\theta(\alpha) \neq \alpha$. Since $\sum_{\sigma\in\Gamma_\theta} \alpha^\sigma = \sum_{\sigma\in\Gamma}(\sigma(\alpha) - \sigma\theta(\alpha)) = 0$ it follows that

$$\sum_{\sigma\in\Gamma} \sigma(\alpha) = \sum_{\sigma\in\Gamma} \sigma\theta(\alpha) = \theta(\sum_{\sigma\in\Gamma} \sigma(\alpha)).$$

We may assume that $\alpha \in \Phi^+$. Since $\succ$ is a Γ-order it follows that $\sum_{\sigma\in\Gamma} \sigma(\alpha) \succ 0$. Similarly since $\succ$ is a θ-order it follows that $\theta(\sum_{\sigma\in\Gamma} \sigma(\alpha)) \prec 0$. It follows that either $\sum_{\sigma\in\Gamma} \alpha^\sigma = 0$ or $\theta(\alpha) = \alpha$. $\square$

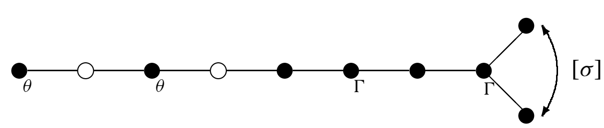
Remarks 5.28. (1) A (Γ, θ)-order, as above, is completely determined by the sextuple

(5.28.1) $$(X,\ \Delta,\ \Delta_0(\Gamma),\ \Delta_0(\theta),\ [\sigma],\ \theta^*).$$

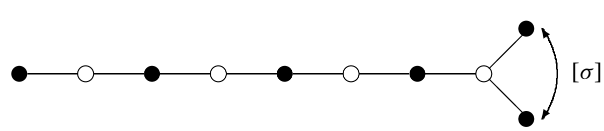
We will call this sextuple an *index of* (Γ, θ) or an (Γ, θ)-index. This terminology follows again Tits [Tit66]. We will also use the name (Γ, θ)-diagram, following the notation in Satake [Sat71, 2.4].

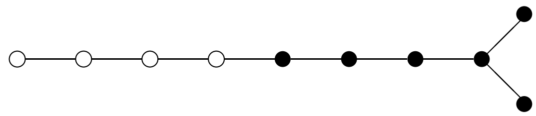
(2) The above index of (Γ, θ) determines the indices of both Γ and θ and vice versa.

(3) We can make a diagrammatic representation of the (Γ, θ)-index by coloring black those vertices of the ordinary Dynkin diagram of Φ, which represent roots in $\Delta_0(\Gamma, \theta)$ and giving the vertices of $\Delta_0(\Gamma) \cup \Delta_0(\theta)$ which are not in $\Delta_0(\Gamma) \cap \Delta_0(\theta)$ a label Γ or θ if $\alpha \in \Delta_0(\Gamma) - \Delta_0(\Gamma) \cap \Delta_0(\theta)$ or $\alpha \in \Delta_0(\theta) - \Delta_0(\Gamma) \cap \Delta_0(\theta)$ respectively. The actions of $[\sigma]$ and θ^* are indicated by arrows. Here is an example with Φ of type D_{10}:



This (Γ, θ)-index is obtained by gluing together the indices





of Γ resp. θ with the above recipe.

(4) A (Γ, θ)-index of Γ_θ may depend again on the choice of the (Γ, θ)-basis of Φ. However if Φ_{Γ_θ} is a root system, then it follows similar as in Proposition 5.6 that the (Γ, θ)-index is independent of the choice of the (Γ, θ)-basis.

The congruence between these indices is defined similar as in 5.3:

Definition 5.29. Let $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ and $\Psi' = (X', \Phi', X'^\vee, \Phi'^\vee)$ be semisimple root data on which Γ acts as in 5.21. Let $\theta_1 \in \mathrm{Aut}(X, \Phi)$ and $\theta_2 \in \mathrm{Aut}(X', \Phi')$ be involutions. A *congruence* φ of the (Γ, θ)-index $(X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta_1), [\sigma], \theta_1^*(\Delta))$ of (Γ, θ_1) onto the (Γ, θ)-index $(X', \Delta', \Delta_0'(\Gamma), \Delta_0(\theta_2), [\sigma]', \theta_2^*(\Delta'))$ of (Γ, θ_2) is an isomorphism which maps $(X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta_1))$ onto $(X, \Delta, \Delta_0'(\Gamma), \Delta_0'(\theta_2))$ and which satisfies:

(5.29.1) $$w\theta_1^*(\Delta)w^{-1} = \theta_2^*(\Delta') \text{ and } w[\sigma]w^{-1} = [\sigma]'.$$

If $\Psi = \Psi'$, then we will call the (Γ, θ)-indices $\mathrm{Aut}(X, \Phi)$-isomorphic respectively $W(\Phi)$-isomorphic if $\varphi \in \mathrm{Aut}(X, \Phi)$ respectively $W(\Phi)$. In the latter case we will also call the indices isomorphic.

6. *k*-automorphisms and *k*-structure

In the previous section we analyzed the actions of the involution and the Galois group on (X, Φ). We need to extend this to the whole group. For this we will use among other things K/k-forms of the related Chevalley group and a realization of the root system Φ in G.

6.1. The fundamental/existence theorems due to Chevalley [Che58] show the existence of a group $\tilde{G}$ (called Chevalley group), unique up to k-isomorphy, corresponding to a pair (G, T), where $T \subset G$ is a maximal torus. This result enables us to consider any k-group as a K/k-form of the related Chevalley group. This is a powerful tool in the study of k-groups. In the following we introduce the notation and briefly review some of the results needed.

Let T be a maximal torus of G, $X = X^*(T)$ and $\Phi = \Phi(T)$. By Chevalleys existence Theorem [Che58] the pair (X, T) (or (G, T)) determines unique up to k-isomorphy a semisimple connected algebraic k-group $\tilde{G} := G(X, \Phi)$, which is k-split (i.e., $\tilde{G}$ contains a maximal torus $\tilde{T}$, which is k-split). Let K be a splitting field for T and $\Gamma = \mathrm{Gal}(K/k)$ the Galois group. Then G is a K/k-form of $\tilde{G}$. In particular there exists a K-isomorphism $\phi : (G, T) \to (\tilde{G}, \tilde{T})$ (see also Theorem 7.10). The Galois group Γ acts on the coefficients of the polynomial mapping ϕ. For $\sigma \in \Gamma$ let $\varphi_\sigma = \phi^\sigma \circ \phi^{-1}$. Then the system of isomorphisms $(\varphi_\sigma)_{\sigma\in\Gamma}$ is a one cocycle, i.e. it satisfies the condition

$$\varphi_\sigma{}^\gamma \circ \varphi_\gamma = \varphi_{\sigma\gamma} \quad \sigma, \gamma \in \Gamma. \tag{6.1.1}$$

The converse is also true by the following well known result:

Proposition 6.2. *The one cocycle* $(\varphi_\sigma)_{\sigma\in\Gamma}$ *of* Γ *in* $\mathrm{Aut}_K(\tilde{G}, \tilde{T})$ *uniquely determines the map* ϕ *as above.*

6.3. Let $X = X^*(T)$ and $\tilde{X} = X^*(\tilde{T})$ be the character modules of T and $\tilde{T}$ respectively, and $\Phi = \Phi(T)$ and $\tilde{\Phi} = \Phi(\tilde{T})$ the root systems in X and $\tilde{X}$ with respect to T and $\tilde{T}$. Then X is a Γ-module, $\tilde{X}$ is a trivial Γ-module and Φ is Γ-invariant. If $\varphi_\sigma = \phi^\sigma \circ \phi^{-1} \in \mathrm{Aut}_K(\tilde{G}, \tilde{T})$ is as above, then ${}^t\varphi_\sigma \in \mathrm{Aut}(\tilde{X}, \tilde{\Phi})$, and if ${}^t\phi(\tilde{\chi}) = \chi$, for $\tilde{\chi} \in \tilde{X}$, $\chi \in X$, then ${}^t\phi \circ {}^t\varphi_\sigma(\tilde{\chi}) = \chi^\sigma$. Thus the automorphisms ${}^t\varphi_\sigma$ transpose the action of Γ on X to an action of Γ on $\tilde{X}$. We will often identify X and $\tilde{X}$ via the isomorphism ${}^t\phi$. In that case, when X is regarded as the character module of T, the action of Γ is non-trivial, and we identify $\chi^\sigma = {}^t\varphi_\sigma(\tilde{X})$, where $\chi = \tilde{\chi}$ under the identification. When X is regarded as the character module of $\tilde{T}$, Γ acts trivially.

The maps $\varphi_\sigma \in \mathrm{Aut}_K(\tilde{G}, \tilde{T})$ and in fact any automorphism in $\mathrm{Aut}(\tilde{G}, \tilde{T})$ can be described by their action on a realization of Φ in G. We discuss this in the following.

6.4. **Realization of $\Phi(T)$ in G.** Let G be a reductive algebraic group, T a maximal k-torus of G, $X = X^*(T)$ and $\Phi = \Phi(T)$. Let $\bar{k}$ denote the algebraic closure of k. For $\alpha \in \Phi(T)$, let x_α be the corresponding one-parameter additive subgroup of G defined by α. This is an isomorphism of the additive subgroup onto a closed subgroup U_α of G, normalized by T, such that

$$t x_\alpha(\xi) t^{-1} = x_\alpha(\alpha(t)\xi), \quad (t \in T, \xi \in \bar{k}) \tag{6.4.1}$$

The x_α may be chosen such that

$$n_\alpha = x_\alpha(1) x_{-\alpha}(-1) x_\alpha(1) \tag{6.4.2}$$

lies in $N_G(T)$ for all $\alpha \in \Phi(T)$, as can be derived using a SL_2-computation. In that case we have

$$x_\alpha(\xi) x_\alpha(-\xi^{-1}) x_\alpha(\xi) = \alpha^\vee(\xi) n_\alpha, \quad (\xi \in \bar{k}), \tag{6.4.3}$$

here $\alpha^\vee \in X_*(T)$ is the coroot of α. Moreover $n_\alpha T$ is the reflection $s_\alpha \in W(T)$ defined by α and $t_\alpha := n_\alpha^2 = \alpha^\vee(-1)$, $n_{-\alpha} = t_\alpha n_\alpha$, $t_{-\alpha} = t_\alpha$.

A family $\{x_\alpha\}_{\alpha\in\Phi(T)}$ with the above properties (6.4.1), (6.4.2) is called a *realization of* $\Phi(T)$ *in* G. Similarly the set of root vectors $X_\alpha = dx_\alpha(1) \in \mathfrak{g}_\alpha$ is called a *realization of* $\Phi(T)$ *in* $\mathfrak{g}$. We then have $\mathrm{Ad}(t)X_\alpha = \alpha(t)X_\alpha$, $(t \in T)$. For these facts see Springer [Spr81, 11.2].

6.5. Assume now that T is a maximal k-torus. The Galois group $\mathrm{Gal}(\bar{k}/k)$ acts on the one-parameter additive subgroups x_α, which are unique up to a scalar multiple. So if we apply $\sigma \in \mathrm{Gal}(\bar{k}/k)$ to both sides of equation (6.4.1) we get for all $\xi \in \bar{k}$:

$$x_\alpha^\sigma(\xi) = x_{\alpha^\sigma}(d_{\alpha,\sigma}\xi), \tag{6.5.1}$$

with $d_{\alpha,\sigma} \in \bar{k}^*$. From (6.5.1), we see that if $\gamma \in \mathrm{Gal}(\bar{k}/k)$, then

$$x_\alpha^{\sigma\gamma}(\xi) = x_{\alpha^\sigma}^\gamma(d_{\alpha,\sigma}^\gamma\xi) = x_{\alpha^{\sigma\gamma}}(d_{\alpha,\sigma}^\gamma d_{\alpha^\sigma,\gamma}\xi),$$

hence the system $\{d_{\alpha,\sigma}\}$ satisfies the condition

$$d_{\alpha,\sigma\gamma} = d_{\alpha,\sigma}^\gamma d_{\alpha^\sigma,\gamma}. \tag{6.5.2}$$

Let K be the smallest Galois extension of k which splits T. We can choose a realization $\{x_\alpha\}_{\alpha\in\Phi}$ of Φ in G such that all x_α are defined over K:

Lemma 6.6. *Let K be the smallest Galois extension of k which splits T and $\Gamma = \mathrm{Gal}(K/k)$. There exists a realization $\{x_\alpha\}_{\alpha\in\Phi}$ of Φ in G such that all x_α are defined over K.*

Proof. Let $\alpha \in \Phi$. Then α is defined over K. Let $\bar{k} \supset K$ be the algebraic closure and let $\Gamma_\alpha = \{\sigma \in \mathrm{Gal}(\bar{k}/k) \mid \alpha^\sigma = \alpha\}$, and let $K_\alpha \subset \bar{k}$ be the fixed field of Γ_α. Since α is defined over K it follows that $\Gamma_\alpha \supset \Gamma$ and $K_\alpha \subset K$. Then by (6.5.2), the system $\{d_{\alpha,\sigma}\}_{\sigma\in\Gamma_\alpha}$ is a one cocycle of $\mathrm{Gal}(\bar{k}/k)$ in $\bar{k}^*$, so by Hilbert's Theorem 90, it follows that there is an element $\xi_\alpha \in \bar{k}^*$ such that $d_{\alpha,\sigma} = \xi_\alpha^\sigma\xi_\alpha^{-1}$. But then if $\tilde{x}_\alpha$ is defined by $\tilde{x}_\alpha(\xi) = x_\alpha(\xi_\alpha^{-1}\xi)$, then $\tilde{x}_\alpha$ satisfies (6.4.1), and $\tilde{x}_\alpha^\sigma = \tilde{x}_\alpha$ for all $\sigma \in \Gamma_\alpha$, hence $\tilde{x}_\alpha$ is defined over K_α. □

Remark 6.7. Let K be the smallest Galois extension of k which splits T and $\Gamma = \mathrm{Gal}(K/k)$. A realization $\{x_\alpha\}_{\alpha\in\Phi}$ will be called a K-realization of Φ in G if all x_α are defined over K. In this case the action of Γ on x_α defines a system of scalars $\{d_{\alpha,\sigma}\}_{\sigma\in\Gamma}$ in K satisfying the condition (6.5.2) above.

The scalars $d_{\alpha,\sigma}$ essentially describe the maps φ_σ as above. To see this we have to look first at the relation between the K-realization and automorphisms.

6.8. **Realization and isomorphisms.** Let G, G' be connected semi-simple algebraic groups defined over k, let T be a maximal k-torus of G, T' a maximal k-torus of G', and assume K is a splitting field for T and T'. Write $\Phi = \Phi(T)$, $X = X^*(T)$, $\Phi' = \Phi(T')$, $X' = X^*(T')$ and let $\{x_\alpha\}_{\alpha\in\Phi}$ be a K-realization of Φ in G and $\{x'_\alpha\}_{\alpha\in\Phi'}$ be a K-realization of Φ' in G'. Suppose there is a K-isomorphism $\varphi : (G, T) \to (G', T')$. Then $\varphi^\star := {}^t(\varphi|T)^{-1}$ is an isomorphism of $(X, \Phi) \to (X', \Phi')$ satisfying $\varphi(U_\alpha) = U'_{\varphi^\star}(\alpha)$. Since $\varphi \circ x_\alpha$ is an isomorphism from the additive subgroup $\bar{k}$ to $U'_{\varphi^\star(\alpha)}$, the uniqueness of $x'_{\varphi^\star(\alpha)}$ implies that there exist $c_{\alpha,\varphi} \in K^*$ such that for $\xi \in \bar{k}$

$$\varphi(x_\alpha(\xi)) = x'_{\varphi^\star(\alpha)}(c_{\alpha,\varphi}\xi) \tag{6.8.1}$$

Since x_α and $x'_{\alpha'}$ are fixed the scalars $c_{\alpha,\varphi} \in K^*$ are uniquely determined by φ. Since G is generated by T and $\{U_\alpha \mid \alpha \in \Phi\}$, equation (6.8.1) shows that φ is uniquely determined by $\varphi^\star$ and $\{c_{\alpha,\varphi}, \alpha \in \Phi\}$. We will denote this correspondence by $\varphi \leftrightarrow \{\varphi^\star,\ c_{\alpha,\varphi}(\alpha \in \Phi)\}$. If there is no ambiguity about the root system involved we will often omit the root system Φ from this notation and denote this correspondence by $\varphi \leftrightarrow \{\varphi^\star,\ c_{\alpha,\varphi}\}$.

Definition 6.9. Let $G,\ G',\ T,\ T'$ etc. be as above, let $\varphi^\star : (X, \Phi) \to (X', \Phi')$ be an isomorphism and let $\{c_{\alpha,\varphi}, \alpha \in \Phi\}$ be a set of scalars in K. We will call a system $\{\varphi^\star,\ c_{\alpha,\varphi}(\alpha \in \Phi)\}$ *admissible* if there exists a K-isomorphism $\varphi : (G, T) \to (G', T')$ (and K-realizations of Φ in G and of Φ' in G') such that $\varphi \leftrightarrow \{\varphi^\star,\ c_{\alpha,\varphi}(\alpha \in \Phi)\}$. In the case that $G = G'$ and $T = T'$ then $\{\varphi^\star,\ c_{\alpha,\varphi}(\alpha \in \Phi)\}$ is admissible if and only if there exist $\varphi \in \operatorname{Aut}(G, T)$ such that $\varphi \leftrightarrow \{\varphi^\star,\ c_{\alpha,\varphi}(\alpha \in \Phi)\}$.

To determine whether an automorphism of (G, T) is k-automorphism of G we have to combine the above with the Γ-action on the K-realization $\{x_\alpha\}_{\alpha\in\Phi}$ of Φ in G as above.

Proposition 6.10. *Let $\varphi \in \operatorname{Aut}(G, T)$ and $\{x_\alpha\}_{\alpha\in\Phi}$ a K-realization of Φ in G. Then we have the following:*

(1) *If $\sigma \in \Gamma$, then $\varphi^\sigma \leftrightarrow \{{\varphi^\star}^\sigma,\ c^\sigma_{\alpha^{\sigma^{-1}},\varphi} \frac{d_{\varphi^\star(\alpha^{\sigma^{-1}}),\sigma}}{d_{\alpha^{\sigma^{-1}},\sigma}}\}$.*
(2) *φ is defined over k if and only if ${\varphi^\star}^\sigma = \varphi^\star$ and $c^\sigma_{\alpha,\varphi} d_{\varphi^\star(\alpha),\sigma} = c_{\alpha^\sigma,\varphi} d_{\alpha,\sigma}$ for all $\sigma \in \Gamma$ and $\alpha \in \Phi$.*
(3) *If $t \in A$, then $\operatorname{Int}(t)$ is a k-automorphism if and only if $\alpha^\sigma(t) = \alpha(t)^\sigma$ for all $\sigma \in \Gamma$.*

Proof. (1). If we apply σ to both sides of equation (6.8.1), then (6.5.1) implies

(6.10.1) $$\varphi^\sigma(x_\alpha^\sigma(\xi)) = x^\sigma_{\varphi^\star(\alpha)}(c^\sigma_{\alpha,\varphi}\xi) = x_{\varphi^{\star^\sigma}(\alpha^\sigma)}(d_{\varphi^\star(\alpha),\sigma}c^\sigma_{\alpha,\varphi}\xi).$$

On the other hand from (6.5.1) and (6.8.1), one gets

(6.10.2) $$\varphi^\sigma(x_\alpha^\sigma(\xi)) = \varphi^\sigma(x_{\alpha^\sigma}(d_{\alpha,\sigma}\xi)) = x_{\varphi^{\sigma^\star}(\alpha^\sigma)}(d_{\alpha,\sigma}c_{\alpha^\sigma,\varphi^\sigma}\xi).$$

Combining the two equations (6.10.1) and (6.10.2), it follows that $\varphi^{\star^\sigma} = \varphi^{\sigma^\star}$, and $c_{\alpha^\sigma,\varphi^\sigma} = c^\sigma_{\alpha,\varphi}\frac{d_{\varphi^\star(\alpha),\sigma}}{d_{\alpha,\sigma}}$. Replacing α by $\alpha^{\sigma^{-1}}$ in the argument, the assertion follows.

(2) and (3) are immediate from (1). □

For isomorphisms of semisimple k-groups a similar result holds.

6.11. Using Proposition 6.10 we can now describe the relation between the scalars $d_{\alpha,\sigma}$ as in (6.5.1) and the maps φ_σ as in 6.1. We use the same notation as in 6.1. In particular let G be defined over k, T a maximal k-torus of G, $X = X^*(T)$, $\Phi = \Phi(T)$, K a splitting field for T, $\Gamma = \mathrm{Gal}(K/k)$ the Galois group and $\tilde{G} := G(X,\Phi)$ a Chevalley group. Let $\phi : (G,T) \to (\tilde{G},\tilde{T})$ be the corresponding K-isomorphism and for $\sigma \in \Gamma$ let $\varphi_\sigma = \phi^\sigma \circ \phi^{-1}$. As in 6.1 we identify X and $\tilde{X} := X^*(\tilde{T})$ by $\chi = \phi^\star(\chi)$ for all $\chi \in X$, and for each $\alpha \in \tilde{\Phi} := \Phi(\tilde{T})$ fix $\tilde{x}_\alpha$ defined over k. If we define $x_\alpha = \phi^{-1}\circ\tilde{x}_\alpha$ for all $\alpha \in \Phi$, then since $\phi \circ x_\alpha$ is an isomorphism from the additive subgroup to $U_{\phi^\star(\alpha)}$, the uniqueness of $\tilde{x}_{\phi^\star(\alpha)}$ implies that

$$\phi(x_\alpha(\xi)) = x'_{\phi^\star(\alpha)}(\xi)$$

for $\alpha \in \Phi$. It follows that $\phi \leftrightarrow \{\phi^\star, c_{\alpha,\phi^\star} = 1(\alpha\in\Phi)\}$ and hence $\phi^{-1} \leftrightarrow \{\phi^{\star^{-1}}, \tilde{c}_{\alpha,\phi^{\star^{-1}}} = 1(\alpha \in \tilde{\Phi})\}$. Since Γ acts trivially on $\tilde{x}_\alpha$ we have $\tilde{x}^\sigma_\alpha(\xi) = \tilde{x}_{\alpha^\sigma}(\xi)$, hence $\tilde{d}_{\alpha,\sigma} = 1$ for all $\sigma \in \Gamma$. With a similar argument as in Proposition 6.10(1) it follows now that

(6.11.1) $$\phi^\sigma \leftrightarrow \{\phi^{\star^\sigma}, d^{-1}_{\alpha^{\sigma^{-1}},\sigma}(\alpha\in\Phi)\}.$$

Since $\varphi_\sigma = \phi^\sigma\circ\phi^{-1}$ and $\varphi^\star_\sigma = \phi^{\sigma^\star}\circ\phi^{-1^\star} = \phi^{\star^\sigma}\circ\phi^{\star^{-1}}$ it follows now that

(6.11.2) $$\varphi_\sigma \leftrightarrow \{\varphi^\star_\sigma,\ c_{\phi^{\star^{-1}}(\alpha),\phi^{\star^\sigma}}\tilde{c}_{\alpha,\phi^{\star^{-1}}} = d^{-1}_{\alpha^{\sigma^{-1}},\sigma}(\alpha\in\tilde{\Phi})\}.$$

Since by Proposition 6.2 ϕ is uniquely determined by the system $(\varphi_\sigma)_{\sigma\in\Gamma}$, it follows that the k-isomorphism classes of G are determined by the systems $\{\varphi^\star_\sigma,\ d^{-1}_{\alpha^{\sigma^{-1}},\sigma}(\alpha\in\tilde{\Phi})\}_{\sigma\in\Gamma}$.

We conclude this section with a result due to Chevalley which reduces the computation of the above structure constants to a basis of Φ (see [Che58, 17-08,09]):

Lemma 6.12. *Let Δ be a basis of Φ, $\theta \in \operatorname{Aut}(X, \Phi)$ an involution and $\varphi \in \operatorname{Aut}(G, T)$ such that $\varphi | T = \theta$. Then φ is uniquely determined by the tuple $\{c_{\alpha,\varphi}\}_{\alpha \in \Delta}$.*

7. Isomorphy classes of involutions and k-groups

In each of the cases of symmetric varieties, symmetric k-varieties and semisimple k-groups there is a natural fine structure associated with these spaces. For a study of these spaces and their representation theory it is important to have a classification of these spaces together with this fine structure of restricted root systems with multiplicities and Weyl groups. This fine structure easily follows from the index as defined in section 5. On the other hand this index can also be used as an invariant to characterize the isomorphy classes. In the case of isomorphy classes of involutions these indices completely characterize the isomorphy classes. In the case of isomorphy classes of semisimple k-groups one needs a second invariant to characterize the isomorphy classes and in the case of isomorphy classes of k-involutions three invariants are needed.

Since the classification of k-involutions depends on the classifications of semisimple k-groups (see [Tit66]) and the classification of involutions over algebraically closed fields (see [Hel88]), we will first briefly review some facts about both these classifications, which will be needed later in the classification of the k-involutions.

7.1. **Characterization of the isomorphy classes of involutions.** The classification of isomorphy classes of involutions can be reduced to a classification of $W(T)$-conjugacy classes of involutions normally related to a maximal torus T (see [Hel88]). In this subsection we briefly review these results. We use the same notation as in 5.11. In particular let G be a reductive algebraic group, $\theta \in \operatorname{Aut}(G)$ an involution and T a maximal torus of G. Write $X = X^*(T)$ and $\Phi = \Phi(T)$. To relate the isomorphy classes of involutions to the indices as in 5.11, we define the following:

Definition 7.2. Let T be a maximal torus of G. An automorphism θ of G of order ≤ 2 is said to be *normally related to T* if $\theta(T) = T$ and T_θ^- is a maximal θ-split torus of G.

Note that, since all maximal tori of G are conjugate under $\operatorname{Int}(G)$, every involutorial automorphism of G is conjugate to one which is normally related to T. The involutions normally related to T can be characterized now as follows (see [Hel88, 3.7]).

Theorem 7.3. *Let $\theta_1, \theta_2 \in \mathrm{Aut}(G)$ be such that $\theta_1^2 = \theta_2^2 = \mathrm{id}$ and assume θ_1, θ_2 are normally related to T. Then we have the following:*

(1) *θ_1 and θ_2 are conjugate under $\mathrm{Int}(G)$ if and only if $\theta_1|T$ and $\theta_2|T$ are conjugate under $W(T)$.*
(2) *θ_1 and θ_2 are conjugate under $\mathrm{Aut}(G)$ if and only if $\theta_1|T$ and $\theta_2|T$ are conjugate under $\mathrm{Aut}(T)$.*

We showed in Proposition 5.15 that the G-isomorphy class determines the θ-index up to congruence. From Theorem 7.3 it follows now that these indices actually completely characterize the isomorphy classes. To formulate this result we need to define first a notion of admissibility.

Definition 7.4. Let $\theta \in \mathrm{Aut}(X, \Phi)$ be an involution. Then θ is called *admissible* if there exists an involution $\tilde{\theta} \in \mathrm{Aut}(G, T)$ such that $\tilde{\theta}|T = \theta$ and $T_{\tilde{\theta}}^-$ is a maximal $\tilde{\theta}$-split torus of G. If X is semisimple, then the indices of admissible involutions of (X, Φ) are called *admissible θ-indices.*

7.5. Recall that an involution θ of (X, Φ) determines a θ-index, but for arbitrary involutions of (X, Φ) this index is not uniquely determined by the W-isomorphy class of θ. However if θ is an involution such that $\bar{\Phi}_\theta$ is a root system with Weyl group $\bar{W}_\theta$, then by Proposition 5.6 the isomorphy class of θ uniquely determines, up to congruence, the θ-index. Conversely a θ-index determines uniquely an involution θ of (X, Φ). Namely define $\theta = -\mathrm{id}\,\theta^* w_0(\theta)$, where θ^* is determined by the θ-index and $w_0(\theta)$ is the opposition involution of $W_0(\theta)$ with respect to $\Delta_0(\theta)$. The latter is completely determined by $\Delta_0(\theta)$. Since admissible involutions of (X, Φ) have the property that $\bar{\Phi}_\theta$ is a root system with Weyl group $\bar{W}_\theta$ we have now the following:

Lemma 7.6. *Let G, T, Φ and $W = W(T)$ be as above. There is a bijective correspondence between the W-isomorphy (resp. $\mathrm{Aut}(\Phi)$-isomorphy) classes of admissible involutions of (X, Φ) and the W-congruence (resp. $\mathrm{Aut}(\Phi)$-congruence) classes of admissible θ-indices.*

Combined with the above result we have now the following characterization of the isomorphy classes of involutions in terms of θ-indices. Note that these θ-indices yield most of the fine structure of the corresponding symmetric variety G/G_θ.

Theorem 7.7. *Let G, T be as above and assume G is semisimple. Then there is a bijection of the set of* $\mathrm{Int}(G)$ (*resp.* $\mathrm{Aut}(G)$) *conjugacy classes of involutorial automorphisms of G and the W-congruence (resp.* $\mathrm{Aut}(\Phi)$*-congruence) classes of indices of admissible involutions of* $(X^*(T), \Phi(T))$.

7.8. **Characterization of the isomorphy classes of semisimple k-groups.** In the remainder of this section we give a characterization of the isomorphy classes of semisimple k-groups. Most of these results can be found in [Tit66] and [Sat71].

7.9. We use the same notation as in 5.17. In particular let G be a connected semisimple groups defined over k and let $A \subset G$ be a maximal k-split torus, $T \supset A$ a maximal k-torus of G, $X = X^*(T)$, $\Phi = \Phi(T)$, K the smallest Galois extension of k which splits T and $\Delta = \Delta(T)$ a Γ-basis of $\Phi(T)$.

In Proposition 5.19 we demonstrated that the Γ-index is an invariant for the isomorphy classes of semisimple k-groups. Another invariant is the following. Let $G_0 = G(\Phi_0)$ denote the connected semisimple subgroup of G generated by $\{U_\alpha \mid \alpha \in \Phi_0\}$. The group G_0 is the semisimple part of $Z_G(A)$ and is k-anisotropic if A is maximal k-split. Let $T_0 = T \cap G_0$. This is a maximal k-torus of G_0. Since all maximal k-split tori of G are conjugate under G_k, it follows that G_0 is uniquely determined (up to k-isomorphism) by the k-isomorphism class of G. We will call G_0 the *k-anisotropic kernel* of G.

We have shown now that the k-isomorphism class of G uniquely determines the Γ-index $(X, \Phi, \Delta_0(\Gamma), [\sigma])$ of G and the k-anisotropic kernel G_0 of G. The following result shows that these two actually suffice to characterize the isomorphy classes (see [Tit66] or [Sat71]).

Theorem 7.10. *Let G, G' be connected semi-simple algebraic groups defined over k. Let T, A, X, G_0, T_0, etc., T', A', X', G_0', T_0' etc. be as defined above, and corresponding to G and G', respectively. There exists a k-isomorphism φ : $(G, T, A) \to (G', T', A')$ if and only if the following conditions are satisfied:*

- (i) *There exists a congruence $\phi : (X, \Delta, \Delta_0(\Gamma), [\sigma]) \to (X', \Delta', \Delta_0'(\Gamma), [\sigma]')$ of the Γ-index of G onto the Γ-index of G'.*
- (ii) *There exists a k-isomorphism $\varphi_0 : (G_0, T_0) \to (G_0', T_0')$ such that the restriction ϕ_0 of ϕ to $(X_0, \Delta_0(\Gamma), [\sigma]|X_0)$ is associated to φ_0 as in 5.9 (i.e., $\varphi_0^{[\star]} = \phi_0$).*

The Γ-indices, which belong to connected semi simple groups will be called admissible. They are defined as follows:

Definition 7.11. If X is a free module of rank n, Δ a fundamental system of a root system Φ in X, $\Delta_0(\Gamma)$ a subset of Δ, and $[\cdot]$ a homomorphism of the Galois group Γ into $\mathrm{Aut}(X, \Delta, \Delta_0(\Gamma))$, we will say that the system $\mathcal{D} = (X, \Delta, \Delta_0(\Gamma), [\sigma])$ is *admissible* if there exists a connected semi-simple group G defined over k having $\mathcal{D}$ as Γ-index.

Remark 7.12. The above result reduces the problem of classifying connected semisimple algebraic groups defined over k to the following two problems:

(1) classification of all admissible Γ-indices.
(2) classification of all k-anisotropic semisimple algebraic groups.

For arbitrary base fields not much is known about the k-anisotropic semisimple algebraic groups. The first problem is discussed in Tits [Tit66]. See also 10.18.

7.13. For $k = \mathbb{R}$ every complex semisimple group contains a compact real form, which is unique up to isomorphism (see [Hel78]). So in this case there is a one to one correspondence between isomorphy classes of k-anisotropic semisimple groups and isomorphy classes of complex semisimple groups. Since the latter (modulo the center) are completely characterized by the corresponding Dynkin diagram, the classification of real semisimple groups reduces to a classification of the admissible Γ-indices.

For a $\mathfrak{p}$-adic field $k = \mathbb{Q}_p$ the only k-anisotropic semisimple groups are $\mathrm{SL}(1, \mathcal{K})$, where $\mathcal{K}/k$ is a normal division algebra . So in particular the Γ-index of a k-anisotropic semisimple group over $\mathbb{Q}_p$ can only consist of copies of the Dynkin diagrams of type A_n.

8. Characterization of the isomorphy classes of k-involutions

In this section we give a characterization of the isomorphy classes of k-involutions of G. The isomorphy classes of involutions of connected semisimple algebraic groups defined over an algebraically closed field are characterized by the θ-index, while the isomorphy classes of connected semi-simple algebraic groups defined over k were determined by the Γ-index and an isomorphy of the k-anisotropic kernels. So to characterize the isomorphy classes of k-involutions one will minimally need the θ-index, Γ-index and an isomorphy of the involutions restricted to the k-anisotropic kernels. In most cases an additional invariant is be needed.

8.1. Let G be a reductive k-group and θ a k-involution of G. We will consider isomorphy classes of k-involutions under the action of $\mathrm{Int}(G_k)$, $\mathrm{Int}_k(G)$ and

$\mathrm{Aut}_k(G)$. We will say that two k-involutions are isomorphic under G_k (or G_k-isomorphic) if they are isomorphic under $\mathrm{Int}(G_k)$. We want to characterize the isomorphism classes in a such a way that we also get a classification of the natural root systems of the symmetric k-varieties. This means we need to characterize the isomorphism classes of the k-involutions on a fixed maximal k-split torus. For this we define the following notion:

Definition 8.2. Let A be a maximal k-split torus of G. A k-involution θ of G is *normally related* to A if $\theta(A) = A$ and A_θ^- is a maximal (θ, k)-split, torus of G.

Lemma 8.3. *Let A be a maximal k-split torus of G. Every k-involution is G_k-isomorphic with one normally related to A.*

Proof. Let A_1 be a maximal k-split torus such that $(A_1)_\theta^-$ is a maximal (θ, k)-split, torus of G. There exists a $g \in G_k$ such that $gA_1g^{-1} = A$. Then $\theta_1 = \mathrm{Int}(g)\theta\,\mathrm{Int}(g^{-1})$ is normally related to A. □

8.4. In the following we let A denote a maximal k-split torus, $T \supset A$ a maximal k-torus of G and we will write $W(A, T) = \{w \in W(T) \mid w(A) \subset A\}$ and $\mathrm{Aut}(A, T) = \{\phi \in \mathrm{Aut}(T) \mid \phi(A) \subset A\}$.

We can either consider congruence classes of (Γ, θ)-indices or conjugacy classes of admissible involutions under $W(A, T)$ or $\mathrm{Aut}(A, T)$ depending on whether we consider inner or outer automorphisms. For this we need to adjust the definition of congruence as follows:

Definition 8.5. Let θ_1, θ_2 be k-involutions of G normally related to a maximal k-split torus A of G, $T_1, T_2 \subset Z_G(A)$ maximal k-split tori such that θ_i is normally related to T_i, $(i = 1, 2)$, let Δ be a (Γ, θ)-basis of T_1, Δ' a (Γ, θ)-basis of T_2 and let $x \in Z_G(A)$ be such that $xT_1x^{-1} = T_2$. A congruence $\phi : (X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta_1), [\sigma], \theta_1^*) \to (X', \Delta', \Delta_0'(\Gamma), \Delta_0'(\theta_2), [\sigma]', \theta_2^*)$ of the (Γ, θ_1)-index of (G, θ_1) onto the (Γ, θ_2)-index of (G, θ_2) will be called an $\mathrm{Int}(G)$-congruence if $\phi\,\mathrm{Int}(x^{-1}) \in W(A, T_2)$ and an $\mathrm{Aut}(G)$-congruence if $\phi\,\mathrm{Int}(x^{-1}) \in \mathrm{Aut}(A, T_2)$.

The admissible involutions are defined as follows.

Definition 8.6. Let G be a reductive k-group, A a maximal k-split torus of G and $T \supset A$ a maximal k-torus of G, $K \supset k$ a splitting extension for T. An involution $\theta \in \mathrm{Aut}(X^*(T), \Phi(T))$ is said to be an *admissible k-involution* (with respect to (G, T, A)) if there exists a k-involution $\tilde{\theta}$ of G, normally related to

A and $x \in Z_{G_K}(A)$ such that $\operatorname{Int}(x)\tilde{\theta}\operatorname{Int}(x^{-1})$ is normally related to T, $x^{-1}Tx$ is a k-torus and $\operatorname{Int}(x)\tilde{\theta}\operatorname{Int}(x^{-1})|T = \theta$.

An admissible k-involution $\theta \in \operatorname{Aut}(X^*(T), \Phi(T))$ is said to be an *special admissible k-involution* (with respect to (G, T, A)) if the pair $(G, \tilde{\theta})$ is a special pair.

Remark 8.7. If A is a maximal k-split torus of G, $T \supset A$ a maximal k-torus of G, $K \supset k$ a splitting extension for T and $\tilde{\theta}$ a k-involution of G, normally related to A, then there exists $x \in Z_{G_K}(A)$ such that $\operatorname{Int}(x)\tilde{\theta}\operatorname{Int}(x^{-1})$ is normally related to T.

Lemma 8.8. *Let A be a maximal k-split torus of G, let θ_1, θ_2 be k-involutions of G normally related to A, let $T_1, T_2 \subset Z_G(A)$ be maximal k-tori such that θ_i is normally related to T_i, $(i = 1, 2)$ and let Δ be a (Γ, θ)-basis of $(X^*(T_1), \Phi(T_1))$ and Δ' a (Γ, θ)-basis of $(X^*(T_2), \Phi(T_2))$. If $\phi : (X^*(T_1), \Delta, \Delta_0(\Gamma), \Delta_0(\theta_1), [\sigma], \theta_1^*) \to (X^*(T_2), \Delta', \Delta'_0(\Gamma), \Delta'_0(\theta_2), [\sigma]', \theta_2^*)$ is a congruence of the (Γ, θ)-index of (G, θ_1) onto the (Γ, θ)-index of (G, θ_2), then there exists $\varphi \in \operatorname{Aut}(G)$ such that $\varphi(A) = A$, $\varphi(T_1) = T_2$, $\varphi(T_1^-) = T_2^-$ and $\varphi(A^-_{\theta_1}) = A^-_{\theta_2}$. Moreover φ is contained in $\operatorname{Int}(G, A) = N_G(A)$ if and only if $\varphi|A \in W(A)$.*

Proof. By Chevalleys existence Theorem [Che58] there exists a map $\varphi \in \operatorname{Aut}(G)$ such that $\varphi(T_1) = T_2$ and $\varphi^\star = \phi$. By Lemma 4.15 $X_0(\Gamma)$ is spanned by $\Delta_0(\Gamma)$ and $\{\alpha^{[\sigma]} - \alpha \mid \alpha \in \Delta - \Delta_0(\Gamma)$ and $\alpha^{[\sigma]} \neq \alpha\}$. So ϕ maps a spanning set of $X_0(\Gamma)$ to a spanning set of $X'_0(\Gamma)$, i.e. $\phi(X_0(\Gamma)) = X'_0(\Gamma)$. Since A is the annihilator of $X_0(\Gamma)$ in T_1 as well as the annihilator of $X'_0(\Gamma)$ in T_2 it follows that $\varphi(A) = A$. Similarly ϕ maps a spanning set of $X_0(\theta)$ to a spanning set of $X'_0(\theta)$ and since T_1^- (resp. T_2^-) is the annihilator of $X_0(\theta_1)$ (resp $X'_0(\theta_2)$) it follows that $\varphi(T_1^-) = T_2^-$. Finally with a similar argument it follows that $\phi(X_0(\Gamma_\theta)) = X'_0(\Gamma_\theta)$, hence $\varphi(A^-_{\theta_1}) = A^-_{\theta_2}$. □

In the following we show that the classification can be split in 3 problems. We first characterize the involutions of $Z_G(A)$ which have the same (Γ, θ)-index.

Theorem 8.9. *Let A be a maximal k-split torus of G, $T \supset A$ a maximal k-torus of G, $K \supset k$ a splitting extension for T, $\theta_1, \theta_2 \in \operatorname{Aut}(X^*(T), \Phi(T))$ admissible k-involutions, $\tilde{\theta}_1, \tilde{\theta}_2 \in \operatorname{Aut}_k(G)$, normally related to A and $x_1, x_2 \in Z_{G_K}(A)$ such that for $(i = 1, 2)$, $\operatorname{Int}(x_i)\tilde{\theta}_i\operatorname{Int}(x_i^{-1})$ is normally related to T, $T_i := x_i^{-1}Tx_i$ the corresponding k-tori and $\operatorname{Int}(x_i)\tilde{\theta}_i\operatorname{Int}(x_i^{-1})|T = \theta_i$. Let Δ be a (Γ, θ)-basis of $(X^*(T_1), \Phi(T_1))$ and Δ' a (Γ, θ)-basis of $(X^*(T_2), \Phi(T_2))$. Then the following are equivalent:*

(1) *θ_1 and θ_2 are conjugate under $W(A, T)$ (resp. $\operatorname{Aut}(A, T)$).*

(2) $\tilde{\theta}_1$ *and* $\tilde{\theta}_2$ *are conjugate under* $N_G(A)$ (*resp.* $\mathrm{Aut}(G, A)$).
(3) *The* (Γ, θ)*-index* $(X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta), [\sigma], \theta_1^*)$ *of* $(G, T_1, A, \tilde{\theta}_1)$ *and the* (Γ, θ)*-index* $(X', \Delta', \Delta_0'(\Gamma), \Delta_0'(\theta), [\sigma]', \theta_2^*)$ *of* $(G, T_2, A, \tilde{\theta}_1)$ *are* $\mathrm{Int}(G)$*-congruent (resp.* $\mathrm{Aut}(G)$*-congruent).*

Proof. We prove the result for $W(A, T)$, $N_G(A)$ and $\mathrm{Int}(G)$-congruence. The proof in the case of outer automorphisms is similar.

(1) $\Longrightarrow$ (2). For $(i = 1, 2)$ let $\bar{\theta}_i = \mathrm{Int}(x_i)\tilde{\theta}_i \,\mathrm{Int}(x_i^{-1})$. Then $\bar{\theta}_i|T = \theta_i$. Let $w \in W(A, T)$ such that $w\theta_1 w^{-1} = \theta_2$ and let $n \in N_G(A, T)$ be a representative. Let $\tilde{\theta} = \mathrm{Int}(n)\bar{\theta}_1 \,\mathrm{Int}(n^{-1})$. Then $\tilde{\theta}|T = \theta_2$. By [Hel88, 3.8] there exists $t \in T_2^-$ such that $\tilde{\theta} = \bar{\theta}_2 \,\mathrm{Int}(t)$. If we take $t_0 \in T_2^-$ such that $t_0^{-2} = t$, then $t_0^{-1}nx_1 \in N_G(A)$ and $\mathrm{Int}(t_0^{-1}nx_1)\tilde{\theta}_1 \,\mathrm{Int}(t_0^{-1}nx_1)^{-1} = \bar{\theta}_2$. But then $x_2^{-1}t_0^{-1}nx_1 \in N_G(A)$ and $\mathrm{Int}(x_2^{-1}t_0^{-1}nx_1)\tilde{\theta}_1 \,\mathrm{Int}(x_2^{-1}t_0^{-1}nx_1)^{-1} = \tilde{\theta}_2$.

(2) $\Longrightarrow$ (3). Let $n \in N_G(A)$ such that $\mathrm{Int}(n)\tilde{\theta}_1 \,\mathrm{Int}(n)^{-1} = \tilde{\theta}_2$. Since $\mathrm{Int}(n)\tilde{\theta}_1$ $\mathrm{Int}(n)^{-1}|A = \tilde{\theta}_2|A$ we have $nA^{\pm}_{\tilde{\theta}_1}n^{-1} = A^{\pm}_{\tilde{\theta}_2}$. Let $T_0 = nT_1n^{-1} \subset Z_G(A)$. This is a θ_2-stable torus of $Z_G(A)$ containing a maximal $\tilde{\theta}_2$-split torus. There exists $h \in G^0_{\tilde{\theta}_2} \cap Z_G(A)$ such that $hT_0h^{-1} = T_2$. Then $hnT_1^-n^{-1}h^{-1} = T_2^-$ and $\mathrm{Int}(hn)$ induces a map ϕ_1 from $(X^*(T_1), \Phi(T_1))$ to $(X^*(T_2), \Phi(T_2))$ mapping Δ to a (Γ, θ)-basis Δ'' of (G, T_2, θ_2). But then there is a unique $w \in W(A, T_2)$ such that $w(\Delta'') = \Delta'$. The map $\phi = w\phi_1$ gives the desired congruence.

(3) $\Longrightarrow$ (1). Assume that $\phi : (X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta), [\sigma], \theta_1^*) \to (X', \Delta', \Delta_0'(\Gamma), \Delta_0'(\theta), [\sigma]', \theta_2^*)$ is a $\mathrm{Int}(G)$-congruence. Let $x = x_2^{-1}x_1 \in Z_{G_K}(A)$. Then $xT_1x^{-1} = T_2$, $\phi \,\mathrm{Int}(x^{-1}) \in W(A, T_2)$ and $\phi \,\mathrm{Int}(x^{-1})$ maps $\mathrm{Int}(x)\tilde{\theta}_1 \,\mathrm{Int}(x^{-1})|T_2$ to $\tilde{\theta}_2|T_2$. Let $\tilde{\phi} = \mathrm{Int}(x_2)\phi \,\mathrm{Int}(x^{-1}) \,\mathrm{Int}(x_2^{-1}) = \mathrm{Int}(x_2)\phi \,\mathrm{Int}(x_1^{-1})$. Since $\phi \,\mathrm{Int}(x^{-1}) \in W(A, T_2)$ it follows that $\tilde{\phi} \in W(A, T)$, what proves the result. □

Since $N_G(A) = N_{G_k}(A).Z_G(A)$ the next step is to analyze when two k-involutions, normally related to A, restricted to $Z_G(A)$ are isomorphic. First we characterize the involutions which coincide on $Z_G(A)$. If we take involutions normally related to A we can also restrict to the centralizer of a maximal (θ, k)-split torus.

Proposition 8.10. *Let A be a maximal k-split torus of G, $\theta_1, \theta_2 \in \mathrm{Aut}_k(G)$ k-involutions, normally related to A satisfying $\tilde{A} = A^-_{\theta_1} = A^-_{\theta_2}$. Then $\theta_1|Z_G(A) = \theta_2|Z_G(A)$ if and only if $\theta_1|Z_G(\tilde{A}) = \theta_2|Z_G(\tilde{A})$.*

Proof. The only if statement being clear we assume $\theta_1|Z_G(A) = \theta_2|Z_G(A)$. Write $Z_G(\tilde{A}) = C \cdot L_1 \cdot L_2$ as the almost direct product of k-groups where $C \supset$

$\tilde{A}$ is the maximal central torus, L_1 is semi-simple anisotropic over k and L_2 has no anisotropic factors over k. By Proposition 2.6 we have $L_1 \subset G_{\theta_1}$ and also $L_1 \subset G_{\theta_2}$, i.e. $\theta_1 | L_1 = \theta_2 | L_1 = \mathrm{id}$. Since $A \subset C \cdot L_1$ and L_2 commutes with C and L_1 it follows that $L_2 \subset Z_G(A)$. From the assumption it follows that $\theta_1 | L_2 = \theta_2 | L_2$. Finally since $\theta_1 | A = \theta_2 | A$ the result follows. □

The following result will be useful in the sequel.

Lemma 8.11. *Let A be a maximal k-split torus, $T \supset A$ a maximal k-torus of G, $K \supset k$ be a minimal Galois extension such that T splits over K and let $\Gamma = \mathrm{Gal}(K/k)$. If $t \in Z(Z_G(A))$ such that $\mathrm{Int}(t)$ is a k-automorphism, then $\sigma(t) \equiv t \mod Z(G)$ for all $\sigma \in \Gamma$.*

Proof. Let $t \in Z(Z_G(A))$ such that $\mathrm{Int}(t)$ is a k-automorphism and let $\sigma \in \Gamma$. By (4.4.1) we have $\mathrm{Int}(t)^\sigma | T = \mathrm{Int}(t) | T$. From Proposition 6.10(3) it follows that $\alpha^\sigma(t) = \alpha(t)^\sigma$ for all $\alpha \in \Phi(T)$ and $\sigma \in \Gamma$. But then $\mathrm{Int}(\sigma(t)) = \mathrm{Int}(t)$, hence $\sigma(t) \equiv t \mod Z(G)$ for all $\sigma \in \Gamma$. □

In the following let θ be a k-involution, A a θ-stable maximal k-split torus such that $\tilde{A} = A_\theta^-$ is maximal (θ, k)-split and $T \supset A$ a θ-stable maximal k-torus of G such that T_θ^- is a maximal θ-split torus. Let $K \supset k$ be a minimal Galois extension such that T splits over K, let $\Gamma = \mathrm{Gal}(K/k)$ and let $\mathcal{E} = \Gamma_\theta$ be as in 5.21.

We will need the following result:

Lemma 8.12. *Let A, T, Γ, Γ_θ be as above and Δ a Γ_θ-basis of $\Phi(T)$. Then we have the following:*

(1) *If $t \in \cap_{\beta \in \Delta_0(\Gamma)} \mathrm{Ker}(\beta)$ then $\alpha^{[\sigma]}(t) = \alpha^\sigma(t) = \alpha(t^{\sigma^{-1}})$ for all $\alpha \in \Phi(T)$ and $\sigma \in \Gamma$.*
(2) *If $t \in \cap_{\beta \in \Delta_0(\Gamma_\theta)} \mathrm{Ker}(\beta)$ and $\theta(t)t \in Z(G)$ then for all $\alpha \in \Phi(T)$ we have $\alpha^{[\sigma]}(t) = \alpha^\sigma(t) = \alpha(t^{\sigma^{-1}})$, $(\sigma \in \Gamma)$ and $\theta^*(\alpha)(t) = \alpha(t)$.*

Proof. If $t \in \cap_{\beta \in \Delta_0(\Gamma)} \mathrm{Ker}(\beta)$ and $\sigma \in \Gamma$, then

$$\begin{aligned} \alpha^{[\sigma]}(t) &= w_0(\sigma)\alpha^\sigma(t) = w_0(\sigma)\alpha(t^{\sigma^{-1}}) \\ &= w_0(\sigma)(\alpha)(t^{\sigma^{-1}}) = \alpha(t^{\sigma^{-1}})\gamma(t^{\sigma^{-1}}) \end{aligned}$$

for some $\gamma \in \mathrm{Span}(\Delta_0(\Gamma))$. Since $\cap_{\beta \in \Delta_0(\Gamma)} \mathrm{Ker}(\beta)$ is Γ-stable and since $\beta(t) = 1$ for all $\beta \in \Phi_0(\Gamma)$ it follows that $\gamma(t) = 1$ and hence $\alpha^{[\sigma]}(t) = \alpha^\sigma(t) = \alpha(t^{\sigma^{-1}})$.

(2). Since $\Delta_0(\Gamma) \subset \Delta_0(\Gamma_\theta)$ it follows from (1) that $\alpha^{[\sigma]}(t) = \alpha^\sigma(t) = \alpha(t^{\sigma^{-1}})$ for all $\alpha \in \Phi(T)$ and $\sigma \in \Gamma$. As for the other statement note that since $\Delta_0(\theta) \subset \Delta_0(\Gamma_\theta)$ we have $\beta(t) = 1$ for all $\beta \in \Phi_0(\theta)$. But then

$$\theta^*(\alpha)(t) = w_0(\theta)\alpha(\theta(t)^{-1}) = w_0(\theta)(\alpha)(t) = \alpha(t)\gamma(t)$$

for some $\gamma \in \mathrm{Span}(\Delta_0(\theta))$. So $\gamma(t) = 1$, hence $\theta^*(\alpha)(t) = \alpha(t)$ for all $\alpha \in \Phi_0(T)$. □

Involutions which coincide on $Z_G(A)$ can be characterized now as follows.

Proposition 8.13. *Let A be a maximal k-split torus of G and let $\theta_1, \theta_2 \in \mathrm{Aut}_k(G)$ be k-involutions normally related to A with $\theta_1|Z_G(A) = \theta_2|Z_G(A)$. Then there exists $a \in A^-_{\theta_2}$ such that $\theta_1 = \theta_2\,\mathrm{Int}(a)$.*

Proof. Let $\theta = \theta_1|Z_G(A) = \theta_2|Z_G(A)$ and $T \supset A$ a θ-stable maximal k-torus of $Z_G(A)$ such that T^-_θ is a maximal θ-split torus of G. By [Hel88, 3.8] there is a $t \in T^-_\theta$ such that $\theta_2 = \theta_1\,\mathrm{Int}(t)$. Let $\tilde{A} = A^-_\theta$. This is a maximal (θ, k)-split torus of $Z_G(A)$. Let $K \supset k$ be a finite splitting field for T and $\Gamma = \mathrm{Gal}(K/k)$ the Galois group. Let $\mathcal{E} = \Gamma_\theta$ and let $\Phi_0(\Gamma_\theta)$ be as in 5.21. Then $\Phi_0(\Gamma_\theta) = \{\alpha \in \Phi(T) \mid \alpha(a) = 1,\ \text{for all } a \in \tilde{A}\}$. This is the root system of $Z_G(\tilde{A})$ with respect to T. Since by Proposition 8.10 we have $\theta_1|Z_G(\tilde{A}) = \theta_2|Z_G(\tilde{A})$ it follows that $\alpha(t) = 1$ for all $\alpha \in \Phi_0(\Gamma_\theta)$ and $t \in Z(Z_G(\tilde{A}))$.

Let Δ be a (Γ, θ)-basis of $\Phi(T)$ and let $\Delta_0(\Gamma, \theta)$, $\bar{\Delta}_{\mathcal{E}}$ be as in 4.8. If $\gamma \in \bar{\Delta}_{\mathcal{E}}$ and $\alpha, \beta \in \Delta, \alpha \neq \beta$, such that $\pi(\alpha) = \pi(\beta)$, then by Lemma 4.17 there exists $\sigma \in \mathcal{E}$ such that $\beta = \alpha^{[\sigma]}$. Since $\mathrm{Int}(t)$ is a k-automorphism it follows from Lemmas 8.11 and 8.12 that $\alpha^{[\sigma]}(t) = \alpha(t)$ for all $\sigma \in \Gamma$.

For each $\gamma \in \bar{\Delta}_{\mathcal{E}}$, take now $\alpha \in \Delta$ such that $\gamma = \pi(\alpha) = \alpha|\tilde{A}$ and choose $u_\gamma \in \tilde{A}$ such that $\lambda(u_\gamma) = 1$ for $\lambda \in \bar{\Delta}_{\mathcal{E}}$, $\lambda \neq \gamma$ and $\gamma(u_\gamma^2) = \alpha(t)$. Let $u = \prod_{\gamma \in \bar{\Delta}_{\mathcal{E}}} u_\gamma$. Then by Lemmas 4.17, 8.11 and 8.12 we find $\alpha(t.u^{-2}) = 1$ for all $\alpha \in \Delta$. So $t.u^{-2} \in Z(G)$ and it follows that $\mathrm{Int}(u)\theta_1\mathrm{Int}(u^{-1}) = \theta_1\,\mathrm{Int}(u^{-2}) = \theta_1\,\mathrm{Int}(t) = \theta_2$. □

We have now the following characterization of the isomorphy classes of the k-involutions of G.

Corollary 8.14. *Let G be a connected semi-simple algebraic group defined over k, A a maximal k-split torus of G and θ_1, θ_2 k-involutions of G, normally related to A. Then θ_1 is G_k-isomorphic to $\theta_2\,\mathrm{Int}(a)$ for some $a \in A^-_{\theta_2}$ if and only if $\theta_1|Z_G(A)$ and $\theta_2|Z_G(A)$ are isomorphic under G_k.*

Proof. This result is immediate from Proposition 8.13. □

The elements $a \in A_\theta^-$ as in Proposition 8.13 and Corollary 8.14 do not need to be contained in $A_\theta^-(k)$ as can be seen from the following example:

Example 8.15. Let $k = \mathbb{R}$, $G = \mathrm{SL}_2(\mathbb{C})$ and $\theta \in \mathrm{Aut}(G)$ defined by $\theta(g) = {}^t g^{-1}$, $g \in G$. Then G and θ are defined over $\mathbb{R}$ and $G_{\mathbb{R}} = \mathrm{SL}_2(\mathbb{R})$. Let $A = \{\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right) \mid x \in \mathbb{C}\}$ be the set of diagonal matrices. Then A is a maximal k-split torus of G, which is a maximal torus as well. Moreover $A = A_\theta^-$. Let $a = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) \in A$. Then $\mathrm{Int}(a)$ is a $\mathbb{R}$-automorphism of G and $\theta\,\mathrm{Int}(a)$ is a $\mathbb{R}$-involution of G which is not isomorphic to θ under $G_{\mathbb{R}}$ (but isomorphic to θ under G).

8.16. ***k*-inner elements.** Denote the set of $a \in A_\theta^-$ such that $\theta\,\mathrm{Int}(a)$ is a k-involution of G by $I_k(A_\theta^-)$. This will be called the *set of k-inner elements of* A_θ^-.

Note that for any $a \in A_\theta^-$ the automorphism $\theta\,\mathrm{Int}(a)$ is an involution of G. So the question is for which $a \in A_\theta^-$ this involution is in fact a k-involution of G. Since θ is a k-automorphism this is equivalent to the condition that $\mathrm{Int}(a)$ is a k-automorphism of G. Combined with Proposition 6.10(3) we get now the following characterization of the set $I_k(A_\theta^-)$:

Lemma 8.17. *Let A be a maximal k-split torus of G, $\theta \in \mathrm{Aut}(G)$ a k-involution normally related to A, $T \supset A$ a θ-stable maximal k-torus of G, $K \supset k$ a finite splitting extension of T and $\Gamma = \mathrm{Gal}(K/k)$. Then $I_k(A_\theta^-) = \{a \in A_\theta^- \mid \alpha^\sigma(a) = \alpha(a)^\sigma$ for all $\alpha \in \Phi(T),\ \sigma \in \Gamma\}$.*

Remark 8.18. In Corollary 8.14 the isomorphy of the involutions of $Z_G(A)$ can be split in an isomorphy of the involutions restricted to a maximal torus containing A (i.e. a congruence of the corresponding (Γ, θ)-indices) and an isomorphy of the k-involutions of the k-anisotropic kernel $G_0 = [Z_G(A), Z_G(A)]$ of G. Unfortunately the isomorphy of the k-involutions of the k-anisotropic kernel G_0 of G is under the action of $\mathrm{Int}(G_k)$ and not under $N_{G_k}(A)$. Similarly for the isomorphy of the involutions $\theta\,\mathrm{Int}(a)$ with $a \in I_k(A_\theta^-)$ we also have to look at the action of G_k on $I_k(A_\theta^-)$ (acting via the θ-twisted action, see 2.3) instead of the action under $N_{G_k}(A)$. Namely if $g \in G$ and $a \in I_k(A_\theta^-)$ then $\mathrm{Int}(g)\theta\,\mathrm{Int}(a)\,\mathrm{Int}(g)^{-1} = \theta\,\mathrm{Int}(\theta(g)ag^{-1})$. If all maximal (θ, k)-split tori of G are H_k-conjugate then one can reduce this in both these cases to an action of $N_{G_k}(A)$ instead of G_k. The action of $N_{G_k}(A)$ can then be split in an action of the Weyl group and an action of the k-anisotropic kernel. This is for example the case when $k = \mathbb{R}$. Unfortunately in general the maximal (θ, k)-split tori are not H_k-conjugate and this creates a major complication in the characterization of these isomorphy classes. For example this means that two k-involutions normally related to A can be G_k-isomorphic, but their restrictions to $Z_G(A)$ are not isomorphic under $N_{G_k}(A)$. It turns out that we can restrict to the action of a slightly larger group then $N_{G_k}(A)$. This group will also be used in section 9 to characterize the isomorphy classes of the k-involutions $\theta\,\mathrm{Int}(a)$ for $a \in I_k(A_\theta^-)$ (or equivalently the θ-twisted orbits in $I_k(A_\theta^-)$). In the following we analyze all these complications and describe this extension of the Weyl group.

8.19. For the remainder of this section we fix a maximal k-split torus A of G, θ a k-involution of G normally related to A and we write $Z = Z_G(A)$ and $N = N_G(A)$. Let $\mathcal{F}(A, \theta)$ be the set of restrictions to $Z_G(A)$ of the k-involutions of G, normally related to A, which are isomorphic to θ under G_k. So $\mathcal{F}(A, \theta) = \{\sigma | Z_G(A) \mid \sigma \in \mathrm{Aut}_k(G),\ \sigma^2 = \mathrm{id},\ \sigma(A) = A,\ \sigma = \mathrm{Int}(g)\theta\,\mathrm{Int}(g)^{-1}$ for some $g \in G_k\}$. Although all these involutions are isomorphic under G_k, their restrictions to $Z_G(A)$ can give a number of isomorphy classes depending on the H_k-conjugacy classes of maximal (θ, k)-split tori. A description of the N_k-isomorphy classes in $\mathcal{F}(A, \theta)$ can be obtained as follows:

Let $\{v_i \mid i \in I\}$ be a set of representatives of $V_1/W(A, H)$ and let $\{x(v_i) \in (Z_G(A)H)_k \mid i \in I\}$ be a set of representatives for the $\{v_i \mid i \in I\}$ in $(Z_G(A)H)_k$. For each $i \in I$ write $x(v_i) = z_i h_i$ with $z_i \in Z_G(A)$ and $h_i \in H$. Then $\mathcal{H}(A, \theta) = \{h_i^{-1} A h_i \mid i \in I\}$ is a set of representatives for the H_k-conjugacy classes of maximal k-split tori containing a maximal (θ, k)-split torus. Let $Z(A, \theta) = \{z_i \mid i \in I\}$ and let $\mathcal{C}(A, \theta) = \{\theta\,\mathrm{Int}(\theta(z_i)z_i^{-1}) \mid i \in I\}$. This is a set of representatives for the N_k-isomorphy classes of k-involutions of G, normally related to A, which are G_k-isomorphic but not N_k-isomorphic. The above observations lead to the following result.

Proposition 8.20. *$\mathcal{C}(A, \theta)$ is a set of representatives for the N_k-isomorphy classes in $\mathcal{F}(A, \theta)$.*

Proof. Let $g \in G_k$ such that $\mathrm{Int}(g)\theta\,\mathrm{Int}(g)^{-1} \in \mathcal{F}(A, \theta)$. Let $S = g^{-1}Ag$. Then S is θ-stable and S_θ^- is maximal (θ_2, k)-split. By Proposition 3.1 there exists $h \in \mathcal{H}(A, \theta)$, $z \in Z(A, \theta)$ such that $zh \in (Z_G(A)G_\theta^0)_k$ and $h^{-1}z^{-1}Azh = g^{-1}Ag$. It follows that $n = gh^{-1}z^{-1} \in N_{G_k}(A)$. So we may assume that $g = zh$. But then $\mathrm{Int}(g)\theta\,\mathrm{Int}(g)^{-1} = \theta\,\mathrm{Int}(\theta(z)z^{-1})$ what proves the result. ☐

8.21. The $Z_{G_k}(A) \times G_\theta(k)$ orbits in $(Z_G(A)G_\theta)_k$ play an important role in the classification of k-involutions isomorphic to θ. However, as follows from Proposition 8.13 we have to consider also the involutions $\theta\,\mathrm{Int}(a)$ with $a \in I_k(A_\theta^-)$. So for these one needs the $Z_{G_k}(A) \times G_{\theta\,\mathrm{Int}(a)}(k)$ orbits in $(Z_G(A)G_{\theta\,\mathrm{Int}(a)})_k$.

In the following we look at the correspondence of these orbits for the involutions θ and $\theta\,\mathrm{Int}(a)$. First we look at the orbits which have a representative contained in $W_H(A)$. In the following let $a \in I_k(A_\theta^-)$ and $s \in A_\theta^-$ such that $s^2 = a^{-1}$. Write $H^a = G_{\theta\,\mathrm{Int}(a)}$. Then $H^a = sHs^{-1}$ and $(Z_G(A)H^a)_k = (Z_G(A)Hs^{-1})_k$. In particular we have the following:

Lemma 8.22. *Let A be a maximal k-split torus of G, θ a k-involution of G normally related to A, $a \in I_k(A_\theta^-)$, $s \in A_\theta^-$ such that $s^2 = a^{-1}$, $w \in W_H(A)$, $h \in N_H(A)$ representative of w and $z \in Z_G(A)$ such that $zh \in (Z_G(A)H)_k$. Then we have the following:*

1. *$h_a = shs^{-1} \in N_{H^a}(A)$ is a representative of w.*
2. *$W_H(A) = W_{H^a}(A)$.*
3. *If $z_a \in Z_G(A)$ such that $z_a h_a = zh \in N_{G_k}(A)$, then $z_a = zw(s)s^{-1}$.*
4. *There exists $h_1 \in H^a$ such that $zh_1 \in (Z_G(A)H^a)_k$ is a representative of w if and only if $w(s)s^{-1} \in Z_{G_k}(A)Z_H(A)$.*

Proof. Let $h_a = shs^{-1}$. Since $h \in N_H(A)$ and $s \in A$ it follows that $\mathrm{Int}(h_a)|A = \mathrm{Int}(h)|A$, so h_a is a representative of w as well.

(2) is immediate from (1).

(3). Let $x = hh_a^{-1} = hsh^{-1}s^{-1}$. Since $h_a = shs^{-1}$ and h are both representatives for w the element $x = w(s)s^{-1} \in Z_G(A)$. Then $z_a = zhh_a^{-1} = zx = zw(s)s^{-1}$, what proves the result.

(4). If $h_1 \in H^a$ such that $zh_1 \in (Z_G(A)H^a)_k$ is a representative of w, then there exists $z_1 \in Z_{G_k}(A)$ such that $z_1 zh_1 = zw(s)s^{-1}h_a$. Since $h_a h_1^{-1} \in Z_H(A)$ the result follows. □

Examples indicate that the sets $Z(A,\theta)$ and $Z(A,\theta\,\mathrm{Int}(a))$ can be identified, what would somewhat simplify the classification of the isomorphy classes of the k-involutions. We conjecture this result in the following:

Conjecture 8.23. *Let A be a maximal k-split torus of G, θ a k-involution of G normally related to A and $a \in I_k(A_\theta^-)$. There exists a set of representatives $\{z_i \mid i \in I\}$ for $Z(A,\theta)$ which is also a set of representatives for $Z(A,\theta\,\mathrm{Int}(a))$, i.e. for each $i \in I$ there exist $h_i^a \in H^a$ such that $x_i^a = z_i h_i^a \in (Z_G(A)H^a)_k$ and $\{(x_i^a)^{-1}Ax_i^a \mid i \in I\}$ is a set of representatives of the H_k^a-conjugacy classes of of maximal k-split tori containing a maximal $(\theta\,\mathrm{Int}(a),k)$-split torus.*

For special pairs (G,θ) the above complication does not occur. In particular from Corollary 3.12 it follows that we have the following result:

Corollary 8.24. *If (G,θ) is a special pair, then $Z(A,\theta) = \{\mathrm{id}\}$ and $\mathcal{C}(A,\theta) = \{\theta\}$.*

8.25. To avoid the above complication with the various sets $Z(A, \theta \operatorname{Int}(a))$ we use the following set instead. Let $\mathfrak{Z}(A, \theta) = \{z \in Z_G(A) \mid \exists\, a \in I_k(A_\theta^-) \text{ and } h \in G_{\theta \operatorname{Int}(a)} \text{ such that } zh \in (Z_G(A) G_{\theta \operatorname{Int}(a)})_k\}$. Then $\mathfrak{Z}(A, \theta) = Z_{G_k}(A)(\cup_{a \in I_k(A_\theta^-)} Z(A, \theta \operatorname{Int}(a)).Z_{G_{\theta \operatorname{Int}(a)}}(A))$. If the above conjecture holds, then $\mathfrak{Z}(A, \theta) = Z_{G_k}(A) Z(A, \theta) \cup_{a \in I_k(A_\theta^-)} (Z_{G_{\theta \operatorname{Int}(a)}}(A))$.

We have now the following characterization of the isomorphy classes of k-involutions:

Theorem 8.26. *Let G be connected semi-simple algebraic group defined over k, A a maximal k-split torus of G, $G_0 = [Z_G(A), Z_G(A)]$ the k-anisotropic kernel of G with respect to A and θ_1, θ_2 k-involutions of G, normally related to A. Then θ_1 is G_k-isomorphic (resp. $\operatorname{Int}_k(G)$ or $\operatorname{Aut}_k(G)$-isomorphic) with $\theta_2 \operatorname{Int}(a)$ for some $a \in I_k(A_{\theta_2}^-)$ if and only if the following conditions are satisfied:*

(1) *If $T_1, T_2 \subset Z_G(A)$ are maximal k-tori such that θ_i is normally related to T_i, $(i = 1, 2)$, $x \in Z_G(A)$ such that $xT_1x^{-1} = T_2$ and $\tilde{\theta}_1 = \operatorname{Int}(x)\theta_1 \operatorname{Int}(x^{-1})$, then $\tilde{\theta}_1|T_2$ and $\theta_2|T_2$ are $W(A, T_2)$ conjugate (resp. $\operatorname{Aut}(A, T_2)$ conjugate).*
(2) *$\theta_1|G_0$ is $N_{G_k}(A)$-isomorphic (resp. $\operatorname{Int}_k(G, A)$ or $\operatorname{Aut}_k(G, A)$-isomorphic) to $\theta_2 \operatorname{Int}(\theta_2(z_i)z_i^{-1})|G_0$ for some $z_i \in \mathfrak{Z}(A, \theta_2)$.*

Proof. Assume first that $\varphi \in \operatorname{Aut}_k(G)$ such that $\varphi\theta_1\varphi^{-1} = \theta_2 \operatorname{Int}(a)$ for some $a \in A_{\theta_2}^-$. Then $\varphi(A)$ is a maximal k-split torus of G containing a maximal $(\theta_2 \operatorname{Int}(a), k)$-split torus of G. By Proposition 3.1 there exist $z \in Z_G(A)$ and $h \in G^0_{\theta_2 \operatorname{Int}(a)}$ such that $zh \in (Z_G(A) G^0_{\theta_2 \operatorname{Int}(a)})_k$ such that $zh\varphi(A)h^{-1}z^{-1} = A$. Now $\operatorname{Int}(zh)\varphi \in \operatorname{Aut}_k(G, A)$ and $\operatorname{Int}(zh)\varphi\theta_1\varphi^{-1} \operatorname{Int}(zh)^{-1} = \theta_2 \operatorname{Int}(a) \operatorname{Int}(\theta_2(z)z^{-1})$. So $\operatorname{Int}(zh)\varphi\theta_1\varphi^{-1} \operatorname{Int}(zh)^{-1}|G_0 = \theta_2 \operatorname{Int}(\theta_2(z)z^{-1})|G_0$ what proves the second condition. Note that $\operatorname{Int}(zh)\varphi \in \operatorname{Int}(G_k, A)$ if and only if $\varphi \in \operatorname{Int}(G_k)$ and $\operatorname{Int}(zh)\varphi \in \operatorname{Int}_k(G, A)$ if and only if $\varphi \in \operatorname{Int}_k(G)$.

As for the first condition let $T_1, T_2 \subset Z_G(A)$ be maximal k-tori such that θ_i is normally related to T_i, $(i = 1, 2)$, $x \in Z_G(A)$ such that $xT_1x^{-1} = T_2$ and $\tilde{\theta}_1 = \operatorname{Int}(x)\theta_1 \operatorname{Int}(x^{-1})$. Write $T = \operatorname{Int}(h)\varphi(T_1)$. The element $\operatorname{Int}(h)\varphi$ is contained in $\operatorname{Aut}(G, A)$, so it follows that T is a $\theta_2 \operatorname{Int}(a)$-stable maximal torus of $Z_G(A)$ with $\theta_2 \operatorname{Int}(a)$ normally related to T. Since $\theta_2 \operatorname{Int}(a)$ is also normally related to T_2 there exists $h_1 \in G_{\theta_2 \operatorname{Int}(a)} \cap Z_G(A)$ such that $h_1 T h_1^{-1} = T_2$. So $\operatorname{Int}(h_1 h)\varphi \in \operatorname{Aut}(G, A)$ maps T_1 to T_2. Let $\varphi_1 = \operatorname{Int}(h_1 h)\varphi \operatorname{Int}(x)^{-1}$. Then $\varphi_1 \in \operatorname{Aut}(G, A, T_2) = \{\phi \in \operatorname{Aut}(G) \mid \phi(A) = A \text{ and } \phi(T_2) = T_2\}$ and $\varphi_1\tilde{\theta}_1|T_2 = \operatorname{Int}(h_1 h)\varphi\theta_1\varphi^{-1} \operatorname{Int}(h_1 h)^{-1}|T_2 = \theta_2 \operatorname{Int}(a)|T_2 = \theta_2|T_2$, what proves the first condition. Clearly $\varphi_1|T_2 \in W(A, T)$ if and only if $\varphi \in \operatorname{Int}(G)$.

Conversely let $\varphi_0 \in \mathrm{Aut}_k(G, A)$ and $z \in \mathfrak{Z}(A, \theta_2)$ such that $\varphi_0 \theta_1 \varphi_0^{-1} | G_0 = \theta_2 \,\mathrm{Int}(\theta_2(z)z^{-1}) | G_0$. From (1) it follows that $\varphi_0 \theta_1 \varphi_0^{-1} | A$ and $\theta_2 | A$ are $\mathrm{Aut}(A)$-conjugate. Let $\varphi_1 \in \mathrm{Aut}_k(G, A)$ such that $\varphi_1 \varphi_0 \theta_1 \varphi_0^{-1} \varphi_1^{-1} | A = \theta_2 | A$. Let $\varphi = \varphi_1 \varphi_0$. Then $\varphi \in \mathrm{Aut}_k(G, A)$ and $\varphi \theta_1 \varphi^{-1} | Z_G(A) = \theta_2 \,\mathrm{Int}(\theta_2(z)z^{-1}) | Z_G(A)$. Let $b \in I_k(A_\theta^-)$ and $h \in G_{\theta_2 \mathrm{Int}(b)}$ such that $x = zh \in (Z_G(A) G_{\theta \mathrm{Int}(b)})_k$. Take $s \in A_\theta^-)$ such that $s^2 = b^{-1}$. Then

$$
\begin{aligned}
\mathrm{Int}(x)^{-1} \varphi \theta_1 \varphi^{-1} \,\mathrm{Int}(x) | Z_G(A) &= \mathrm{Int}(h^{-1} z^{-1}) \theta_2 \,\mathrm{Int}(\theta_2(z) z^{-1}) \,\mathrm{Int}(zh) | Z_G(A) \\
&= \mathrm{Int}(h^{-1}) \theta_2 \,\mathrm{Int}(h) | Z_G(A) \\
&= \theta_2 \,\mathrm{Int}(\theta_2(h^{-1})) \,\mathrm{Int}(h) | Z_G(A) \\
&= \theta_2 \,\mathrm{Int}(b^{-1} h^{-1} b h) | Z_G(A) = \theta_2 | Z_G(A).
\end{aligned}
$$

But then by Proposition 8.13 there exists $a \in A_{\theta_2}^-$ such that $\mathrm{Int}(x)^{-1} \varphi \theta_1 \varphi^{-1} \,\mathrm{Int}(x) = \theta_2 \,\mathrm{Int}(a)$. Clearly $\mathrm{Int}(x)^{-1} \varphi \in \mathrm{Int}(G_k)$ if and only if $\varphi_0, \varphi_1 \in \mathrm{Int}(G_k, A)$ and $\mathrm{Int}(x)^{-1} \varphi \in \mathrm{Int}_k(G)$ if and only if $\varphi_0, \varphi_1 \in \mathrm{Int}_k(G, A)$. This proves the result. □

Theorem 8.26 shows that we cannot restrict to conjugation of the involutions under $N_{G_k}(A)$. We need to extend this group with a set of representatives for $Z(A, \theta)$. This then leads to another characterization of the isomorphy classes of k-involutions of G. We discuss this in the following:

Notation 8.27. Let A be a maximal k-split torus of G, θ a k-involution of G, normally related to A. Let $Z(A, \theta)$ be as in 8.19 and let

$$\mathcal{N}(A, \theta) = N_{G_k}(A).Z(A, \theta).$$

Similarly let

$$
\begin{aligned}
\mathrm{Int}(A, \theta) &= \mathrm{Int}_k(G, A).\,\mathrm{Int}(Z(A, \theta)), \\
\mathrm{Aut}(A, \theta) &= \mathrm{Aut}_k(G, A).\,\mathrm{Int}(Z(A, \theta)) \text{ and} \\
\mathcal{Z}(A, \theta) &= Z_{G_k}(A).Z(A, \theta).
\end{aligned}
$$

We note that $\mathcal{Z}(A, \theta) \subset \mathcal{N}(A, \theta) \subset N_G(A)$ and $\mathrm{Int}(A, \theta) \subset \mathrm{Aut}(A, \theta) \subset \mathrm{Aut}(G, A)$.

Remark 8.28. From Corollary 8.24 it follows that for special pairs (G, θ) we have $\mathcal{N}(A, \theta) = N_{G_k}(A)$. This is in particular the case for $k = \mathbb{R}$, where all pairs (G, θ) are special.

We have now the following equivalent characterizations of the isomorphy classes of k-involutions which induce the same k-involution of $Z_G(A)$:

Theorem 8.29. *Let A be a maximal k-split torus of G and θ_1, θ_2 k-involutions of G, normally related to A. The following are equivalent.*

(1) $\theta_2|Z_G(A)$ *and* $\theta_1|Z_G(A)$ *are isomorphic under* $\mathcal{N}(A,\theta_2)$ (*resp.* $\operatorname{Int}(A,\theta_2)$ *or* $\operatorname{Aut}(A,\theta_2)$).

(2) $\theta_2|Z_G(A)$ *and* $\theta_1|Z_G(A)$ *are isomorphic under* G_k (*resp.* $\operatorname{Int}_k(G)$ *or* $\operatorname{Aut}_k(G)$).

(3) θ_2 *is isomorphic under* G_k (*resp.* $\operatorname{Int}_k(G)$ *or* $\operatorname{Aut}_k(G)$) *with* $\theta_1\operatorname{Int}(a)$ *for some* $a \in A^-_{\theta_1}$.

(4) θ_2 *is isomorphic under* $\mathcal{N}(A,\theta_2)$ (*resp.* $\operatorname{Int}(A,\theta_2)$ *or* $\operatorname{Aut}(A,\theta_2)$) *with* $\theta_1\operatorname{Int}(a)$ *for some* $a \in A^-_{\theta_1}$.

Proof. We prove the result for G_k-isomorphy. The proof for isomorphy under $\operatorname{Int}_k(G))$ or $\operatorname{Aut}_k(G))$ follows with a similar argument.

(1) $\Longrightarrow$ (2). Assume $\theta_2|Z_G(A)$ and $\theta_1|Z_G(A)$ are isomorphic under $\mathcal{N}(A,\theta_1)$. Let $n \in N_{G_k}(A)$ and $z \in Z(A,\theta_2)$ be such that $\operatorname{Int}(nz)\theta_2\operatorname{Int}(nz)^{-1}|Z_G(A) = \theta_1|Z_G(A)$. This implies that

$$\operatorname{Int}(n)\theta_2\operatorname{Int}(\theta_2(z)z^{-1})\operatorname{Int}(n)^{-1}|Z_G(A) = \theta_1|Z_G(A).$$

Let $h \in \mathcal{H}(A,\theta_2)$ such that $zh \in (Z_G(A)G_{\theta_2})_k$. Then $nzh \in G_k$ satisfies

$$\operatorname{Int}(nzh)\theta_2\operatorname{Int}(nzh)^{-1}|Z_G(A) = \operatorname{Int}(nz)\theta_2\operatorname{Int}(nz)^{-1}|Z_G(A) = \theta_1|Z_G(A).$$

(2) $\Longrightarrow$ (3). Let $g \in G_k$ such that $\operatorname{Int}(g)\theta_2\operatorname{Int}(g)^{-1}|Z_G(A) = \theta_1|Z_G(A)$. From Proposition 8.13 it follows now that there exists an element $a \in A^-_{\theta_1}$ such that $\operatorname{Int}(g)\theta_2\operatorname{Int}(g)^{-1} = \theta_1\operatorname{Int}(a)$, what shows (3).

(3) $\Longrightarrow$ (4). Let $g \in G_k$ such that $\operatorname{Int}(g)\theta_2\operatorname{Int}(g)^{-1} = \theta_1\operatorname{Int}(a)$ for some $a \in A^-_{\theta_1}$. Let $S = g^{-1}Ag$. Then S is θ_2-stable and $S^-_{\theta_2}$ is maximal (θ_2,k)-split. By Proposition 3.1 there exists $h \in G^0_{\theta_2}$, $z \in Z_G(A)$ such that $hz \in (G^0_{\theta_2}Z_G(A))_k$ and $hzAz^{-1}h^{-1} = g^{-1}Ag$. By 8.19 we may assume that $z \in Z(A,\theta_2)$. It follows that $n = ghz \in N_{G_k}(A)$. Let $n_1 = nz^{-1} \in \mathcal{N}(A,\theta_2)$. Then $\operatorname{Int}(n_1)\theta_2\operatorname{Int}(n)^{-1} = \operatorname{Int}(gh)\theta_2\operatorname{Int}(gh)^{-1} = \operatorname{Int}(g)\theta_2\operatorname{Int}(g)^{-1} = \theta_1\operatorname{Int}(a)$.

Finally, since (4) $\Longrightarrow$ (1) is immediate, the result follows. ☐

From Theorem 8.29 and Corollary 8.24 it follows now that for special pairs we have the following result:

Corollary 8.30. *Let A be a maximal k-split torus of G and θ_1, θ_2 k-involutions of G, normally related to A. If the pairs (G,θ_1) and (G,θ_2) are special, then the following are equivalent.*

(1) $\theta_2|Z_G(A)$ *and* $\theta_1|Z_G(A)$ *are isomorphic under* $N_{G_k}(A)$.

(2) $\theta_2|Z_G(A)$ *and* $\theta_1|Z_G(A)$ *are isomorphic under* G_k.
(3) θ_2 *is isomorphic under* G_k *with* $\theta_1 \operatorname{Int}(a)$ *for some* $a \in A^-_{\theta_1}$.
(4) θ_2 *is isomorphic under* $N_{G_k}(A)$ *with* $\theta_1 \operatorname{Int}(a)$ *for some* $a \in A^-_{\theta_1}$.

8.31. Using the above results we can divide the characterization of the isomorphy classes of k-involutions in 3 parts. This can be seen as follows. Fix a maximal k-split torus A of G and write $Z = Z_G(A)$, $N = N_G(A)$. Denote the family of all k-involutions of G by $\mathcal{F}_k$ and the family of all k-involutions of G, which are normally related to A by $\mathcal{F}_k(A)$. Denote the set of G_k-isomorphism classes in $\mathcal{F}_k$ by $\mathcal{C}_k$. From Proposition 2.6 and the conjugacy of the maximal k-split tori of G it follows that every k-involution of G is G_k-isomorphic to one normally related to A, so every class in $\mathcal{C}_k$ has a representative in $\mathcal{F}_k(A)$.

Let $T \supset A$ be a maximal k-torus of G, $W(A, T) = \{w \in W(T) \mid w(A) = A\}$, $\mathcal{T}$ the set of $W(T)$-isomorphy classes of involutions of $(X^*(T), \Phi(T))$ and $\mathcal{T}(A)$ the set of $W(A, T)$-isomorphy classes of involutions of $(X^*(T), \Phi(T), \Phi(A))$. By Theorem 8.9 the N-isomorphy classes are related to conjugacy classes of admissible k-involutions. Denote the set of N-isomorphy classes of k-involutions in $\mathcal{F}_k(A)$ by $\mathcal{C}_k(A, G)$.

From the conjugacy of the maximal θ-split tori of G it follows then that every involution in $\mathcal{F}_k(A)$ is isomorphic under $Z_G(A)$ with one normally related to T. So we have a natural map

$$\rho_N : \mathcal{C}_k(A, G) \longrightarrow \mathcal{T}(A).$$

From Theorem 8.9 it follows that ρ_N is one to one. Denote the image of ρ_N by $\mathcal{T}_0(A)$. These are the $W(A, T)$-isomorphy classes of admissible k-involutions, which by Theorem 8.9 can be described by a (Γ, θ)-index (see also 10.35).

8.32. The next step is to determine when two k-involutions in $\mathcal{F}_k(A)$ which are N-isomorphic are in fact G_k-isomorphic. For this we note first that there exist also a natural map of $\mathcal{C}_k$ into $\mathcal{C}_k(A, G)$. This can be seen as follows. If $\theta_1, \theta_2 \in \mathcal{F}_k(A)$ are isomorphic under G_k, then $\theta_1 = \operatorname{Int}(g)\theta_2 \operatorname{Int}(g)^{-1}$ for some $g \in G_k$. The torus $\tilde{A} = gAg^{-1}$ is maximal k-split, θ_1-stable and $\tilde{A}^-_{\theta_1}$ is maximal (θ_1, k)-split. By Proposition 3.1 there exists $x \in (ZG_{\theta_1})_k$ such that $x\tilde{A}x^{-1} = A$. Write $x = zh$ with $z \in Z$ and $h \in \theta_1$. Then $hgAg^{-1}h^{-1} = A$, so $hg \in N_G(A)$. Since $\theta_1 = \operatorname{Int}(hg)\theta_2 \operatorname{Int}(hg)^{-1}$ it follows that θ_1 and θ_2 are N-conjugate. This defines a natural map of $\mathcal{C}_k$ into $\mathcal{C}_k(A, G)$.

Using Theorem 8.29 we can split this map into two parts. Let $\mathcal{F}_k(A, Z) = \{\theta|Z \in \operatorname{Aut}(Z, G) \mid \theta \in \mathcal{F}_k(A)\}$ the restrictions of the k-involutions in $\mathcal{F}_k(A)$ to $Z_G(A)$. So we essentially identify all the involutions $\theta \operatorname{Int}(a)$, $(a \in A^-_\theta)$. Let

$\mathcal{C}_k(Z, G)$ denote the isomorphy classes of the involutions in $\mathcal{F}_k(A, Z)$, which are isomorphic under G_k. By Theorem 8.29 these are exactly the $\mathcal{N}(A, \theta)$-isomorphy classes of involutions in $\mathcal{F}_k(A, Z)$. We have now natural maps from $\mathcal{C}_k$ to $\mathcal{C}_k(Z, G)$ and from $\mathcal{C}_k(Z, G)$ to $\mathcal{C}_k(A, G)$. This can be seen as follows. If $\theta_1, \theta_2 \in \mathcal{F}_k(A, Z)$ are $\mathcal{N}(A, \theta)$-isomorphic, then by Theorem 8.29 there exists $g \in \mathcal{N}(A, \theta_1)$ and $a \in A^-_{\theta_1}$ such that $\operatorname{Int}(g)\theta_2 \operatorname{Int}(g)^{-1} = \theta_1 \operatorname{Int}(a)$. Let $s \in A^-_{\theta_1}$ with $s^2 = a$. Then $\operatorname{Int}(s)\theta_1 \operatorname{Int}(a) \operatorname{Int}(s)^{-1} = \theta_1 \operatorname{Int}(s^{-2}a) = \theta_1$. Since $g \in \mathcal{N}(A, \theta_1)$ and $a \in A$, it follows that θ_1 and θ_2 are N-conjugate, hence we have a natural map $\nu : \mathcal{C}_k(Z, G) \to \mathcal{C}_k(A, G)$. This map is clearly surjective and its fibers are essentially the G_k-isomorphy classes of k-involutions of $Z_G(A)$ (coming from involutions of G), which give the same N-isomorphy class.

Finally we also have a natural map from $\mathcal{C}_k$ to $\mathcal{C}_k(Z, G)$ by taking restrictions of k-involutions in $\mathcal{F}_k(A)$ to $Z_G(A)$ (i.e. the restriction map from $\mathcal{F}_k(A)$ to $\mathcal{F}_k(A, Z)$). Denote this map by μ. The fibers of μ can be characterized by a set of k-inner elements $\{a_i \in I_k(A^-_\theta) \mid i \in I\}$. Summarized we have now the following sequence

$$\mathcal{C}_k \xrightarrow{\mu} \mathcal{C}_k(Z, G) \xrightarrow{\nu} \mathcal{C}_k(A, G) \xrightarrow{\rho_N} \mathcal{T}(A) \tag{8.32.1}$$

For a k-involution θ of G, normally related to A we denote its G_k-isomorphy class (or equivalently $\mathcal{N}(A, \theta)$-isomorphy class) in $\mathcal{C}_k$ (resp. $\mathcal{C}_k(Z, G)$) by $[\theta]$ (resp. $[\theta]_Z$) and its N-isomorphy class in $\mathcal{C}_k(A, G)$ by $[\theta]_N$. For an admissible k-involution θ we denote the k-involution in $\operatorname{Aut}_k(G)$ representing the isomorphy class $\rho_N^{-1}(\theta) = [\theta]_N$ in $\mathcal{C}(A, G)$ also by θ. Denote the fiber of ν above $[\theta]_N = \rho_N^{-1}(\theta)$ by $\mathcal{C}(\theta) = \nu^{-1}\rho_N^{-1}(\theta)$. Finally for an isomorphy class $[\theta]_Z \in \mathcal{C}_k(Z, G)$ denote the fiber of μ by $\mathcal{C}_A(\theta)$. For isomorphy classes of k-involutions under $\operatorname{Int}_k(G)$ (resp. $\operatorname{Aut}_k(G)$) we have a similar characterization and we write $\bar{\mathcal{C}}$, $\bar{\mathcal{F}}$, $\bar{\mathcal{T}}(A)$ (resp. $\tilde{\mathcal{C}}$, $\tilde{\mathcal{F}}$, $\tilde{\mathcal{T}}(A)$) instead of $\mathcal{C}$, $\mathcal{F}$, $\mathcal{T}(A)$.

The above results give us now the following characterization of the isomorphy classes of k-involutions.

Theorem 8.33. *Let A be a maximal k-split torus of G and $T \supset A$ a maximal k-torus of G. Write $Z = Z_G(A)$, $N = N_G(A)$, $X = X^*(T)$ and $\Phi = \Phi(T)$.*

(1) *There is a bijection between the $W(A, T)$-isomorphy classes of admissible k-involutions of $(X, \Phi, \Phi(A))$ and the N-isomorphy classes of k-involutions in $\mathcal{C}_k(A, G)$.*
(2) *The G_k-isomorphy classes in $\mathcal{C}(\theta)$ (θ an admissible k-involution of $(X, \Phi, \Phi(A))$) consist of $\{[\theta_i]_Z \mid i \in I\}$, where the θ_i are representatives of the G_k-isomorphy classes (or $\mathcal{N}(A, \theta)$-isomorphy classes) of k-involutions of Z, which are N-isomorphic to θ.*

(3) *The isomorphy classes in* $\mathcal{C}_A(\theta_i)$ ($[\theta_i]_Z \in \mathcal{C}(\theta)$*, with* θ *an admissible* k*-involution*) *are represented by a set of* k*-inner elements* $\{a_{i,j} \in I_k(A_\theta^-) \mid j \in J_i\}$.

Remark 8.34. The above result reduces the classification of k-involutions of G to the following 3 problems.

(1) a classification of admissible k-involutions.
(2) a classification of k-involutions of k-anisotropic semisimple groups.
(3) for each k-involution of $(G, Z_G(A))$ a classification of the k-inner elements characterizing the isomorphy classes in $\mathcal{C}_A(\theta)$.

Corollary 8.35. *Let* A *be a maximal* k*-split torus of* G *and* $T \supset A$ *a maximal* k*-torus of* G*. Write* $Z = Z_G(A)$, $N = N_G(A)$, $X = X^*(T)$ *and* $\Phi = \Phi(T)$*. If* $\theta \in \mathrm{Aut}(X, \Phi)$ *is a special admissible* k*-involution, then the* G_k*-isomorphy classes in* $\mathcal{C}(\theta)$ *consist of* $\{[\theta_i]_Z \mid i \in I\}$*, where the* θ_i *are representatives of the* $N_{G_k}(A)$*-isomorphy classes of* k*-involutions of* Z*, which are* N*-isomorphic to* θ*.*

Remarks 8.36. (1) The isomorphy classes of admissible k-involutions can be represented by a (Γ, θ)-index. A classification of these for a number of base fields, including finite fields, number fields, $\mathfrak{p}$-adic fields and the real numbers will be discussed in more detail in 10.35.

(2) A classification of the k-inner elements in $I_k(A_\theta^-)$ representing the isomorphy classes in $\mathcal{C}_A(\theta_i)$ (see Theorem 8.33(3)) depends on the base field k and for general k a classification of these is a difficult problem. The group G_k acts on $I_k(A_\theta^-)$ with the θ-twisted action as in 2.3. A characterization of these θ-twisted orbits will be discussed in the next section. For most fields k the group G_k has infinitely many orbits in $I_k(A_\theta^-)$ (see for example 9.9). If k is a finite field, $\mathfrak{p}$-adic field or the real numbers then there are only finitely many G_k-orbits in $I_k(A_\theta^-)$ and a classification is visible. For $k = \mathbb{R}$ the G_k-orbits in $I_k(A_\theta^-)$ were classified in [Hel88, §8]. The classification of the G_k-orbits in $I_k(A_\theta^-)$ for $k = \mathbb{Q}_p$ will be dealt with in a future paper.

(3) The classification of the k-inner elements is somewhat simpler in a number of cases. This includes the case when G_θ is k-anisotropic, k-split, θ-split or (θ, k)-split (i.e. a maximal (θ, k)-split torus of G is also maximal θ-split). The main reason for this is that the description of the $Z_k \times H_k$-orbits in $(Z_G(A)H)_k$ is simpler and the underlying geometry is more transparent. For more details on these cases, see [HW93].

8.37. **Involutions of compact real groups.** For $k = \mathbb{R}$ there is a one to one correspondence between isomorphy classes of k-anisotropic semisimple groups and isomorphy classes of complex semisimple groups (see 7.13). For involutions of compact groups we have a similar correspondence. This can be seen as follows. If θ is an involution of a complex group G, then there exists a conjugation σ of a compact real form U of G such that $\theta\sigma = \sigma\theta$ (see [Hel88, 10.3]). Then $\theta|U$ is an involution of U. Conversely any involution of U can be lifted to an involution of G by extending the base field. It is easy to show then that there exists a one to one correspondence between isomorphy classes of involutions of k-anisotropic semisimple groups and isomorphy classes of involutions of complex semisimple groups (see [Hel78, Chap. X, 1.4]). By Theorem 7.3 the latter are characterized by isomorphy classes of admissible involutions. This means that the classification of the k-involutions reduces to the first and third problem in 8.34. A classification of the isomorphy classes of k-involutions, for $k = \mathbb{R}$, together with all the fine structure, can be found in [Hel88].

9. Isomorphy of k-involutions related to an admissible involution

In this section we analyze the isomorphy of the involutions $\theta\,\mathrm{Int}(a)$, $a \in I_k(A_\theta^-)$, which characterize the isomorphy classes in the set $\mathcal{C}_A(\theta)$ as in 8.32.

9.1. Let A be a maximal k-split torus of G and θ a k-involution of G, normally related to A. Let $\mathcal{F}_A(\theta) = \{\theta\,\mathrm{Int}(a) \in \mathcal{F}_k(A) \mid a \in I_k(A_\theta^-)\}$. By Proposition 8.13 every class in $\mathcal{C}_A(\theta)$ has a representative in $\mathcal{F}_A(\theta)$.

If $g \in G_k$ such that $\mathrm{Int}(g)\theta\,\mathrm{Int}(a_1)\,\mathrm{Int}(g)^{-1} = \theta\,\mathrm{Int}(a_2)$ for $a_1, a_2 \in I_k(A_\theta^-)$, then $\theta\,\mathrm{Int}(\theta(g)a_1g^{-1}) = \theta\,\mathrm{Int}(a_2)$. So modulo the center of G we have $\theta(g)a_1g^{-1} = a_2$. In order to characterize the G_k-isomorphy classes in $\mathcal{F}_A(\theta)$ one needs to find representatives for the θ-twisted G_k-orbits in $I_k(A_\theta^-)$. Unfortunately $I_k(A_\theta^-)$ is not stable under the θ-twisted action of G_k. It can happen that $a \in I_k(A_\theta^-)$, $g \in G_k$ and the element $\theta(g)ag^{-1} \not\in A_\theta^-$, so also not in $I_k(A_\theta^-)$. The $g \in G_k$ which stabilize $I_k(A_\theta^-)$ under the θ-twisted action are essentially contained in $(Z_G(A)H^0)_k$ as follows from the following result:

Proposition 9.2. *Let A be a maximal k-split torus of G, θ a k-involution of G, normally related to A and $g \in G_k$ such that* $\mathrm{Int}(g)\theta\,\mathrm{Int}(g)^{-1} = \theta\,\mathrm{Int}(a)$ *for some $a \in Z(Z_G(A))$. Then $g \in (Z_G(A)H^0)_k$.*

Proof. Since $\mathrm{Int}(g)\theta\,\mathrm{Int}(g)^{-1} = \theta\,\mathrm{Int}(\theta(g)g^{-1}) = \theta\,\mathrm{Int}(a)$ it follows that $\theta(g)g^{-1} \in Z_{G_k}(A)$. Let $P \supset A$ be a minimal θ-split parabolic k-subgroup of G and $P_0 \subset P$ a minimal parabolic k-subgroup containing A. Then by [HW93, 4.9 and 9.2] $P_0H \subset G$ open. Let $P_1 = gPg^{-1}$ and $A_1 = gAg^{-1}$. If $x \in A$, then $\theta(gxg^{-1}) = \theta(g)\theta(x)\theta(g)^{-1} = g\theta(x)g^{-1}$. It follows that $(A_1)_\theta^- = gA_\theta^-g^{-1}$ is a maximal (θ, k)-split torus of G. With a similar argument it follows that

$\theta(P_1) \cap P_1 = Z_G(A_1)$. Hence P_1 is also θ-split and $gP_0g^{-1} \subset P_1$ is a minimal parabolic k-subgroup containing A_1. Again by [HW93, 4.9 and 9.2] $gP_0g^{-1}H \subset G$ open. But then $P_0g^{-1}H = P_0H$ and hence $g^{-1} \in P_0H$. With a similar argument one shows that also $g \in P_0H$.

On the other hand we also have $\tau(g) = \theta(g)g^{-1} \in Z_{G_k}(A)$, hence $g \in \tau^{-1}(N_G(A))_k$. Let $U = R_u(P)$ be the unipotent radical of P. Write $g = uzh$ with $u \in U$, $z \in Z_G(A)$ and $h \in H$. Then $\theta(g)g^{-1} = \theta(u)\theta(z)z^{-1}u^{-1} = n \in N_G(A)$, hence $Uz\theta(z)^{-1}\theta(U) = Un\theta(U)$. By [BT65, 5.15], $z\theta(z)^{-1} = n$ and as a consequence $z^{-1}uz \in U^\theta$. Since $U^\theta \subset H^0$ it follows that $g = z(z^{-1}uz)h \in Z_G(A)H$. □

This result leads to the following characterization of when a k-involution in $\mathcal{F}_A(\theta)$ is G_k-isomorphic to θ.

Corollary 9.3. *Let A be a maximal k-split torus of G, θ a k-involution of G normally related to A and $a \in I_k(A_\theta^-)$. Then the following are equivalent:*

(1) *θ and $\theta \operatorname{Int}(a)$ are isomorphic under G_k.*
(2) *θ and $\theta \operatorname{Int}(a)$ are isomorphic under $(Z_G(A)G_\theta)_k$.*
(3) *There exists $z \in Z(G)$ such that $az \in \tau(G_k)$.*

Proof. The equivalence of (1) and (2) is immediate from Proposition 9.2, so we show the equivalence of (1) and (3). If θ and $\theta \operatorname{Int}(a)$ are isomorphic under G_k, then $\operatorname{Int}(g)\theta \operatorname{Int}(g)^{-1} = \theta \operatorname{Int}(a)$ for some $g \in G_k$. But then $\operatorname{Int}(\theta(g)g^{-1}) = \operatorname{Int}(a)$, hence $\theta(g)g^{-1} = az$ for some $z \in Z(G)$.

Conversely assume there exists $z \in Z(G)$ such that $az \in \tau(G_k)$. Let $g \in G_k$ such that $\tau(g) = az$. Then $\operatorname{Int}(g)\theta \operatorname{Int}(g)^{-1} = \theta \operatorname{Int}(\theta(g)g^{-1}) = \theta \operatorname{Int}(az) = \theta \operatorname{Int}(a)$, what proves the result. □

To determine the isomorphy of involutions $\theta \operatorname{Int}(a)$ and $\theta \operatorname{Int}(b)$ with $a, b \in I_k(A_\theta^-)$ one needs to consider the action of the set $(Z_G(A)G_{\theta \operatorname{Int}(a)})_k$ and not $(Z_G(A)G_\theta)_k$. Combined with the above result we get the following:

Corollary 9.4. *Let A be a maximal k-split torus of G, θ a k-involution of G normally related to A and $a, b \in I_k(A_\theta^-)$. Then the following are equivalent:*

(1) *$\theta \operatorname{Int}(a)$ and $\theta \operatorname{Int}(b)$ are isomorphic under G_k.*
(2) *$\theta \operatorname{Int}(a)$ and $\theta \operatorname{Int}(b)$ are isomorphic under $(Z_G(A)G_{\theta \operatorname{Int}(a)})_k$.*
(3) *There exists $z \in Z(G)$ such that $bz \in \tau_{\theta \operatorname{Int}(a)}(G_k)$.*

For a special pair the above result can be sharpened as follows:

Corollary 9.5. *Let (G, θ) be a special pair, A a maximal k-split torus of G, such that θ is normally related to A and assume $Z(G) = \{e\}$. If $a, b \in I_k(A_\theta^-)$, then the following are equivalent:*

(1) *$\theta\,\mathrm{Int}(a)$ and $\theta\,\mathrm{Int}(b) \in \mathcal{F}_A(\theta)$ are G_k-isomorphic.*
(2) *$\theta\,\mathrm{Int}(a)$ and $\theta\,\mathrm{Int}(b) \in \mathcal{F}_A(\theta)$ are isomorphic under $(Z_G(A)G_{\theta\,\mathrm{Int}(a)})_k \cap N_{G_k}(A)$.*
(3) *There exists $w \in W_{G_{\theta\,\mathrm{Int}(a)}}(A)$ such that $b = \tau_{\theta\,\mathrm{Int}(a)}(zh) = \tau_\theta(zh)w(a)a^{-1} \in \tau_\theta(G_k)w(a)a^{-1}$ where $zh \in N_{G_k}(A)$ is representative for $w \in W_{G_{\theta\,\mathrm{Int}(a)}}(A)$.*

Proof. If $\theta\,\mathrm{Int}(a)$ and $\theta\,\mathrm{Int}(b) \in \mathcal{F}_A(\theta)$ are G_k-isomorphic, then by Proposition 9.2 they are also isomorphic under $(Z_G(A)G_{\theta\,\mathrm{Int}(a)})_k$. Let $x = zh \in (Z_G(A)G_{\theta\,\mathrm{Int}(a)})_k$ such that $\mathrm{Int}(x)\theta\,\mathrm{Int}(a)\,\mathrm{Int}(x)^{-1} = \theta\,\mathrm{Int}(b)$. Here $z \in Z_G(A)$ and $h \in G_{\theta\,\mathrm{Int}(a)}$. Then since (G, θ) is a special pair there exists $h_1 \in G_{\theta\,\mathrm{Int}(a)}(k)$ such that $hh_1 \in W_{G_{\theta\,\mathrm{Int}(a)}}(A)$. But then $xh_1 \in (Z_G(A)G_{\theta\,\mathrm{Int}(a)})_k \cap N_{G_k}(A)$ and $\mathrm{Int}(xh_1)\theta\,\mathrm{Int}(a)\,\mathrm{Int}(xh_1)^{-1} = \theta\,\mathrm{Int}(b)$, what proves (1) implies (2).

Assume next that $x = zh \in (Z_G(A)G_{\theta\,\mathrm{Int}(a)})_k \cap N_{G_k}(A)$ such that $\theta\,\mathrm{Int}(b) = \mathrm{Int}(x)\theta\,\mathrm{Int}(a)\,\mathrm{Int}(x)^{-1}$. Here $z \in Z_G(A)$ and $h \in N_{G_{\theta\,\mathrm{Int}(a)}}(A)$. Let $w \in W_H(A)$ be the corresponding Weyl group element. From Corollary 9.3 it follows that $b = \tau_{\theta\,\mathrm{Int}(a)}(zh) = \theta(z)z^{-1}$. Since $\theta(h) = aha^{-1}$ we get

$$\tau_\theta(zh) = \theta(z)\theta(h)h^{-1}z^{-1} = \theta(z)aha^{-1}h^{-1}z^{-1} = \theta(z)z^{-1}aw(a)^{-1},$$

what proves (2) implies (3).

Finally if $x = zh \in (Z_G(A)G_{\theta\,\mathrm{Int}(a)})_k \cap N_{G_k}(A)$ such that $b = \tau_{\theta\,\mathrm{Int}(a)}(zh)$, then $\mathrm{Int}(zh)\theta\,\mathrm{Int}(a)\,\mathrm{Int}(zh)^{-1} = \theta\,\mathrm{Int}(b)$ what proves (3) implies (1). □

The equivalence of (2) and (3) also follows from Lemma 8.22.

Remark 9.6. The classification of the G_k-isomorphy classes in $\mathcal{F}_A(\theta)$ is independent of the center of G. This can be seen as follows. Let A be a maximal k-split torus of G, θ a k-involution of G normally related to A and let $a \in A_\theta^-$ such that $\theta\,\mathrm{Int}(a)$ is a k-involution of G. Let $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ denote the adjoint representation of G, $\tilde{G} = \mathrm{Ad}(G)$, $\tilde{A} = \mathrm{Ad}(A)$, $\tilde{\theta}$ the induced k-involution of $\tilde{G}$ and $\tilde{a} = \mathrm{Ad}(a)$. Then θ and $\theta\,\mathrm{Int}(a)$ are isomorphic under G_k if and only if $\tilde{\theta}$ and $\tilde{\theta}\,\mathrm{Int}(\tilde{a})$ are isomorphic under $\tilde{G}_k$.

For the remainder of this section we will assume that G is adjoint, i.e. $Z(G) = \{e\}$.

Although the isomorphy of the k-inner elements depends on the involutions $\theta\,\mathrm{Int}(a)$ we can limit the possible representatives to a set which does not depend on $\theta\,\mathrm{Int}(a)$:

Lemma 9.7. *Let A be a maximal k-split torus of G, θ a k-involution of G normally related to A. Then $A_\theta^-(k)^2 \subset A_\theta^-(k) \cap \tau_{\theta\,\mathrm{Int}(a)}(G_k)$ for all $a \in I_k(A_\theta^-)$.*

Proof. Let $b \in A_\theta^-(k)$, then $\theta \operatorname{Int}(a)(b)b^{-1} = \theta(b)b^{-1} = b^{-2} \in A_\theta^-(k) \cap \tau(G_k)$, what proves the result. □

Remarks 9.8. (1) It follows from this result that one can find a set of representatives for the G_k-isomorphy classes in $\mathcal{F}_A(\theta)$ in the set $I_k(A_\theta^-)/A_\theta^-(k)^2$. For many base fields k, like the real numbers and $\mathfrak{p}$-adic numbers this set is finite so a classification becomes feasible.

(2) If $Z(G) = \{e\}$, then an involution $\theta \operatorname{Int}(a)$ with $a \in I_k(A_\theta^-)$ can only be conjugate to θ if $a \in A_\theta^-(k)$. The other k-inner elements in $I_k(A_\theta^-)$ definitely give involutions which are not isomorphic to θ. So for the isomorphy of the involutions $\theta \operatorname{Int}(a)$ and θ it suffices to consider $A_\theta^-(k)/A_\theta^-(k)^2$. We note that essentially $A_\theta^-(k)/(A_\theta^-(k))^2 \simeq (k^*/(k^*)^2)^n$, where $n = \operatorname{rank}(A_\theta^-)$. If k is a real closed field (i.e. k is formally real, but has no formally real proper algebraic extension field, see [Pre84, 3.2]), then $k^*/(k^*)^2 \simeq \{\pm 1\}$. Recall that a field is called formally real if -1 is not the sum of squares (see [Bec82, Pre84]). Of course $k = \mathbb{R}$ is real closed.

If $k = \mathbb{Q}_p$ is a $\mathfrak{p}$-adic field and p is odd, then $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ contains four elements represented by $1, \epsilon, p, \epsilon p$ where $1 < \epsilon < p$ and ϵ is not a square modulo p. If $p = 2$, then $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$ consists of eight elements represented by $1, 3, 5, 7, 2, 6, 10, 14$.

If $k = \mathbb{F}_{p^n}$ is a finite field of order p^n with p prime and n odd, then $k^*/(k^*)^2$ consists of 2 elements (see [Sch85, Lemma 3.7]).

In most cases it does not suffice to only mod out $A_\theta^-(k)^2$ to determine representatives for the isomorphy classes in $\mathcal{F}_A(\theta)$ as follows from the following example.

Example 9.9. Let $G = \mathrm{SL}_2(k)$, $\theta(x) = {}^t x^{-1}$ and A the group of diagonal matrices. Then G, A and θ are defined over k and A a maximal (θ, k)-split torus of G, which is also maximal k-split. The fixed point group of θ is $H = G_\theta = \mathrm{SO}_2(k) = \{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \mid a, b \in k,\ a^2 + b^2 = 1\}$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(k)$, then $\theta(g) = (\det g)^{-1}\left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix}\right)$ and

$$\theta(g)g^{-1} = (\det g)^{-2} \begin{pmatrix} c^2+d^2 & -(ac+bd) \\ -(ac+bd) & a^2+b^2 \end{pmatrix}.$$

So if $g \in \mathrm{SL}_2(k)$, then $\theta(g)g^{-1} \in A^- = A = N_{Z_G(A^-)}(A)$ if and only if $ac + bd = 0$. If $ac \neq 0$ then g is of the form $\left(\begin{smallmatrix} a & at \\ -dt & d \end{smallmatrix}\right)$ with $t = \frac{b}{a} = -\frac{c}{d} \in k$ and $ad(1 + t^2) = \det g = 1$. In this case we have:

$$\theta(g)g^{-1} = \begin{pmatrix} d^2(1+t^2) & 0 \\ 0 & a^2(1+t^2) \end{pmatrix}. \tag{9.9.1}$$

If $ac = 0$ then either $g = \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}$ and $\theta(g)g^{-1} = \begin{pmatrix} 0 & b^{-2} \\ 0 & b^2 \end{pmatrix}$ or $g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $\theta(g)g^{-1} = \begin{pmatrix} 0 & a^{-2} \\ 0 & a^2 \end{pmatrix}$.

Let $q = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in I_k(A_\theta^-)$. Then $\mathrm{Int}(q)(g) = \begin{pmatrix} a & bx^2 \\ cx^{-2} & d \end{pmatrix} \in \mathrm{GL}_2(k)$ if and only if $x^2 \in k$. By Corollary 9.3 the involution $\theta\,\mathrm{Int}(q)$ is isomorphic to θ under G_k if and only if there exists $g \in \mathrm{SL}_2(k)$ such that $\theta(g)g^{-1} = qz$ with $z \in Z(G) = \{\pm\,\mathrm{id}\}$, i.e. $\theta(g)g^{-1}q^{-1} \in Z(G)$. Since by (9.9.1) $\theta(g)g^{-1} = \begin{pmatrix} d^2(1+t^2) & 0 \\ 0 & a^2(1+t^2) \end{pmatrix} \begin{pmatrix} d^2(1+t^2) & 0 \\ 0 & a^2(1+t^2) \end{pmatrix}$ it follows that $\pm x \in k$ is a sum of two squares in k.

If $k = \mathbb{R}$, then $\mathbb{R}/\mathbb{R}^2 \simeq \{\pm 1\}$. Take $g = \begin{pmatrix} |x| & 0 \\ 0 & |x^{-1}| \end{pmatrix}$. Since $x \in \mathbb{R}$ it follows that $\theta(g)g^{-1}q^{-1} = \mathrm{id} \in Z(G)$ if $x > 0$ and $\theta(g)g^{-1}q^{-1} = -\,\mathrm{id} \in Z(G)$ if $x < 0$. So $\theta\,\mathrm{Int}(q)$ is G_k-isomorphic to θ. If $x \notin \mathbb{R}$ then $\theta\,\mathrm{Int}(q)$ is not G_k-isomorphic to θ. In that case, since $x^2 \in \mathbb{R}$, we have $x = iy$ for some $y \in \mathbb{R}$. In particular $\theta(g)g^{-1}q^{-1} = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

If $k = \mathbb{F}_3$, then $(k^*)^2 = \{1\}$. Let $q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Then $q \notin A_k^2$, but if we take $a = 1$, $d = 2$ and $t = 1$ in 9.9.1, then $g = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $q = \theta(g)g^{-1} \in \tau(G_k) \cap A_\theta^-$.

If $k = \mathbb{F}_5$, then $(k^*)^2 = \{1, 4\}$. Let $q = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Then $q \notin A_k^2$, but $q = \theta(g)g^{-1} \in \tau(G_k) \cap A_\theta^-$, where $g = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$.

If $k = \mathbb{F}_7$, then $(k^*)^2 = \{1, 4, 2\}$. Let $q_1 = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$ and $q_2 = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. Then $q_1, q_2 \notin A_k^2$, but $q_1 = \theta(g_1)g_1^{-1}$ and $q_2 = \theta(g_2)g_2^{-1}$, where $g_1 = \begin{pmatrix} -1 & 2 \\ 1 & 4 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$.

In general for $k = \mathbb{F}_p$, with p prime, one can show that $A_\theta^-(k) = \tau(G_k) \cap A_\theta^-$, since every element in $k = \mathbb{F}_p$ can be written as a sum of 2 squares (see e.g. [Sch85, Lemma 3.7]).

If $k = \mathbb{Q}$ one can get involutions $\theta\,\mathrm{Int}(q)$ with $q \in \tau(G_k).Z(G) \cap A_\theta^-(k)$, which are not G_k-isomorphic to θ. Take $x \in \mathbb{Q}$ such that x is not the sum of 2 squares. Then $\theta\,\mathrm{Int}(q)$ and θ are not G_k-isomorphic. Moreover if $q_1 = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \in I_k(A_\theta^-)$ is another k-inner element of A_θ^-, then $\theta\,\mathrm{Int}(q)$ and $\theta\,\mathrm{Int}(q_1)$ are G_k-isomorphic if and only if xy^{-1} is the sum of 2 squares in $\mathbb{Q}$. So there are infinitely many $q \in I_k(A_\theta^-) \cap A_\theta^-(k)$, which give non isomorphic k-involutions $\theta\,\mathrm{Int}(q)$ of G.

9.10. **Commuting involutions.** The involutions θ and $\theta\,\mathrm{Int}(a)$ for $a \in I_k(A_\theta^-)$ commute if and only if $\mathrm{Int}(a) = \mathrm{Int}(a^{-1})$, i.e. $a^2 \in Z(G)$. We will call these elements in $I_k(A_\theta^-)$ the *quadratic elements of* A_θ^- and we will write $Q(A_\theta^-) = \{a \in I_k(A_\theta^-) \mid a^2 \in Z(G)\}$.

In many cases one can actually find a set of representatives for the G_k-isomorphy classes in $\mathcal{F}_A(\theta)$ in $Q(A_\theta^-)$. For example if $k = \mathbb{R}$ we get:

Proposition 9.11. *Let $k = \mathbb{R}$ and G, A, θ be as above. Every G_k-isomorphy class in $\mathcal{F}_A(\theta)$ has a representative $\theta\operatorname{Int}(a)$ with $a \in Q(A_\theta^-)$.*

Proof. Any real reductive group has a Cartan involution, unique up to G_k-isomorphy. In fact one can choose a Cartan involution σ of G such that $\sigma\theta = \theta\sigma$ and such that A is a maximal σ-split torus of G (see for example [HW93, 11.17]). If $a \in I_k(A_\theta^-)$, then $\theta\operatorname{Int}(a)$ is another k-involution of G, which does not need to commute with the Cartan involution σ. Since all Cartan involutions of G are G_k-isomorphic there exists $g \in G_\mathbb{R}$ such that $\operatorname{Int}(g)\sigma\operatorname{Int}(g)^{-1}$ commutes with $\theta\operatorname{Int}(a)$ (see for example [Hel78, Ch. III, Theorem 7.2]). One can in fact choose the conjugating element g in $A_\theta^-(k)$, what can be seen as follows.

Consider the involutions $\theta\operatorname{Int}(a)\sigma\theta\operatorname{Int}(a) = \theta\sigma\theta\operatorname{Int}(a^2) = \sigma\operatorname{Int}(a^2)$ and σ, which are both Cartan involutions of G. By [HW93, 11.15], there is $g \in \operatorname{Int}(G)_k$ such that $\sigma\operatorname{Int}(a^2) = g\sigma g^{-1}$. But then

$$(\theta\operatorname{Int}(a)\sigma)^2 = \sigma\operatorname{Int}(a^2)\sigma = \operatorname{Int}(a^{-2}) = g\sigma g^{-1}\sigma \in \tau_\sigma(\operatorname{Int}(G)_k) \cap \operatorname{Int}(A_\theta^-).$$

Let $S = \operatorname{Int}(A_\theta^-)$. Since $(\theta\operatorname{Int}(a)\sigma)^2 = \operatorname{Int}(a^{-2}) \in S_k^2$ it follows from [HW93, Lemma 11.14] that there is a unique $x \in S_k^2$ with $\operatorname{Int}(a^{-2}) = x^4$. Let $t \in A_\theta^-$ such that $\operatorname{Int}(t) = x$. Let $\sigma_1 = \operatorname{Int}(t)\sigma\operatorname{Int}(t)^{-1} = \sigma\operatorname{Int}(t^{-2})$. Then

$$\begin{aligned}\sigma_1\theta\operatorname{Int}(a) &= \sigma\operatorname{Int}(t^{-2})\theta\operatorname{Int}(a) = \sigma\theta\operatorname{Int}(t^2a) = \theta\sigma\operatorname{Int}(t^2a)\\ &= \theta\operatorname{Int}(a)\operatorname{Int}(a)^{-1}\sigma\operatorname{Int}(t^2a) = \theta\operatorname{Int}(a)\sigma\operatorname{Int}(t^2a^2)\\ &= \theta\operatorname{Int}(a)\sigma\operatorname{Int}(t^{-2}) = \theta\operatorname{Int}(a)\sigma_1.\end{aligned}$$

From [Hel78, Ch. III, Theorem 7.2] it follows that $\operatorname{Int}(t) = \operatorname{Int}(\sigma(g)g^{-1}) \in \operatorname{Int}(A_\theta^-(k)^2)$ for some $g \in G_\mathbb{R}$, since $\operatorname{Int}(t)$ maps the compact real form related to σ to the compact real form related to σ_1. It follows that there exists $z \in Z(G)$ such that $y = tz \in (A_\theta^- \cdot Z(G))_k$ and $x = \operatorname{Int}(t) = \operatorname{Int}(y) \in \operatorname{Int}(G_k, A)$. Let $b = y^2a$. Then $\operatorname{Int}(b) = \operatorname{Int}(y^2a) = \operatorname{Int}(t^2a) = \operatorname{Int}(t^{-2}a^{-1}) = \operatorname{Int}(y^{-2}a^{-1}) = \operatorname{Int}(b^{-1})$, so $b \in Q(A_\theta^-)$. Since $\theta\operatorname{Int}(a)$ and $\theta\operatorname{Int}(b)$ are isomorphic under $\operatorname{Int}(y) \in \operatorname{Int}(G_k, A)$ the result follows. □

9.12. To determine whether the isomorphy classes in $\mathcal{F}_A(\theta)$ have a set of representatives $\theta\operatorname{Int}(a)$ with $a \in Q(A_\theta^-)$ it is useful to consider first the question when $\theta\operatorname{Int}(a)$ and $\theta\operatorname{Int}(a^{-1})$ are isomorphic. For G_k-isomorphy we can show the following:

Proposition 9.13. *Let G, A, θ be as above and $a \in I_k(A_\theta^-)$. The involutions $\theta\,\mathrm{Int}(a)$ and $\theta\,\mathrm{Int}(a^{-1})$ are G_k-isomorphic if and only if $a^2 \in \tau_{\theta\,\mathrm{Int}(a)}(G_k)\cdot Z(G)$.*

Proof. Assume first that $g \in G_k$ such that $\mathrm{Int}(g)\theta\,\mathrm{Int}(a)\,\mathrm{Int}(g)^{-1} = \theta\,\mathrm{Int}(a^{-1})$. By Corollary 9.4 we may assume $g = zh \in (Z_G(A)G_{\theta\,\mathrm{Int}(a)})_k$, where $z \in Z_G(A)$ and $h \in G_{\theta\,\mathrm{Int}(a)}$. Then $\theta\,\mathrm{Int}(a)(g)g^{-1} = \theta\,\mathrm{Int}(a)(z)z^{-1} = \theta(z)z^{-1}$. Moreover

$$\begin{aligned}\theta\,\mathrm{Int}(\theta(g)ag^{-1}) &= \mathrm{Int}(g)\theta\,\mathrm{Int}(a)\,\mathrm{Int}(g)^{-1} = \mathrm{Int}(z)\theta\,\mathrm{Int}(a)\,\mathrm{Int}(z)^{-1}\\ &= \theta\,\mathrm{Int}(\theta(z)az^{-1}) = \theta\,\mathrm{Int}(\theta(z)z^{-1}a)\\ &= \theta\,\mathrm{Int}(\theta\,\mathrm{Int}(a)(g)g^{-1})\,\mathrm{Int}(a) = \theta\,\mathrm{Int}(a^{-1}).\end{aligned}$$

It follows that $\mathrm{Int}(\theta\,\mathrm{Int}(a)(g)g^{-1}) = \mathrm{Int}(a^{-2})$, hence $a^2 \in \tau_{\theta\,\mathrm{Int}(a)}(G_k)\cdot Z(G)$.

If conversely $a^2 \in \tau_{\theta\,\mathrm{Int}(a)}(G_k)\cdot Z(G)$, then also $a^{-2} \in \tau_{\theta\,\mathrm{Int}(a)}(G_k)\cdot Z(G)$. Let $g \in G_k$ such that $\theta\,\mathrm{Int}(a)(g)g^{-1}a^2 \in Z(G)$. Then

$$\begin{aligned}\mathrm{Int}(g)\theta\,\mathrm{Int}(a)\,\mathrm{Int}(g)^{-1} &= \theta\,\mathrm{Int}(a)\,\mathrm{Int}(\theta\,\mathrm{Int}(a)(g)g^{-1})\\ &= \theta\,\mathrm{Int}(a)\,\mathrm{Int}(a^{-2}) = \theta\,\mathrm{Int}(a)^{-1},\end{aligned}$$

what proves the result. □

Remark 9.14. If we consider isomorphy classes under $\mathrm{Int}_k(G)$ instead of G_k-isomorphy, then the involutions $\theta\,\mathrm{Int}(a)$ and $\theta\,\mathrm{Int}(a^{-1})$ are always isomorphic. Namely $\mathrm{Int}(a)$ is contained in $\mathrm{Int}_k(G)$, hence $\mathrm{Int}(a)\theta\,\mathrm{Int}(a)\,\mathrm{Int}(a)^{-1} = \theta\,\mathrm{Int}(a^{-1})$. If $a \in A_\theta^-(k)$, then $\theta\,\mathrm{Int}(a)$ and $\theta\,\mathrm{Int}(a^{-1})$ are always G_k-isomorphic.

9.15. The remaining question is now when a k-involution $\theta\,\mathrm{Int}(a)$ is G_k-isomorphic with $\theta\,\mathrm{Int}(q)$ for some $q \in Q(A_\theta^-)$. Using the above result we get the following result:

Corollary 9.16. *Let G, A, θ be as above and $a \in I_k(A_\theta^-)$. The involution $\theta\,\mathrm{Int}(a)$ is G_k-isomorphic with $\theta\,\mathrm{Int}(q)$ for some $q \in Q(A_\theta^-)$ if and only if $a^2 \in \tau_{\theta\,\mathrm{Int}(a)}(G_k)^2\cdot Z(G)$.*

Proof. Assume first that $g \in G_k$ such that $\mathrm{Int}(g)\theta\,\mathrm{Int}(a)\,\mathrm{Int}(g)^{-1} = \theta\,\mathrm{Int}(q)$ for some $q \in Q(A_\theta^-)$. By Corollary 9.4 we may assume $g = zh \in (Z_G(A)G_{\theta\,\mathrm{Int}(a)})_k$, where $z \in Z_G(A)$ and $h \in G_{\theta\,\mathrm{Int}(a)}$. Then $\theta\,\mathrm{Int}(a)(g)g^{-1} = \theta\,\mathrm{Int}(a)(z)z^{-1} = \theta(z)z^{-1}$, so $\theta(g)ag^{-1} = a\theta(z)z^{-1}$. Now $\mathrm{Int}(\theta(g)ag^{-1}) = \mathrm{Int}(q)$, so $\theta(g)ag^{-1}q^{-1} \in Z(G)$. Since $q^2 \in Z(G)$ we get $(\theta(g)ag^{-1}q^{-1})^2 = (a\theta(z)z^{-1})^2 \in Z(G)$. It follows that $a^2 = (z\theta(z)^{-1})^2 \mod Z(G) = (g\theta\,\mathrm{Int}(a)(g)^{-1})^2 \mod Z(G)$, hence $a^2 \in \tau_{\theta\,\mathrm{Int}(a)}(G_k)^2\cdot Z(G)$.

Conversely assume $a^2 \in \tau_{\theta \operatorname{Int}(a)}(G_k)^2 \cdot Z(G)$. Let $g \in G_k$ and $z \in Z(G)$ such that $a^2(\theta \operatorname{Int}(a)(g)g^{-1})^2 = z \in Z(G)$. Let $x = \theta(g)ag^{-1}$. Then $a^2(a^{-1}\theta(g)ag^{-1})^2 = axa^{-1}x = z$, so $x^2 = (\theta(g)ag^{-1})^2 = z \in Z(G)$. It follows that $x \in Q(A_\theta^-)$. Now $\operatorname{Int}(g)\theta \operatorname{Int}(a) \operatorname{Int}(g)^{-1} = \theta \operatorname{Int}(\theta(g)ag^{-1}) = \theta \operatorname{Int}(x)$, what proves the result. □

9.17. **Weyl group action on $I_k(A_\theta^-)$.** The above result gives a first characterization of the G_k-isomorphy classes in $\mathcal{F}_A(\theta)$. By Lemma 9.7 we know that we can find a set of representatives of the G_k-isomorphy classes in $\mathcal{F}_A(\theta)$ in the set $I_k(A_\theta^-)/A_\theta^-(k)^2$. The Weyl group $W(A_\theta^-)$ acts on $I_k(A_\theta^-)$ and $A_\theta^-(k)^2$ by the usual conjugation action and it would be natural to try and use this Weyl group to reduce the set of representatives to a Weyl chamber or even a fundamental domain. Unfortunately we do not have the usual conjugation action but the θ-twisted action. This means that if $g \in N_{G_k}(A)$ is a representative of $w \in W(A_\theta^-)$ and $a \in I_k(A_\theta^-)$, then

$$\theta(g)ag^{-1} = \theta(g)g^{-1}gag^{-1} = \theta(g)g^{-1}w(a). \tag{9.17.1}$$

So besides the action of w there is an additional translation factor $\theta(g)g^{-1}$ which could push $w(a)$ out of $I_k(A_\theta^-)$. We will show next that $W(A_\theta^-)$ does act on $I_k(A_\theta^-)$.

We note first that the full Weyl group $W_{G_k}(A)$ does not act on $\mathcal{F}_A(\theta)$, since an element of $W_G(A)/W_H(A)$ could map a k-inner element in A_θ^- to A_θ^+. In particular from Proposition 9.2 we get the following result:

Corollary 9.18. *Let A be a maximal k-split torus of G, θ a k-involution of G, normally related to A, $A_0 = A_\theta^-$, $w \in W(A)$, $n \in N_{G_k}(A)$ a representative and $a \in I_k(A_\theta^-)$. If $\operatorname{Int}(n)\theta \operatorname{Int}(a) \operatorname{Int}(n)^{-1} \in \mathcal{F}_A(\theta)$, then $w \in W_H(A)$.*

Proof. Assume $\operatorname{Int}(n)\theta \operatorname{Int}(a) \operatorname{Int}(n)^{-1} = \theta \operatorname{Int}(b)$ for some $b \in A_0$. Since

$$\begin{aligned} \operatorname{Int}(n)\theta \operatorname{Int}(a) \operatorname{Int}(n)^{-1} &= \operatorname{Int}(n)\theta \operatorname{Int}(n)^{-1} \operatorname{Int}(n) \operatorname{Int}(a) \operatorname{Int}(n)^{-1} \\ &= \operatorname{Int}(n)\theta \operatorname{Int}(n)^{-1} \operatorname{Int}(w(a)) \end{aligned}$$

it follows that $\operatorname{Int}(n)\theta \operatorname{Int}(n)^{-1} = \theta \operatorname{Int}(b) \operatorname{Int}(w(a))^{-1}$. But then from Proposition 9.2 it follows that $n \in (Z_G(A)G_\theta)_k$. Write $n = zh$ with $z \in Z_G(A)$ and $h \in G_\theta$. By Proposition 3.5 $h \in N_H(A)$ is a representative of w as well, hence $w \in W_H(A)$. □

9.19. The group $W_H(A)$ is essentially the Weyl group of A_θ^-. Before we describe this relation we give first another description of the group $W_H(A)$. Let $X = X^*(A)$, $X_0(\theta) = \{\chi \in X^*(A) \mid \theta(\chi) = \chi\}$ and $\Phi_0(\theta) = \{\alpha \in \Phi(A) \mid \theta(\alpha) = \alpha\}$ be as in 5.11. Similarly as in 4.5 write

$$W^\theta(A) = W_1(A, \theta) = \{w \in W(A) \mid w(X_0(\theta)) \subset X_0(\theta)\} \tag{9.19.1}$$

and $W_0(\theta) = W_0(A, \theta) = W(\Phi_0(\theta))$. Then by Proposition 4.11 we have

$$W(A_\theta^-) \simeq W^\theta(A)/W_0(\theta). \tag{9.19.2}$$

The group $W_H(A)$ corresponds with $W^\theta(A)$ due to the following result.

Proposition 9.20. *Let A be a maximal k-split torus of G, θ a k-involution of G, normally related to A and $A_0 = A_\theta^-$. Then we have the following.*

(*i*) *Any $w \in W(A_0)$ has a representative in $(H^0 Z_G(A))_k \cap N_G(A_0)$.*
(*ii*) $N_G(A_0) = N_{H^0}(A_0) Z_G(A_0)$.

Proof. (*i*) Let $n \in N_{G_k}(A_0)$ be a representative for $w \in W(A_0)$ and P a minimal θ-split parabolic k-subgroup of G. Then $P_1 = nPn^{-1}$ is also a minimal θ-split parabolic k-subgroup of G containing A. By [HW93, 4.9] there exists $x \in (H^0 P)_k$ such that $xPx^{-1} = P_1$. Let P_0 be a minimal parabolic k-subgroup of P containing A and let $U = R_u(P_0)$ be the unipotent radical of P_0. Then $H^0 P_0 = H^0 P$ (see [HW93, 4.8]). On the other hand by [HW93, 10.2] we have $(H^0 P)_k = (H^0 Z_G(A))_k U_k$. It follows that $x = hzu$ with $h \in H^0$, $z \in Z_G(A)$ and $u \in U_k$. If we take $g = hz \in (H^0 Z_G(A))_k$, then $gPg^{-1} = P_1$ and $gA_0 g^{-1}$ is (θ, k)-split. Moreover $gA_0 g^{-1} \subset P_1 \cap \theta(P_1) = Z_G(A_0)$, so $gA_0 g^{-1} A_0$ is a (θ, k)-split torus of G. Since A_0 is maximal (θ, k)-split it follows that $gA_0 g^{-1} = A_0$, what proves the result.

(*ii*) follows immediately from (*i*). □

Corollary 9.21. *Let A be a maximal k-split torus of G, θ a k-involution of G, normally related to A and $A_0 = A_\theta^-$. Then $W^\theta(A) = W_H(A)$.*

Proof. Clearly $W(A, H) \subset W^\theta(A)$. As for the other inclusion let $w \in W^\theta(A)$ and let $\tilde{w}$ be the corresponding element of $W(A_0)$. By Proposition 9.20 there exists a representative $h \in N_H(A_0)$ of $\tilde{w}$. Moreover there exists $z \in Z_G(A)$ such that $x = hz \in (HZ_G(A))_k$. Then $\tilde{A} = xAx^{-1}$ is a maximal k-split torus of $Z_G(A_0)$. Now A_θ^+ and $\tilde{A}_\theta^+$ are maximal k-split tori of $Z_G(A_0) \cap H$, hence there exists $h_1 \in (Z_G(A_0) \cap H)_k$ such that $h_1 \tilde{A} h_1^{-1} = A$. But then $h_1 h \in N_H(A) \cap N_H(A_0)$. Let $w_1 \in W^\theta(A)$ be the corresponding Weyl group element. Since both w and w_1 induce $\tilde{w}$ in $W(A_0)$ it follows from (9.19.2) that

$ww_1^{-1} \in W_0(\theta)$. So it suffices to show that $W_0(\theta) = W_0(A, \theta) \subset W(A, H)$. Since A_0 is maximal (θ, k)-split it follows from Proposition 2.6 that for every root $\alpha \in \Phi_0(\theta) \subset \Phi(A)$ the group G_α as in 2.1 is contained in H. This proves the result. ☐

Remark 9.22. It follows from the above results that instead of the action of $W_H(A)$ on $I_k(A_\theta^-)$ it suffices to consider the action of $W_H(A_\theta^-)$ on $I_k(A_\theta^-)$. So instead of showing that $W_H(A)$ acts on $\mathcal{F}_A(\theta)$ it suffices to show that $W_H(A_\theta^-)$ acts on $\mathcal{F}_A(\theta)$. This would be immediate if $W(A_\theta^-)$ has representatives in H_k, but as the proof of Proposition 9.20 indicates, this will not always be true. However $\mathcal{F}_A(\theta)$ always contains a k-involution for which this is true. We define this as follows:

Definition 9.23. Let A be a maximal k-split torus of G, θ a k-involution of G, normally related to A and $A_0 = A_\theta^-$. The pair (G, θ) is called a *weakly standard pair* if $W(A_0)$ has representatives in H_k. In this case we will also call θ a weakly standard k-involution.

Remark 9.24. For $k = \mathbb{R}$ one can define these standard pairs also using the signatures of the roots of a basis of $\Phi(A_\theta^-)$. For this one can modify similar definitions in [HS97] and [Hel88]. They are defined as follows: Let $T \supset A$ be a maximal k-torus of G and $\sigma \in \Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, $\sigma \neq \mathrm{id}$. Denote the conjugation of G corresponding to σ also by σ. There exists a conjugation τ of a compact real form of G, which commutes with σ. Let $\mathfrak{g}(A_0, \lambda)$ denote the root space corresponding to $\lambda \in \Phi(A_0)$ and let $\Delta = \Delta(T)$ be a basis of $\Phi(T)$. Since $\sigma(\lambda) = \lambda$, $\tau(\lambda) = -\lambda$ and $\theta(\lambda) = -\lambda$, $\tau\sigma\theta$ stabilizes $\mathfrak{g}(A_0, \lambda)$. Set

$$\mathfrak{g}(A_0, \lambda)_\pm^{\tau\sigma\theta} = \{X \in \mathfrak{g}(A_0, \lambda) \mid \tau\sigma\theta(X) = \pm X\}$$
$$m^\pm(\lambda, \sigma\theta) = \dim \mathfrak{g}(A_0, \lambda)_\pm^{\tau\sigma\theta}$$

For $\lambda \in \Phi(A_0)$ call $(m^+(\lambda, \sigma\theta), m^-(\lambda, \sigma\theta))$ the *signature* of λ. Following [Hel88, 6.11] we say that (G, θ) is a *standard pair* (resp. *weakly-standard pair*) if $m^+(\lambda, \sigma\theta) \geq m^-(\lambda, \sigma\theta)$ (resp. $m^+(\lambda, \sigma\theta) \neq 0$ or $m^+(2\lambda, \sigma\theta) \neq 0$) for any $\lambda \in \Delta$.

The two notions of weakly-standard pairs are the same as follows from the following result (see [HS97]):

Proposition 9.25 ([HS97])**.** *Let G, θ, T, A, A_0 and Δ be as above and assume $k = \mathbb{R}$. Then $W(A_0)$ has representatives in H_k if and only if $m^+(\lambda, \sigma\theta) \neq 0$ or $m^+(2\lambda, \sigma\theta) \neq 0$ for any $\lambda \in \Delta$.*

For general fields k one can show that every set $\mathcal{F}_A(\theta)$ contains a weakly-standard pair.

Proposition 9.26. *Let A be a maximal k-split torus of G, θ a k-involution of G, normally related to A and $A_0 = A_\theta^-$. There exists $a \in A_0$ such that $\theta \operatorname{Int}(a) \in \mathcal{F}_A(\theta)$ and $W(A_0)$ has representatives in $G_{\theta \operatorname{Int}(a)}(k)$.*

For a proof of this result we refer to [Hel99], where we classify the k-inner elements for symmetric k-varieties over the $\mathfrak{p}$-adic numbers. This result is very useful in the analysis of the k-inner elements in A_k.

Remark 9.27. In general there can be more than one isomorphy class of weakly standard pairs in $\mathcal{F}_A(\theta)$. However for $k = \mathbb{R}$ one can show that there is a unique isomorphy class of standard pairs. For this see [Hel88, 8.21].

Corollary 9.28. *Let A be a maximal k-split torus of G, θ a k-involution of G, normally related to A and $A_0 = A_\theta^-$. The Weyl group $W(A_0)$ acts on $\mathcal{F}_A(\theta)$.*

Proof. By Proposition 9.26 we may assume that (G, θ) is a weakly-standard pair. Then by (6.15) every $w \in W(A)$ has a representative $h \in (G_\sigma \cap G_\theta)^0$. So if $\theta \operatorname{Int}(a) \in \mathcal{F}_A(\theta)$, $w \in W(A_0)$ and $h \in G_\theta(k)$ a representative of w, then $\operatorname{Int}(h)\theta \operatorname{Int}(a) \operatorname{Int}(h)^{-1} = \theta \operatorname{Int}(hah^{-1}) = \theta \operatorname{Int}(w(a))$. ☐

Remark 9.29. The set $I_k(A_\theta^-)$ depends on the base field k, so a classification of the $W(A_\theta^-)$-conjugacy classes in $I_k(A_\theta^-)$ depends on the base field k as well. For $k = \mathbb{R}$ a classification of the $W(A_\theta^-)$-conjugacy classes in $I_k(A_\theta^-)$ was given in [Hel88, §8]. For $k = \mathbb{Q}_p$ a classification will be given in [Hel99].

9.30. If $x \in \mathcal{N}(A, \theta)$ acts on $\mathcal{F}_A(\theta)$, then this action can be split in an action of $W_H(A)$ on $\mathcal{F}_A(\theta)$ and an action of $\mathcal{Z}(A, \theta) := Z_{G_k}(A).Z(A, \theta)$ on $\mathcal{F}_A(\theta)$. Above we described the action of $W_H(A)$. In the remainder of this section we analyze the other question when two pairs in $\mathcal{F}_A(\theta)$ are isomorphic under $\mathcal{Z}(A, \theta)$. We note that if (G, θ) is a special pair then $\mathcal{Z}(A, \theta) = Z_{G_k}(A)$.

Remark 9.31. For $a \in A$ the restrictions of θ and $\theta \operatorname{Int}(a)$ are the same, so in particular $Z_H(A) = Z_{G_{\theta \operatorname{Int}(a)}}(A)$.

Proposition 9.32. *Let A be a maximal k-split torus of G, θ a k-involution of G, normally related to A and $T \supset A$ a θ-stable maximal k-torus of $Z_G(A)$ such that T_θ^- is a maximal θ-split torus of $Z_G(A)$. Then two k-involutions $\theta \operatorname{Int}(a)$ and $\theta \operatorname{Int}(b)$ in $\mathcal{F}_A(\theta)$ are isomorphic under $\mathcal{Z}(A, \theta \operatorname{Int}(a))$ if and only if there exists $t \in T$ and $h \in Z_H(A) = Z_{G_{\theta \operatorname{Int}(a)}}(A)$ such that $th \in \mathcal{Z}(A, \theta \operatorname{Int}(a))$ and $a^{-1}b = \theta(t)t^{-1}$.*

Proof. If $th \in \mathcal{Z}(A, \theta\operatorname{Int}(a))$ satisfies the above conditions, then there exists $h_0 \in \mathcal{H}(A, \theta\operatorname{Int}(a))$ such that $thh_0 \in (Z_G(A)G_{\theta\operatorname{Int}(a)})_k$ and $\operatorname{Int}(thh_0)$ maps $\theta\operatorname{Int}(a)$ to $\theta\operatorname{Int}(b)$. So assume there is an element $g \in \mathcal{Z}(A, \theta\operatorname{Int}(a))$ such that $\operatorname{Int}(g)\theta\operatorname{Int}(a)\operatorname{Int}(g)^{-1} = \theta\operatorname{Int}(b)$. Let $\tilde{T} = g^{-1}Tg$. Then

$$\theta(\tilde{T}) = \theta\operatorname{Int}(a)(\tilde{T}) = \theta(g)^{-1}\theta(T)\theta(g) = g^{-1}\theta(T)g = \tilde{T}$$

and $\tilde{T}_\theta^-$ is a maximal θ-split torus of $Z_G(A)$. By [Vus74, §1] there exists $h_1 \in Z_H(A)$ such that $h_1\tilde{T}_\theta^- h_1^{-1} = T_\theta^-$. Then $h_1g^{-1} \in N_G(T_\theta^-) \cap Z_G(A)$. By [Ric82, Proposition 4.7] there exists $h_2 \in N_G(T_\theta^-) \cap Z_H(A)$ such that $h_2h_1g^{-1} \in Z_G(AT_\theta^-)$. Now $h_2h_1\tilde{T}_\theta^- h_1^{-1}h_2^{-1}$ and T are maximal tori in $Z_G(AT_\theta^-)$. Since $Z_G(AT_\theta^-)$ does not contain any non central θ-split tori it follows from [Vus74, §1] that $[Z_G(AT_\theta^-), Z_G(AT_\theta^-)] \subset H$, hence there exists $h_3 \in Z_G(AT_\theta^-) \cap H$ such that $h_3h_2h_1\tilde{T}_\theta^- h_1^{-1}h_2^{-1}h_3^{-1} = T$. Finally since $[Z_G(AT_\theta^-), Z_G(AT_\theta^-)] \subset H$ the Weyl group $W(T, Z_G(AT_\theta^-))$ has representatives in H, so there exists $h_4 \in Z_G(AT_\theta^-) \cap H$ such that $h_4h_3h_2h_1g^{-1} \in T$. It follows that $g = th$ with $t \in T$ and $h \in Z_H(A)$, which proves the result. □

For special pairs (G, θ) we can sharpen this result as follows:

Corollary 9.33. *Let (G, θ) be a special pair and T, A be as above. Then two k-involutions $\theta\operatorname{Int}(a)$ and $\theta\operatorname{Int}(b)$ in $\mathcal{F}_A(\theta)$ are isomorphic under $Z_{G_k}(A)$ if and only if there exists $t \in T$ and $h \in Z_H(A)$ such that $th \in (TZ_H(A))_k$ and $a^{-1}b = \theta(t)t^{-1}$.*

For $k = \mathbb{R}$ we even get a much stronger result:

Corollary 9.34. *Assume $k = \mathbb{R}$ and let θ, T, A be as above. Then two k-involutions $\theta\operatorname{Int}(a)$ and $\theta\operatorname{Int}(b)$ in $\mathcal{F}_A(\theta)$ are isomorphic under $Z_{G_k}(A)$ if and only if there exists $t \in T_k$ such that $a^{-1}b = \theta(t)t^{-1}$.*

Proof. Since (G, θ) is a special pair $\mathcal{Z}(A, \theta\operatorname{Int}(a)) = Z_{G_k}(A)$ and since $[Z_{G_k}(A), Z_{G_k}(A)]$ is compact, the four conjugating elements h_1, h_2, h_3, h_4 in the proof of Proposition 9.32 can be chosen in H_k instead of H, but then $h_4h_3h_2h_1g^{-1} \in T_k$, what proves the result. □

For a weakly-standard pair we even get a characterization for G_k-isomorphy instead of $\mathcal{Z}(A, \theta\operatorname{Int}(a))$-isomorphy:

Corollary 9.35. *Assume (G, θ) is a weakly-standard pair. Let T, A be as above and $a \in I_k(A_\theta^-)$. Then we have the following:*

(1) *θ and $\theta\operatorname{Int}(a)$ are G_k-isomorphic if and only if there exists $t \in T$ and $h \in Z_H(A)$ such that $th \in \mathcal{Z}(A, \theta)$ and $a = \theta(t)t^{-1}$.*

(2) *If (G,θ) is a special pair, then θ and $\theta\,\mathrm{Int}(a)$ are G_k-isomorphic if and only if there exists $t\in T$ and $h\in Z_H(A)$ such that $th\in (TZ_H(A))_k$ and $a=\theta(t)t^{-1}$.*
(3) *If $k=\mathbb{R}$, then θ and $\theta\,\mathrm{Int}(a)$ are G_k-isomorphic if and only if there is $t\in T_k$ such that $a=\theta(t)t^{-1}$.*

Proof. By Corollary 9.4 it suffices to consider isomorphy under $(Z_G(A)H)_k$. If θ and $\theta\,\mathrm{Int}(a)$ are isomorphic under $x=zh\in (Z_G(A)H)_k$, then modulo elements of H_k, $Z_H(A)$ and $W(A_\theta^-)$ the element $h\in H$ is contained in $\mathcal{H}(A,\theta)$. Since (G,θ) is a weakly-standard pair $W(A_\theta^-)$ has representatives in H_k as well, so there exists $h_1\in H_k$ and $h_2\in Z_H(A)$ such that $h_2hh_1\in\mathcal{H}(A,\theta)$. So we may assume that $x=zh$ with $z\in\mathcal{Z}(A,\theta)$ and $h\in\mathcal{H}(A,\theta)$. Now the result follows from Proposition 9.32, Corollary 9.33 and Corollary 9.34. □

We conclude this section with a description of the conjugating element $t\in T$ as in Proposition 9.32 in the case that $k=\mathbb{R}$. We will let A denote a maximal k-split torus of G, θ a k-involution of G, normally related to A and $T\supset A$ a θ-stable maximal k-torus of $Z_G(A)$ such that T_θ^- is a maximal θ-split torus of $Z_G(A)$. Let S be the anisotropic part of T. Then S is θ-stable, $T=SA$ and $A\cap S$ is finite, see 2.1.

Proposition 9.36. *Let θ, A, T, S be as above and assume $k=\mathbb{R}$. Let $a\in I_k(A_\theta^-)$. Then the following statements are equivalent:*

(1) *There is a $t\in T_k$ such that $a=\theta(t)t^{-1}$.*
(2) *$a=xy$ where $x\in S_\theta^-(k)\cap A_\theta^-(k)$ and $y\in\tau_\theta(A_k)$.*

Proof. (1) $\Longrightarrow$ (2): Assume $t\in T_k$ such that $a=\theta(t)t^{-1}$. Write $t=sa_0$, where $s\in S_k$ and $a_0\in A_k$. Since both S and A are θ-stable we can write $s=s_1s_2$ and $a_0=a_1a_2$ with $s_1\in S_\theta^+$, $s_2\in S_\theta^-$, $a_1\in A_\theta^+$ and $a_2\in A_\theta^-$. Then $t=s_1s_2a_1a_2$ and $a=\theta(t)t^{-1}=s_2^{-2}a_2^{-2}$. Now $a_2^{-2}=\tau_\theta(a_0)\in A_\theta^-(k)$ and $s_2^{-2}=\tau_\theta(s)=aa_2^2\in S_\theta^-(k)\cap A_\theta^-(k)$. Taking $x=s_2^{-2}$ and $y=a_2^{-2}$ the result follows.

(2) $\Longrightarrow$ (1): Assume now that $a=xy$ as in (2). Since S_k is compact it follows that $S_\theta^-(k)=\tau_\theta(S_k)$ (see for example [HS97, 11.4]). So there exists $s\in S_k$ such that $x=\tau(s)$. Let $a_0\in A_k$ such that $y=\tau_\theta(a_0)$. Then $t=sa_0\in T_k$ and $a=\theta(t)t^{-1}$, which proves (1). □

10. Characterization of the admissible indices

In this section we will show that an admissible k-involution of a semisimple root datum Ψ can be represented by a (Γ,θ)-index and that there is a one to one correspondence between the congruence classes of these (Γ, θ)-indices and the isomorphy classes of admissible k-involutions. We will also characterize the admissible (Γ, θ)-indices. Using this characterization we will classify the admissible (Γ, θ)-indices in section 11 for k the real numbers, $\mathfrak{p}$-adics numbers, finite fields and number fields. To be able to determine whether an involution of a root datum is admissible we need to determine first whether it can be lifted to an involution of the group. For this we use a realization of the root system in G as in 6.4.

10.1. Let G be a reductive algebraic group, T a maximal k-torus of G, $X = X^*(T)$ and $\Phi = \Phi(T)$. Let $\bar{k}$ denote the algebraic closure of k. We say that an involution $\theta \in \operatorname{Aut}(X, \Phi)$ can be *lifted* if there exists an involutorial automorphism $\varphi \in \operatorname{Aut}(G, T)$ inducing θ on (X, Φ). From the isomorphism theorem it follows that there exists always a possibly non involutorial $\varphi \in \operatorname{Aut}(G, T)$, inducing θ on (X, Φ). So the question is when φ is an involution or even a k-involution. This is again a matter of structure constants:

Proposition 10.2. *Let $\theta \in \operatorname{Aut}(X, \Phi)$ be an involution and $\varphi \in \operatorname{Aut}(G, T)$ such that $\varphi^\star = \theta$. Then we have the following:*

(1) *φ is an involution if and only if $(\varphi|T)^2 = \mathrm{id}_T$ and $c_{\alpha,\varphi}c_{\theta(\alpha),\varphi} = 1$ for all $\alpha \in \Phi(T)$.*
(2) *φ is a k-involution if and only if it satisfies the conditions in* (1) *and for all $\sigma \in \Gamma$ and $\alpha \in \Phi$ we have $\varphi^{\star^\sigma} = \varphi^\star$ and $c^\sigma_{\alpha,\varphi} d_{\varphi^\star(\alpha),\sigma} = c_{\alpha^\sigma,\varphi} d_{\alpha,\sigma}$.*

Proof. (1) follows from (6.8.1) and (2) is immediate from this and Proposition 6.10(2). ☐

10.3. The above result describes when an involution $\theta \in \operatorname{Aut}(X, \Phi)$ can be lifted to an involution or k-involution. It remains to verify when this involution is admissible, i.e. if it satisfies the normality condition 7.2 for an involution and for a k-involution additionally the normality condition 8.2. The normality condition 7.2 for an involution is again a matter of structure constants. Recall first from [Hel91] that a root $\alpha \in \Phi(T)$ is called *θ-singular* if $\theta(\alpha) = \pm\alpha$ and $\theta | Z_G((\mathrm{Ker}\alpha)^0) \neq \mathrm{id}$. If $\theta(\alpha) = -\alpha$ we say that α is *real* with respect to θ. If $\theta(\alpha) = \alpha$ and α is θ-singular, then α is also called *noncompact imaginary* with respect to θ. In that case $c_{\alpha,\theta} = -1$, as follows by simple computation in SL_2. If $\theta(\alpha) = \alpha$ and α is not θ-singular, then $c_{\alpha,\theta} = 1$. These roots are called *compact imaginary* with respect to θ.

Proposition 10.4 ([Hel91, 4.12])**.** *Let $\theta \in \operatorname{Aut}(X, \Phi)$ be an involution and $\varphi \in$* $\operatorname{Aut}(G, T)$ *an involution with $\varphi^\star = \theta$. Then θ is admissible if and only if $c_{\alpha,\varphi} =$*

1 *for all* $\alpha \in \Phi_0(\theta)$, *i.e.* $\Phi(T)$ *has no roots, which are noncompact imaginary with respect to* θ.

Similarly a necessary and sufficient condition for an involution of (X, Φ) to be an admissible k-involution follows from this result and Proposition 5.26(3).

Corollary 10.5. *Let* A *be a maximal* k*-split torus of* G, $T \supset A$ *a maximal* k*-torus* $\theta \in \mathrm{Aut}(X^*(T), \Phi(T), \Phi(A))$ *an involution and* $\varphi \in \mathrm{Aut}_k(G, T, A)$ *a* k*-involution with* $\varphi^\star = \theta$. *Then* θ *is admissible if and only if* $c_{\alpha,\varphi} = 1$ *for all* $\alpha \in \Phi_0(\theta)$ *and for every irreducible component* $\Phi_1 \subset \Phi_0(\Gamma_\theta)$ *we have* $\Phi_1 \subset \Phi_0(\theta)$ *or* $\Phi_1 \subset \Phi_0(\Gamma)$.

Proof. From Proposition 10.4 it follows that φ is normally related to T. Since $\varphi \in \mathrm{Aut}_k(G, T, A)$ we have $\varphi(A) = A$ and A_θ^- is the annihilator of $X_0(\Gamma_\theta)$. From the condition that for every irreducible component $\Phi_1 \subset \Phi_0(\Gamma_\theta)$ we have $\Phi_1 \subset \Phi_0(\theta)$ or $\Phi_1 \subset \Phi_0(\Gamma)$ it follows that $Z_G(A_\theta^-)/A_\theta^-$ contains no non trivial (θ, k)-split torus. But then φ is normally related to A. □

10.6. Although the above result characterizes which involutions of (X, Φ) can be lifted to an admissible involution of G, it is still difficult to determine for which involutions there exists a suitable set of structure constants satisfying the conditions in Proposition 10.2, Proposition 10.4 and Corollary 10.5. In the following we derive a few more properties of these structure constants, what will simplify the actual classification. First we note that up to a sign the above structure constants $c_{\alpha,\theta}$ are of the form $\alpha(t)$ for some $t \in T$. To see this we use the following automorphism defined by Steinberg (see [Ste68, Theorem 29]):

Definition 10.7. Let Δ be a basis of Φ. For an involution $\theta \in \mathrm{Aut}(X, \Phi)$ let $\theta_\Delta \in \mathrm{Aut}(G, T)$ denote the unique automorphism of G such that

(10.7.1) $$\theta_\Delta(x_\alpha(\xi)) = x_{\theta(\alpha)}(\xi) \text{ for all } \alpha \in \Delta, \xi \in \bar{k}.$$

From [Ste68, Theorem 29] it follows that $c_{\alpha,\theta_\Delta} = \pm 1$ for all $\alpha \in \Phi$.

By the isomorphy theorem any automorphism of (G, T) inducing θ on (X, Φ) is now of the form $\theta_\Delta \,\mathrm{Int}(t)$ for some $t \in T$. The question is then when $\theta_\Delta \,\mathrm{Int}(t)$ is an involution of G and when it is admissible. Combining (10.7.1) with the above results we get the following:

Proposition 10.8. *Let* $\theta \in \mathrm{Aut}(X, \Phi)$ *be an involution and* Δ *a basis of* Φ. *Then the following are equivalent:*

(1) θ *can be lifted.*
(2) *There is a* $t \in T$ *such that* $\theta_\Delta \mathrm{Int}(t)$ *is an involution.*
(3) *There is a* $t \in T^+$ *such that* $\theta_\Delta \mathrm{Int}(t)$ *is an involution.*

(4) *There is a* $t \in T$ *such that* $c_{\theta(\alpha),\theta_\Delta} = \alpha(\theta(t)t)$ *for all* $\alpha \in \Delta$.
(5) *There is a* $t \in T_\theta^+$ *such that* $c_{\theta(\alpha),\theta_\Delta} = \alpha(t)$ *for all* $\alpha \in \Delta$.

Proof. Let $\{x_\alpha\}_{\alpha\in\Phi(T)}$ be a realization of $\Phi(T)$ in G and let $\{c_{\alpha,\theta_\Delta}\}_{\alpha\in\Phi(T)}$ be as in (6.8.1). Then for $\xi \in \bar{k}$ we get:

$$(\theta_\Delta \operatorname{Int}(t))^2(x_\alpha(\xi)) = x_\alpha(c_{\alpha,\theta_\Delta} c_{\theta(\alpha),\theta_\Delta} \alpha(t)\theta(\alpha)(t)\xi).$$

The equivalence of (1), (2), (4) and (5) follows now from this, the definition of θ_Δ and Lemma 6.12.

So it suffices to prove (2) implies (3). Let $t \in T$ such that $\theta_\Delta \operatorname{Int}(t)$ is an involution. Write $t = t_+t_-$ with $t_+ \in T^+$ and $t_- \in T^-$. Let $s \in T^-$ such that $s^2 = t_-$. Now $\operatorname{Int}(s)\theta_\Delta \operatorname{Int}(t)\operatorname{Int}(s)^{-1} = \theta_\Delta \operatorname{Int}(t_+)$ is an involution as well, what proves the result. □

Remarks 10.9. (1). If $\varphi \in \operatorname{Aut}(G, T)$ is an involution, then for any $t \in T_\varphi^-$ the automorphism $\varphi \operatorname{Int}(t)$ is an involution as well.

(2). If $t \in T_\theta^+$ such that $\theta_\Delta \operatorname{Int}(t)$ is an involution, then, since $c_{\alpha,\theta_\Delta} = \pm 1$ for all $\alpha \in \Phi$, we have by (5) that $\alpha(t^4) = 1$ for all $\alpha \in \Phi$, hence $t^4 \in Z(G)$.

Combining this result with Proposition 6.10 we get the following characterization of the involutions of (X, Φ) which can be lifted to k-involutions of (G, T).

Corollary 10.10. *Let* $\theta \in \operatorname{Aut}(X, \Phi)$ *be an involution and assume* Γ *acts on* (X, Φ) *as in 5.21. Let* Δ *be a basis of* Φ*. There exists a* $t \in T$ *such that* $\varphi = \theta_\Delta \operatorname{Int}(t)$ *is a k-involution of* (G, T) *if and only if the following conditions are satisfied:*

(1) $c_{\theta(\alpha),\theta_\Delta} = \alpha(\theta(t)t)$ *for all* $\alpha \in \Delta$.
(2) $\theta^\sigma = {\theta_\Delta^*}^\sigma = \theta_\Delta^* = \theta$ *for all* $\sigma \in \Gamma$.
(3) $c^\sigma_{\alpha,\varphi} d_{\theta(\alpha),\sigma} = \alpha(t)^\sigma c^\sigma_{\alpha,\theta_\Delta} d_{\theta(\alpha),\sigma} = \alpha^\sigma(t) c_{\sigma(\alpha),\theta_\Delta} d_{\alpha,\sigma} = c_{\sigma(\alpha),\varphi} d_{\alpha,\sigma}$ *for all* $\alpha \in \Delta$ *and* $\sigma \in \Gamma$.

Combining Propositions 10.8 and 10.4 we get the following characterization of the involutions of (X, Φ) which can be lifted to admissible involutions of (G, T).

Corollary 10.11. *Let* $\theta \in \operatorname{Aut}(X, \Phi)$ *be an involution and let* Δ *be a* θ*-basis of* Φ*. Then* θ *is admissible if and only if there is a* $t \in T$ *such that*

(1) $c_{\theta(\alpha),\theta_\Delta} = \alpha(\theta(t)t)$ *for all* $\alpha \in \Delta - \Delta_0(\theta)$
(2) $\alpha(t) = 1$ *for all* $\alpha \in \Delta_0(\theta)$

In Theorem 10.45 we will give a detailed characterization of the involutions $\theta \in \mathrm{Aut}(X, \Phi)$ which can be lifted to an admissible k-involution of G. In fact we will characterize these involutions by their admissible (Γ, θ)-index. The characterization of these depends on the classifications of the underlying θ-indices and Γ-indices. Therefor before we characterize the admissible (Γ, θ)-indices, we review in the following briefly a few facts of the classifications of the admissible θ-indices and Γ-indices.

10.12. **Admissible θ-indices.** In this subsection we discuss the classification of admissible θ-indices related to conjugacy classes of involutions of G. Our notations remain as in 5.11 and 7.1. In particular let G be a reductive algebraic group, T a maximal torus of G, $X = X^*(T)$ and $\Phi = \Phi(T)$.

The first step is to determine when a quadruple $\mathcal{D} = (X, \Delta, \Delta_0(\theta), \theta^*)$ is a θ-index. This follows from the following result:

Proposition 10.13. *A quadruple $\mathcal{D} = (X, \Delta, \Delta_0(\theta), \theta^*)$ is a θ-index of an involution $\theta \in \mathrm{Aut}(X, \Phi)$ if and only if the restriction index $\mathcal{D}_0 = (X_0, \Delta_0(\theta), \Delta_0(\theta), \theta^*|\Delta_0(\theta))$ is a θ-index for* id *and $\Delta_0(\theta)$ is θ^*-stable.*

Proof. This result is immediate from Lemma 5.13. □

Remarks 10.14. (1). A θ-index for id is always admissible. For Γ-indices this is not the case.

(2). A complete list of θ-indices for id follows from Remark 5.12.

Using this result one can easily obtain a list of the possible θ-indices. The next step is to determine which of these θ-indices are admissible. For this we can use a rank one reduction:

10.15. **Rank one restriction.** Recall that the *restricted rank* of an involution $\theta \in \mathrm{Aut}(X, \Phi)$ is defined as the rank of the set of restricted roots $\bar{\Phi}_\theta$. The classification of admissible involutions can be reduced to admissible involutions of restricted rank one as follows. For each $\lambda \in \bar{\Phi}_\theta$ such that $\frac{1}{2}\lambda \notin \bar{\Phi}_\theta$, let $\Phi(\lambda)$ denote the set of all roots $\beta \in \Phi$ such that the restriction of β to $\bar{X}_\lambda$ is an integral multiple of λ. Then $\Phi(\lambda)$ is a θ-stable closed subsystem of Φ (See [BT65, p.71]). Let $X(\lambda)$ denote the projection of $X = X^*(T)$ on the subspace of $E = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by $\Phi(\lambda)$.

Proposition 10.16. *Let $\theta \in \mathrm{Aut}(X, \Phi)$ be an involution and Δ a θ-basis of Φ. Then θ is admissible if and only if $\theta|X(\lambda) \in \mathrm{Aut}(X(\lambda), \Phi(\lambda))$ is admissible for all $\lambda \in \bar{\Delta}_\theta$.*

For a proof of this result see [Hel88, 4.5]. This reduces the classification of admissible θ-indices to θ-indices of restricted rank one. From 5.12 it follows that it suffices to consider irreducible θ-indices. The irreducible, but not absolutely irreducible θ-indices are those for which $\theta^* = -\theta$ exchanges the connected components (see 5.12). There are 17 absolutely irreducible θ-indices of restricted rank one and the question which of these are admissible is a matter of manipulating the structure constants satisfying the conditions in Propositions 10.2(1) and 10.4. For more details, see [Hel88, §4].

Remark 10.17. The classification of isomorphy classes of involutions automorphisms of G is independent of the base field k. For G of adjoint type it is equivalent to the classification of real forms of a semisimple Lie algebra over $\mathbb{C}$, as is carried out by Araki [Ara62]. See also Sugiura [Sat71, appendix] for a simplification of this method. A further simplification of Araki's classification can be found in [Hel88, §4].

10.18. **Admissible Γ-indices.** In this subsection we discuss the admissible Γ-indices related to the isomorphy classes of semisimple k-groups. Our notations remain as in 5.17 (unless specified otherwise).

If $\mathcal{D} = (X, \Delta, \Delta_0(\Gamma), [\sigma])$ is a admissible Γ-index and (G, T) is a k-group corresponding to $\mathcal{D}$, then we will write $(G, T) \leftrightarrow \mathcal{D}$ to indicate the relation between $\mathcal{D}$ and (G, T).

10.19. Similar as in the case of involutions, the classification of admissible Γ-indices can be reduced to the case of absolutely irreducible indices of k-rank $= 1$. That it suffices to consider absolutely irreducible indices can be seen as follows. Suppose $\mathcal{D} = (X, \Delta, \Delta_0(\Gamma), [\sigma])$ and X is simply connected. If $\mathcal{D}$ is k-irreducible, but not absolutely irreducible, then $\Delta = \Delta_1 \cup \ldots \cup \Delta_s$, where the Δ_i are mutually disjoint connected components of Δ and correspondingly one has $X = X_1 + \cdots + X_s$. Define $\Gamma_1 = \{\sigma \in \Gamma \mid \Delta_1^{[\sigma]} = \Delta_1\}$. Then $\Gamma = \bigcup_{i=1}^s \Gamma_1\sigma_i$, where $\Delta_i = \Delta_1^{[\sigma_i]}$. Let $\mathcal{D}_1 = (X_1, \Delta_1, \Delta_1 \cap \Delta_0(\Gamma), [\sigma])$, where $\sigma \in \Gamma_1$ and let k_1 be the fixed field of Γ_1. Now we have the following:

Lemma 10.20. *$\mathcal{D}$ is admissible as a Γ-index if and only if $\mathcal{D}_1$ is admissible as a Γ_1-index.*

Note that if $(G_1, T_1)/k_1 \leftrightarrow \mathcal{D}_1$ then $(G, T) = R_{k_1/k}(G_1, T_1) \leftrightarrow \mathcal{D}$. Here $R_{k_1/k}$ is the functor of "descent" from the field k_1 to the field k (see [Wei61]).

This reduces the classification of admissible Γ-indices to absolutely irreducible indices.

10.21. To classify the admissible Γ-indices one needs to classify first the admissible Γ-indices for k-anisotropic groups. This can be seen as follows. If $\mathcal{D} = (X, \Delta, \Delta_0(\Gamma), [\sigma])$ is an admissible Γ-index, then one obtains a subsystem $\mathcal{D}_0 = \{X_0, \Delta_0(\Gamma), [\sigma]\}$ where X_0 is the projection of X on $\Delta_0(\Gamma)_{\mathbb{Q}}$ (one may write $X_{\mathbb{Q}} = \Delta_0(\Gamma)_{\mathbb{Q}}$ with respect to some W-invariant metric), and $\mathcal{D}_0$ is just the Γ-index of (G_0, T_0), the k-anisotropic kernel of the group (G, T) having $\mathcal{D}$ as Γ-index. So a necessary condition for an Γ-index $\mathcal{D}$ to be admissible is that the subindex $\mathcal{D}_0$ is a admissible Γ-index of a k-anisotropic group.

For general base fields not much is known about the k-anisotropic groups. However for a number of base fields, like the real numbers, finite fields, $\mathfrak{p}$-adic fields and number fields, the k-anisotropic groups are known. For the remainder of this subsection we will assume that the classification of admissible indices of k-anisotropic groups is known for the base field k we consider. So we assume that $\mathcal{D}$ is a Γ-index, for which the subindex $\mathcal{D}_0$ is admissible and corresponds to a k-anisotropic group (G_0, T_0) where T_0 splits over K. We redefine the notion of admissibility for these indices now as follows.

Definition 10.22. The Γ-index $\mathcal{D}$ is said to be *admissible over* $(G_0, T_0) \leftrightarrow \mathcal{D}_0$ if there is a connected semi-simple algebraic group G defined over k and a maximal torus T defined over k such that (G_0, T_0) is the k-anisotropic kernel of (G, T), and $\mathcal{D}$ is the Γ-index of G.

From Theorem 7.10 it follows now that if $\mathcal{D}$ is admissible over $(G_0, T_0) \to \mathcal{D}_0$, then the group (G, T) as described in the above definition is unique up to k-isomorphism.

10.23. We still need a condition for when a Γ-index $\mathcal{D} = \{X, \Delta, \Delta_0(\Gamma), [\sigma]\}$ is admissible over $(G_0, T_0) \leftrightarrow \mathcal{D}_0$. For this we can use the one cocycle (φ_σ) of Γ in $\mathrm{Aut}_K(\tilde{G}, \tilde{T})$ as in 6.1. Recall that the maps φ_σ are completely determined by the system $\varphi_\sigma \leftrightarrow \{\varphi^\star_\sigma, d^{-1}_{\alpha^{\sigma^{-1}},\sigma}\}$ (see 6.8). So similar as in the case of involutions the problem of determining whether a Γ-index $\mathcal{D}$ is admissible over $(G_0, T_0) \leftrightarrow \mathcal{D}_0$ comes down to a question about structure constants. The following result gives a necessary and sufficient condition:

Proposition 10.24. *Let $\mathcal{D}$ be a Γ-index and for $\sigma \in \Gamma$ let $\varphi^\star_\sigma = [\sigma]^{-1} w_\sigma^{-1}$. Then $\mathcal{D}$ is admissible over $(G_0, T_0) \leftrightarrow \mathcal{D}_0$ if and only if $\{d^{-1}_{\alpha^{\sigma^{-1}},\sigma} \mid \alpha \in \Phi_0,\ \sigma \in \Gamma\}$ can be extended to a set of scalars $\{c^{-1}_{\alpha^{\sigma^{-1}},\sigma} \mid \alpha \in \Phi,\ \sigma \in \Gamma\}$ satisfying*

(10.24.1) $$d_{\alpha,\sigma\gamma} = d^{\gamma}_{\alpha,\sigma} d_{\alpha^\sigma,\gamma} \ \textit{for } \sigma, \gamma \in \Gamma.$$

and for each $\sigma \in \Gamma$ the map defined by $\{\varphi^\star_\sigma, d^{-1}_{\alpha^{\sigma^{-1}},\sigma}\}$ is admissible in the sense of 6.8.

Proof. Let $\mathcal{D} = \{X, \Delta, \Delta_0(\Gamma), [\sigma]\}$ and suppose $\mathcal{D}$ is admissible over $(G_0, T_0) \to \mathcal{D}_0$. Assume $\mathcal{D}$ is the Γ-index of (G, T), a connected semi-simple group defined over k having (G_0, T_0) as k-compact kernel. The Dynkin diagram (X, Δ) uniquely determines (up to k-isomorphism) a Chevalley group $(\tilde{G}, \tilde{T})$ defined over k. Then (G, T) is a K/k-form of $(\tilde{G}, \tilde{T})$. There exists a K-isomorphism $\phi : (G, T) \to (\tilde{G}, \tilde{T})$ which is uniquely determined by the system $(\varphi_\sigma = \phi^\sigma \circ \phi^{-1})_{\sigma\in\Gamma}$ of automorphisms in $\mathrm{Aut}_K(\tilde{G}, \tilde{T}))$. From 6.11 it follows that $\varphi_\sigma \leftrightarrow \{\varphi^\star_\sigma, d^{-1}_{a^{\sigma^{-1}},\sigma}\}$. The cocycle condition $\varphi_\sigma{}^\gamma \circ \varphi_\gamma = \varphi_{\sigma\gamma}$ implies that $\varphi^\star_\sigma \varphi^\star_\gamma = \varphi^\star_{\sigma\gamma}$ and from (6.5.2) it follows that the scalars $\{d_{\alpha,\sigma}\}$ satisfy the condition

$$d_{\alpha,\sigma\gamma} = d^\gamma_{\alpha,\sigma} d_{\alpha^\sigma,\gamma} \ \text{ for } \sigma,\ \gamma \in \Gamma.$$

Since $(G_0, T_0) \leftrightarrow \mathcal{D}_0$, the scalars $\{d^{-1}_{\alpha^{\sigma^{-1}},\sigma},\ \alpha \in \Phi_0,\ \sigma \in \Gamma\}$ are given, and so is $\varphi^\star_\sigma | X_0$. In fact the isomorphisms $\varphi^\star_\sigma$ are determined by the Γ-index $\mathcal{D}$ and the restrictions $\varphi^\star_\sigma | X_0$. Namely, using the identification of X with $\tilde{X}$ as in 6.1 we have $\chi^{[\sigma]} = w^{-1}_\sigma \chi^\sigma = w^{-1}_{\sigma^\star}(\chi)$ so $\varphi^\star_\sigma = [\sigma]^{-1} w^{-1}_\sigma$. But the diagram automorphism $[\sigma]$ is given by the Γ-index $\mathcal{D}$, and since $w_\sigma \in W_0$, $\varphi^\star_\sigma | X_0$ determines w_σ. On the other hand, since $\mathcal{D}$ is admissible it follows from Theorem 7.10 (applied to the case $G' = \tilde{G}$) that the set of scalars $\{d^{-1}_{\alpha^{\sigma^{-1}},\sigma},\ \alpha \in \Phi,\ \sigma \in \Gamma\}$ is determined by the subset $\{d^{-1}_{\alpha^{\sigma^{-1}},\sigma},\ \alpha \in \Phi_0,\ \sigma \in \Gamma\}$. It follows that for each $\sigma \in \Gamma$ the set of scalars $\{\varphi^\star_\sigma := [\sigma]^{-1} w^{-1}_\sigma, d^{-1}_{\alpha^{\sigma^{-1}},\sigma}\}$ is admissible in the sense of 6.8.

Conversely if $\{\varphi^\star_\sigma, d^{-1}_{\alpha^{\sigma^{-1}},\sigma}\}$ is admissible for each σ, then they determine a system of automorphisms $\{\varphi_\sigma\}_{\sigma\in\Gamma}$ in $\mathrm{Aut}_K(\tilde{G}, \tilde{T})$. From equation (10.24.1) and the definition of $\varphi^\star_\sigma$, it follows than that the system $\{\varphi_\sigma\}$ is a one-cocycle of Γ in $\mathrm{Aut}_K(\tilde{G}, \tilde{T})$, hence $\{\varphi_\sigma\}$ determines a K/k-form of $(\tilde{G}, \tilde{T})$. Thus $\mathcal{D}$ is admissible over $(G_0, T_0) \leftrightarrow \mathcal{D}_0$. □

10.25. **Restriction to k-rank = 1.** Similar as in the case of involutions we want to reduce the problem of classifying admissible Γ-indices to the case of Γ-indices of groups having k-rank $= 1$. Recall that the k-rank of G is just the number of restricted fundamental roots of $\bar{\Phi}_\Gamma$.

First we give a condition for when a subgroup of G, generated by a subset of a fundamental basis, is a k-subgroup of G.

Lemma 10.26. *Let $\mathcal{D} = \{X, \Delta, \Delta_0(\Gamma), [\sigma]\}$ be an admissible Γ-index of (G, T), Δ' a $[\sigma]$-invariant subset of Δ and $G(\Delta') \subset G$ the connected semi-simple subgroup generated by $\{U_\alpha \mid \alpha \in \Phi \cap \Delta'_{\mathbb{Q}}\}$. Then $G(\Delta')$ is defined over k if the following two conditions are satisfied:*

(1) $\Delta'^{[\sigma]} = \Delta'$,
(2) *if $\alpha \in \Delta'$ and $\beta \in \Delta_0(\Gamma)$ and $\langle \alpha, \beta \rangle \neq 0$, then $\beta \in \Delta'$.*

10.27. Let $\mathcal{D} = \{X, \Delta, \Delta_0(\Gamma), [\sigma]\}$. If Δ' is a $[\sigma]$-invariant subset of Δ, then we can define a subsystem $\mathcal{D}_{\Delta'} = \{X', \Delta', \Delta'_0(\Gamma), [\sigma]'\}$ of $\mathcal{D}$ where X' is the projection of X on $\Delta'_{\mathbb{Q}}$, $\Delta'_0(\Gamma) = \Delta' \cap \Delta_0(\Gamma)$, $[\sigma]' = [\sigma]|X'$. The system $\mathcal{D}_{\Delta'}$ will be called a *canonical subsystem* of $\mathcal{D}$.

If $\mathcal{D}$ is admissible, $(G, T) \leftrightarrow \mathcal{D}$, and Δ' is a subset of Δ which satisfies conditions (i), (ii) of Lemma 10.26, then the canonical subsystem $\mathcal{D}_{\Delta'}$ is clearly admissible, and $(G(\Delta'), T') \leftrightarrow \mathcal{D}_{\Delta'}$, where $T' = T \cap G(\Delta')$ is a maximal torus of $G(\Delta')$ and $A' = A \cap T'$ is a maximal k-split torus in $G(\Delta')$.

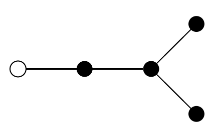
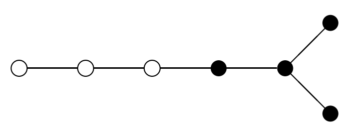
The following result gives a criterion for how one can combine admissible Γ-indices to obtain other admissible Γ-indices.

Proposition 10.28. *Let $\mathcal{D} = (X, \Delta, \Delta_0(\Gamma), [\sigma])$, and suppose $\Delta = \Delta' \cup \Delta''$, where Δ' and Δ'' satisfy conditions* (1), (2) *of Lemma 10.26 and $\Delta' \cap \Delta'' \subset \Delta_0(\Gamma)$. If the canonical subsystems $\mathcal{D}_{\Delta'}$, $\mathcal{D}_{\Delta''}$ are admissible over $G_0(\Delta'_0(\Gamma))$, $G_0(\Delta''_0(\Gamma))$, respectively, where $\Delta'_0(\Gamma) = \Delta' \cap \Delta_0(\Gamma)$, $\Delta''_0(\Gamma) = \Delta'' \cap \Delta_0(\Gamma)$, then $\mathcal{D}$ is admissible over $(G_0, T_0) \leftrightarrow \mathcal{D}_0$.*

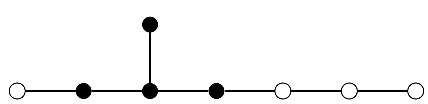
Remarks 10.29. (1) The condition $\Delta' \cap \Delta'' \subset \Delta_0(\Gamma)$ in Proposition 10.28 implies that $\Delta'_0(\Gamma)$ and $\Delta''_0(\Gamma)$ consist of unions of connected component of $\Delta_0(\Gamma)$, hence $G_0(\Delta'_0(\Gamma))$ and $G_0(\Delta''_0(\Gamma))$ are normal subgroups of G_0.

(2) This result reduces the classification to indices of k-rank $= 1$. By Lemma 10.20 we may further assume that the index is absolutely irreducible.

Example 10.30. The following Γ-indices are admissible over $\mathbb{R}$:

Δ' :

Δ'' :


Thus by Proposition 10.28, the following Γ-index for E_8 is admissible:



Remark 10.31. From the above discussion it follows that the classification of admissible Γ-indices reduces to a classification of absolutely irreducible Γ-indices of k-rank $= 1$. For a number of base fields the semisimple algebraic k-groups have been classified. For $k = \mathbb{R}$ the classification was already known to Cartan (see [Car72]). In this case the Γ-indices were classified by Araki [Ara62]. See also [Sat71] and [Hel88] for simplifications of this classification. The admissible Γ-indices have also been classified for $\mathfrak{p}$-adic fields, finite fields and number fields. For more details see [Tit66] and [Sat71]. To complete the classification for these fields one needs to classify all k-anisotropic semisimple algebraic groups. A classification of these basically reduces to determining the first cohomology group of Γ in $\mathrm{Aut}(G)$. For $\mathfrak{p}$-adic fields this classification was studied by a number of people, including Tits [Tit66], Satake [Sat63] and Veisfeiler [Vei64]. The classification in the case of number fields was completed only recently. For simply connected semisimple algebraic groups Kneser [Kne65] and Harder [Har65, Har66] determined the first cohomology group of Γ in $\mathrm{Aut}(G)$. The problem of constructing all central simple Lie algebras of a given type over a number field was solved by a number of people, including Jacobson, Ferrar and Allison, see [Jac79, Fer76, Fer78, Fer88, All92].

10.32. In the characterization of admissible indices for k-involutions we will need the following result about the admissible Γ-indices.

Lemma 10.33. *Let $\mathcal{D} = (X, \Delta, \Delta_0(\Gamma), [\sigma])$ be an admissible Γ-index. Write* $\mathrm{id} = -\mathrm{id} \circ \mathrm{id}^* \circ w_0(\mathrm{id})$ *as in 5.12. Then $\Delta_0(\Gamma)$ is id^*-stable.*

Proof. Let Δ be a Γ-basis of (X, Φ). Since $-\Delta$ is also a Γ-basis of (X, Φ) it follows that $-\bar{\Delta}_\Gamma$ is a basis of $\bar{\Phi}_\Gamma$, hence there exists $w_1 \in W^\Gamma$ such that $w_1(-\bar{\Delta}_\Gamma) = \bar{\Delta}_\Gamma$. Then $w_1 \circ -\mathrm{id}(\Delta_0(\Gamma))$ is a basis of $\Delta_0(\Gamma)$, hence there exists $w \in W_0(\Gamma)$ such that $ww_1 \circ -\mathrm{id}(\Delta_0(\Gamma)) = \Delta_0(\Gamma)$. It follows that $ww_1 \circ -\mathrm{id}(\Delta) = \Delta$, hence $ww_1 = w_0(\mathrm{id})$. Since $\mathrm{id}^* = -\mathrm{id}\, w_0(\mathrm{id})$ and $ww_1 \circ -\mathrm{id}(\Delta_0(\Gamma)) = \Delta_0(\Gamma)$ the result follows. □

10.34. **Admissible *k*-involutions.** In 6.8 we characterized the indices of involutions which can be lifted to admissible involutions of the group. Even if these admissible involutions are k-involutions then they are not necessarily admissible k-involutions. In the remainder of this section we give a characterization of the admissible k-involutions. We first note that the notion of admissibility of the k-involutions can be induced to (Γ, θ)-indices, which will be easier to classify. Recall that in Theorem 8.9 we showed that the $W(A, T)$-isomorphy classes of admissible k-involutions of $(X^*(T), \Phi(T), \Phi(A))$ correspond with congruence classes of certain (Γ, θ)-indices. If we use the same notations as in 5.21 and 8.31, then we can define admissible (Γ, θ)-indices as follows:

Definition 10.35. Let G be a reductive k-group, T a maximal k-torus of G, $X = X^*(T)$, $\Phi = \Phi(T)$, A the subtorus of T annihilated by $X_0(\Gamma)$, $K \supset k$ a splitting extension for T and $\theta \in \mathrm{Aut}(X, \Phi)$ an involution. If $\succ$ is a (Γ, θ)-order on (X, Φ) and $\mathcal{D} = (X,\ \Delta,\ \Delta_0(\Gamma),\ \Delta_0(\theta),\ [\sigma],\ \theta^*)$ the corresponding (Γ, θ)-index, then $\mathcal{D}$ is said to be an *admissible* (Γ, θ)*-index* (with respect to $(G,\ T)$) if A is a maximal k-split torus of G and if there exists a k-involution $\tilde{\theta}$ of G, normally related to A and $x \in Z_{G_K}(A)$ such that $\mathrm{Int}(x)\tilde{\theta}\,\mathrm{Int}(x^{-1})$ is normally related to T, $x^{-1}Tx$ is a k-torus and $\mathrm{Int}(x)\tilde{\theta}\,\mathrm{Int}(x^{-1})|T = \theta$.

From Theorem 8.9 it follows now that the $W(A, T)$-isomorphy classes of admissible k-involutions of $(X^*(T),\ \Phi(T),\ \Phi(A))$ correspond with the congruence classes of admissible (Γ, θ)-indices:

Proposition 10.36. *Let A be a maximal k-split torus of G and $T \supset A$ a maximal k-torus of G. There is a bijection between the $W(A, T)$-isomorphy classes of admissible k-involutions of $(X^*(T), \Phi(T), \Phi(A))$ and the congruence classes of admissible (Γ, θ)-indices of $(X^*(T), \Phi(T))$.*

10.37. To classify the admissible (Γ, θ)-indices we can first determine all the possible (Γ, θ)-indices. Recall that from Proposition 5.26 it follows that a Γ_θ-index is a (Γ, θ)-index if and only if

(10.37.1)
$$\text{If } \Phi_1 \subset \Phi_0(\Gamma_\theta) \text{ irreducible component, then } \Phi_1 \subset \Phi_0(\theta) \text{ or } \Phi_1 \subset \Phi_0(\Gamma).$$

The problem which remains then is to determine which of these (Γ, θ)-indices are admissible. An admissible (Γ, θ)-index also satisfies the following conditions:

Proposition 10.38. *Let $\mathcal{D} = (X,\ \Delta,\ \Delta_0(\Gamma),\ \Delta_0(\theta),\ [\sigma],\ \theta^*)$ be an admissible (Γ, θ)-index. Then $\mathcal{D}$ satisfies the following conditions:*

(1) *$\Phi_0(\theta)$ is w_σ-stable for each $\sigma \in \Gamma$.*
(2) *$\Delta_0(\theta)$ is $[\sigma]$-stable for each $\sigma \in \Gamma$.*
(3) *$\Phi_0(\Gamma)$ is $w_0(\theta)$-stable .*
(4) *$\Delta_0(\Gamma)$ is θ^*-stable.*
(5) *$w_0(\theta)$ and $[\sigma]$ commute for each $\sigma \in \Gamma$.*
(6) *θ^* and $[\sigma]$ commute for each $\sigma \in \Gamma$.*
(7) *w_σ commutes with θ for each $\sigma \in \Gamma$.*

Proof. By (4.4.1) $\sigma\theta = \theta\sigma$ for all $\sigma \in \Gamma$. Let $\sigma \in \Gamma$ and $w_\sigma \in W_0(\Gamma)$ such that $\sigma(\Delta_0(\Gamma)) = w_\sigma(\Delta_0(\Gamma))$. Then by Proposition 4.9(2) we have $\sigma(\Delta) = w_\sigma(\Delta)$. Since $\sigma(\Phi_0(\theta)) = \Phi_0(\theta)$ and $[\sigma] = w_\sigma^{-1}\sigma$ statement (2) is equivalent to $w_\sigma(\Phi_0(\theta)) = \Phi_0(\theta)$. We will show the latter. Since by Proposition 5.26 $\Phi_0(\Gamma,\theta) = \Phi_0(\Gamma) \cup \Phi_0(\theta)$ we may restrict to the case that $\Phi = \Phi_0(\Gamma,\theta)$. Write $\Phi_0(\Gamma,\theta) = \Phi_1 \cup \cdots \cup \Phi_n$, where each Φ_i is irreducible ($i = 1,\ldots,n$), $\Phi_1,\ldots,\Phi_r \not\subset \Phi_0(\theta)$ and $\Phi_{r+1},\ldots,\Phi_n \subset \Phi_0(\theta)$. If $i > r$, then $w_\sigma(\Phi_i) \subset \Phi_0(\theta)$. So we may assume $\Phi_0(\Gamma,\theta) = \Phi_0(\Gamma)$. Since $\sigma\theta = \theta\sigma$ it follows that $\sigma(\Delta_0(\Gamma))$ is also a θ-basis of $\Phi_0(\Gamma)$. Moreover since $\mathcal{D}$ is admissible it follows that $\theta|\Phi_0(\Gamma)$ is admissible as well, hence $\bar{\Phi}_\theta$ is a root system with basis $\bar{\Delta}_\theta$. Then $\sigma(\bar{\Delta}_\theta)$ is a basis of $\bar{\Phi}_\theta$ as well, so by Proposition 4.11(3) there exists $w \in W^\theta$ such that $w(\sigma(\bar{\Delta}_\theta)) = \bar{\Delta}_\theta$. Now $w\sigma(\Delta_0(\theta))$ and $\Delta_0(\theta)$ are bases of $\Phi_0(\theta)$, so there exists $w_0 \in W_0(\theta)$ such that $w_0 w\sigma(\Delta_0(\theta)) = \Delta_0(\theta)$. But then also $w_0 w\sigma(\Delta) = \Delta$. It follows that $w_\sigma^{-1} = w_0 w$, but then $w_\sigma(\Phi_0(\theta)) = \Phi_0(\theta)$, what proves (1) and (2).

(3) and (4). Write $\theta = -\mathrm{id}\,\theta^* w_0(\theta)$ as in 5.11. Since $\theta(\Phi_0(\Gamma)) = \Phi_0(\Gamma)$ it follows that (4) is equivalent to $w_0(\theta)(\Phi_0(\Gamma)) = \Phi_0(\Gamma)$. With a similar argument as in the proof of (2) we can reduce to the case that $\Phi_0(\Gamma) \subset \Phi_0(\theta)$ and $\Phi = \Phi_0(\theta)$. Note that since $\mathcal{D}$ is admissible the restriction of $\mathcal{D}_k$ to $\Phi_0(\theta)$ is also an admissible Γ-index. For each irreducible component of $\Phi_0(\theta)$ we have that either $\theta^* = \mathrm{id}$, $\theta^* = \mathrm{id}^*$ or θ^* exchanges to irreducible components. Now the result follows from Lemma 10.33.

(5). Recall that $w_0(\theta)$ is the unique Weyl group element in $W_0(\theta)$ such that $w_0(\theta)(\Phi_0(\theta)^+) = \Phi_0(\theta)^-$. Since by (2) $[\sigma](\Delta_0(\theta)) = \Delta_0(\theta)$ it follows that $[\sigma]w_0(\theta)[\sigma]^{-1}(\Phi_0(\theta)^+) = \Phi_0(\theta)^-$, hence $[\sigma]w_0(\theta)[\sigma]^{-1} = w_0(\theta)$.

(6). Let $\sigma \in \Gamma$. Recall that by (4) the diagram automorphism θ^* leaves $\Delta_0(\Gamma)$-stable. But then $\theta^*[\sigma]\theta^*(\Phi_0(\Gamma)^+) = \theta^*[\sigma](\Phi_0(\Gamma)^+) = \Phi_0(\Gamma)^+$. From the definition of $[\sigma]$ it follows now that $\theta^*[\sigma]\theta^* = [\sigma]$.

Finally as for (7) note that from $\sigma\theta = \theta\sigma$ for all $\sigma \in \Gamma$ we get $\theta w_\sigma^{-1}[\sigma] = w_\sigma^{-1}[\sigma]\theta = w_\sigma^{-1}\theta[\sigma]$. From this it follows that $\theta w_\sigma = w_\sigma\theta$, what proves the result. □

Corollary 10.39. *Let Ψ be a semisimple root datum, let Γ, θ act on (X,Φ) as in 5.21, assume that for all $\sigma \in \Gamma$ we have $\theta^\sigma = \theta$, let $\succ$ be a Γ_θ-order on (X,Φ) with basis Δ such that condition* (10.37.1) *is satisfied and assume that $\Delta_0(\Gamma)$ is* id**-stable. Then the following are equivalent:*

(1) *$[\sigma]$ commutes with θ^* and $w_0(\theta)$.*

(2) *$\Delta_0(\Gamma)$ is θ^*-stable and $\Delta_0(\theta)$ is $[\sigma]$-stable for each $\sigma \in \Gamma$.*

Proof. From the proof of Proposition 10.38 it follows that (2) implies (1).

Write $\theta = -\operatorname{id}\theta^* w_0(\theta)$ as in 5.11. If $[\sigma]$ commutes with θ^* and $w_0(\theta)$, then $[\sigma]$ also commutes with θ, hence $\Delta_0(\theta)$ is $[\sigma]$-stable for each $\sigma \in \Gamma$. Since $\theta(\Phi_0(\Gamma)) = \Phi_0(\Gamma)$ and $\theta^*(\Delta_0(\Gamma)) \subset \Delta$ it follows that $\Delta_0(\Gamma)$ is θ^*-stable if and only if $w_0(\theta)(\Phi_0(\Gamma)) = \Phi_0(\Gamma)$. We will show the latter. By (10.37.1) we have that for each irreducible component $\Phi_1 \subset \Phi_0(\Gamma_\theta)$ either $\Phi_1 \subset \Phi_0(\theta)$ or $\Phi_1 \subset \Phi_0(\Gamma)$. Write $\Phi_0(\Gamma, \theta) = \Phi_1 \cup \cdots \cup \Phi_n$, where each Φ_i is irreducible ($i = 1, \dots, n$) and such that $\Phi_1, \dots, \Phi_r \not\subset \Phi_0(\theta)$ and $\Phi_{r+1}, \dots, \Phi_n \subset \Phi_0(\theta)$. Since $\Phi_0(\theta) \subset \Phi_0(\Gamma_\theta)$ it follows that $w_0(\theta)$ can be written as $w_0(\theta) = w_1 \dots w_n$ with $w_i \in W(\Phi_i)$ for $i = 1, \dots, n$. Since $\Delta_0(\Gamma)$ is id*-stable it follows that $w_i(\Phi_0(\Gamma)) = \Phi_0(\Gamma)$ for $i = r+1, \dots n$. Since $w_i \in W_0(\Gamma)$ for $i = 1, \dots, r$ it follows that also $w_i(\Phi_0(\Gamma)) = \Phi_0(\Gamma)$ for $i = 1, \dots r$, hence $w_0(\theta)(\Phi_0(\Gamma)) = \Phi_0(\Gamma)$. This proves the result. □

This leads to the following definition.

Definition 10.40. Let Ψ be a semisimple root datum, let Γ, θ act on (X, Φ) as in 5.21 and assume $\theta^\sigma = \theta$ for all $\sigma \in \Gamma$. The involution θ of (X, Φ) is called a basic Γ_θ-involution if there exists a Γ_θ-order $\succ$ on (X, Φ) with basis Δ such that condition (10.37.1) is satisfied, $\Delta_0(\Gamma)$ is id*-stable, $\Delta_0(\Gamma)$ is θ^*-stable and $\Delta_0(\theta)$ is $[\sigma]$-stable for each $\sigma \in \Gamma$. In this case we call the corresponding 6-tuple $(X,\ \Delta,\ \Delta_0(\Gamma),\ \Delta_0(\theta),\ [\sigma],\ \theta^*)$ a *basic* Γ_θ*-index*.

Notation 10.41. A basic Γ_θ-index $\mathcal{D} = (X,\ \Delta,\ \Delta_0(\Gamma),\ \Delta_0(\theta),\ [\sigma],\ \theta^*)$ contains both a Γ-index and a θ-index. Denote the Γ-index by $\mathcal{D}_k = (X,\ \Delta, \Delta_0(\Gamma),\ [\sigma])$ and the θ-index by $\mathcal{D}_\theta = (X,\ \Delta,\ \Delta_0(\theta),\ \theta^*)$.

Remark 10.42. If $\mathcal{D}$ is a basic Γ_θ-index such that the corresponding Γ-index $\mathcal{D}_k$ is admissible, then the condition "$\Delta_0(\Gamma)$ is id*-stable" is automatically satisfied (see Lemma 10.33) and would not be needed in the definition of basic Γ_θ-index. However this condition is needed to prove Corollary 10.39, which is independent of the condition that $\mathcal{D}_k$ is admissible and will be useful in the classification.

10.43. Similarly as for admissible Γ-indices one can restrict an admissible (Γ, θ)-index $\mathcal{D} = (X,\ \Delta,\ \Delta_0(\Gamma),\ \Delta_0(\theta),\ [\sigma],\ \theta^*)$ to the k-anisotropic kernel G_0 and obtain an admissible (Γ, θ)-index $\mathcal{D}_0 = (X_0(\Gamma),\ \Delta_0(\Gamma),\ \Delta_0(\theta),\ [\sigma]|\Delta_0(\Gamma),\ \theta^*|\Delta_0(\Gamma))$ of the pair $(X_0(\Gamma), \Phi_0(\Gamma))$. Similar as in 10.21, the admissibility of a (Γ, θ)-index $\mathcal{D}$ depends on the admissibility of the restriction index $\mathcal{D}_0$. In Theorem 10.45 we will see that one can always extend an admissible restriction index to an admissible index for $X_0(\Gamma_\theta)$.

We will need the following result:

Lemma 10.44. *Let $\mathcal{D}$ be a basic Γ_θ-index, $A = \{t \in T \mid \chi(t) = e$ for all $\chi \in X_0(\Gamma)\}$ the annihilator of $X_0(\Gamma)$ and $\varphi \in \operatorname{Aut}(G, T)$ such that $\varphi^\star = \theta$. Then $\varphi(A) = A$.*

Proof. Since $X_0(\Gamma)$ is θ-stable it follows that for $a \in A$ and $\chi \in X_0(\Gamma)$ we have $\chi(\varphi(a)) = \theta^{-1}(\chi)(a) = \theta(\chi)(a) = e$, hence $\varphi(a) \in A$. □

We can now characterize when a basic Γ_θ-index is an (Γ, θ)-index and when these are admissible. These problems can be solved simultaneously.

Theorem 10.45. *Let (X, Φ) be as above, let Γ, θ act on (X, Φ) as in 5.21 and assume $\theta^\sigma = \theta$ for all $\sigma \in \Gamma$. Let Δ be a Γ_θ-fundamental basis of (X, Φ) and let $\mathcal{D} = (X,\ \Delta,\ \Delta_0(\Gamma),\ \Delta_0(\theta),\ [\sigma],\ \theta^*)$. Then the 6-tuple $\mathcal{D}$ is an admissible (Γ, θ)-index if and only if the following conditions are satisfied*

(1) *$\mathcal{D}$ is a basic Γ_θ-index.*
(2) *$\mathcal{D}_k$ is an admissible Γ-index.*
(3) *$\mathcal{D}_\theta$ is an admissible θ-index.*
(4) *$\mathcal{D}_0$ is an admissible (Γ, θ)-index.*

Proof. If $\mathcal{D}$ is an admissible (Γ, θ)-index, then both $\mathcal{D}_k$ and $\mathcal{D}_\theta$ are admissible and clearly the restriction of $\mathcal{D}$ to $\mathcal{D}_0$ is admissible. By (10.37.1) and Proposition 10.38 $\mathcal{D}$ is a basic Γ_θ-index. So it suffices to show the "if" statement.

Assume $\mathcal{D}$ is a basic Γ_θ-index, such that $\mathcal{D}_k$ is an admissible Γ-index, $\mathcal{D}_\theta$ is an admissible θ-index and $\mathcal{D}_0$ is an admissible (Γ, θ)-index. Let $\{x_\alpha\}_{\alpha \in \Phi}$ be a K-realization of Φ in G as in 6.7, Δ a (Γ, θ)-basis and $\theta_\Delta \in \operatorname{Aut}(G, T)$ the automorphism as in 10.7. Since $\mathcal{D}_\theta$ is an admissible θ-index there exists by Proposition 10.8(3) a $t \in T_\theta^+$ such that $\tilde{\theta} = \theta_\Delta \operatorname{Int}(t)$ is an involution. Let A be the annihilator of $X_0(\Gamma)$. Since $\mathcal{D}_k$ is admissible A is a maximal k-split torus of G. Moreover since $\mathcal{D}$ is a basic Γ_θ-index it follows from Lemma 10.44 that $\tilde{\theta}(A) = A$, hence $\tilde{\theta}|Z_G(A)$ is an involution as well. Since $\mathcal{D}_0$ is an admissible (Γ, θ)-index there exists a k-involution $\theta_1 \in \operatorname{Aut}(Z_G(A), T)$ such that $\theta_1^* = \theta|X_0(\Gamma)$. Let $T_1 = T \cap [Z_G(A), Z_G(A)]$. By [Hel88, 3.8] there exists $t \in (T_1)_\theta^-$ such that $\theta_1 = \tilde{\theta}|Z_G(A) \operatorname{Int}(t)$. Let $\theta_2 = \tilde{\theta} \operatorname{Int}(t) \in \operatorname{Aut}(G, T)$. Since $(T_1)_\theta^- \subset T_\theta^-$ it follows from Remark 10.9 that θ_2 is an involution. Since $\theta_2|Z_G(A) = \theta_1$ is a k-involution it follows from Proposition 6.10(2) that for all $\sigma \in \Gamma$ and $\alpha \in \Delta_0(\Gamma)$:

$$c^\sigma_{\alpha,\theta_2} d_{\theta(\alpha),\sigma} = c_{\alpha^\sigma,\theta_2} d_{\alpha,\sigma}. \tag{10.45.1}$$

Similarly if $\alpha \in \Delta_0(\theta)$, then since $\mathcal{D}_\theta$ is admissible we have $c_{\alpha,\theta_2} = c_{\alpha^\sigma,\theta_2} = 1$, hence $c^\sigma_{\alpha,\theta_2} d_{\theta(\alpha),\sigma} = c_{\alpha^\sigma,\theta_2} d_{\alpha,\sigma}$. Combined with (10.45.1) it follows now that $\theta|\Phi_0(\Gamma, \theta)$ is an admissible k-involution, i.e. if A_0 is the annihilator of

$X_0(\Gamma,\theta)$, then $\theta_2|Z_G(A_0)$ is an k-involution. We must show now that there exists $t\in A_0$ such that $\theta_2\operatorname{Int}(t)\in\operatorname{Aut}(G,T)$ is a k-involution. Let $t\in A_0$ and write $\varphi=\theta_2\operatorname{Int}(t)$. Since $A_0\subset T_\theta^-$ it follows from Remark 10.9 that φ is an involution. Define

$$e_{\alpha,\sigma}=\frac{c^\sigma_{\alpha,\varphi}}{c_{\alpha^\sigma,\varphi}}\frac{d_{\theta(\alpha),\sigma}}{d_{\alpha,\sigma}}=\frac{\alpha(t)^\sigma}{\alpha^\sigma(t)}\frac{c^\sigma_{\alpha,\theta_2}}{c_{\alpha^\sigma,\theta_2}}\frac{d_{\theta(\alpha),\sigma}}{d_{\alpha,\sigma}}. \tag{10.45.2}$$

We need to show now that we can find $t\in A_0$ such that $e_{\alpha,\sigma}=1$ for all $\alpha\in\Phi$, $\sigma\in\Gamma$. Note that since $\varphi|Z_G(A_0)$ is a k-involution, we have by Proposition 6.10(2) $e_{\alpha,\sigma}=1$ for all $\alpha\in\Phi_0(\Gamma,\theta)$, $\sigma\in\Gamma$. So it suffices to show that the $c_{\alpha_i,\varphi}$, $\alpha_i\in\Delta-\Delta_0(\Gamma,\theta)$ (or equivalently the $\alpha_i(t)$, $\alpha_i\in\Delta-\Delta_0(\Gamma,\theta)$) can be modified so that $e_{\alpha,\sigma}=1$ for all $\alpha\in\Phi$, $\sigma\in\Gamma$.

Since $\theta=\varphi^\star$ and $\theta^\sigma=\theta$ for all $\sigma\in\Gamma$, Proposition 6.10 and equation (10.45.2) imply $\varphi^\sigma\leftrightarrow\{\theta,e_{\alpha^{\sigma^{-1}},\sigma}c_{\alpha,\varphi}\}$ and $\varphi^\sigma\circ\varphi^{-1}\leftrightarrow\{\mathrm{id},e_{\alpha^{\sigma^{-1}},\sigma}\}$. It follows from [Hel88, 3.8], that for each $\sigma\in\Gamma$ there exists $t_\sigma\in T_\theta^-$ such that $\varphi^\sigma\circ\varphi^{-1}=\operatorname{Int}(t_\sigma)$. But then $e_{\alpha^{\sigma^{-1}},\sigma}=\alpha(t_\sigma)$. From this it easily follows that

$$\begin{aligned} e_{-\alpha,\sigma}&=e^{-1}_{\alpha,\sigma}\\ e_{\alpha+\beta,\sigma}&=e_{\alpha,\sigma}e_{\beta,\sigma}.\end{aligned} \tag{10.45.3}$$

From these equations, it is clear that if we can choose the scalars $c_{\alpha_i,\varphi}$, $\alpha_i\in\Delta-\Delta_0(\Gamma,\theta)$ so that $e_{\alpha_i,\sigma}=1$ for all $\sigma\in\Gamma$, then $e_{\alpha,\sigma}=1$ for all $\alpha\in\Phi$, $\sigma\in\Gamma$. The relation $e_{\alpha^{\sigma^{-1}},\sigma}=\alpha(t_\sigma)$ shows that $e_{\alpha,\sigma}$ depends only on $\alpha\mod(\Phi_0(\Gamma,\theta))_{\mathbb{Z}}$. Since by (6.5.2) $d_{\alpha,\sigma\gamma}=d^\gamma_{\alpha,\sigma}d_{\alpha^\sigma,\gamma}$ for $\sigma,\gamma\in\Gamma$ it follows from equation (10.45.2) that the scalars $e_{\alpha,\sigma}$ satisfy a similar condition:

$$e_{\alpha,\sigma\gamma}=e^\gamma_{\alpha,\sigma}e_{\alpha^\sigma,\gamma}\quad\text{for all }\sigma,\gamma\in\Gamma. \tag{10.45.4}$$

Fix $\lambda_r\in\bar\Delta_{\Gamma_\theta}$, $\alpha_i\in\Delta-\Delta_0(\Gamma,\theta)$ such that $\lambda_r=\pi(\alpha_i)\in\Phi(A_0)$. Let $\gamma\in\Gamma$ and $w_\gamma\in W_0(\Gamma,\theta)$ such that $\gamma=w_\gamma[\gamma]$. Then $\alpha_i^{[\gamma]}=w_\gamma^{-1}\alpha_i^\gamma=\alpha_i^\gamma+\chi_0$ for some $\chi_0\in\Phi_0(\Gamma,\theta)_{\mathbb{Z}}$. Since $\chi_0(t_\sigma)=1$ it follows that $\alpha_i^\gamma(t_\sigma)=\alpha_i^{[\gamma]}(t_\sigma)$, hence $e_{\alpha_i^\gamma,\sigma}=e_{\alpha_i^{[\gamma]},\sigma}$. Note that since $\varphi|Z_G(A_0)$ is a k-involution it follows that $\chi(t_\sigma)=1$ for $\chi\in X_0(\Gamma_\theta)$. Since $\alpha_i^{[\gamma]}|A_0=\alpha_i|A_0=\lambda_r$ it follows that $\alpha_i^{[\gamma]}-\alpha\in X_0(\Gamma_\theta)$, hence $\alpha_i(t_\sigma)=\alpha_i^\gamma(t_\sigma)=\alpha_i^{[\gamma]}(t_\sigma)$. But then also

$$e_{\alpha_i^\gamma,\sigma}=e_{\alpha_i^{[\gamma]},\sigma}=e_{\alpha_i,\sigma}\quad\text{for all }\sigma,\gamma\in\Gamma. \tag{10.45.5}$$

Combined with (10.45.4) it follows that

$$e_{\alpha,\sigma\gamma}=e^\gamma_{\alpha,\sigma}e_{\alpha,\gamma}\quad\text{for all }\sigma,\gamma\in\Gamma, \tag{10.45.6}$$

hence the system of scalars $(e_{\alpha_i,\sigma})$ is a one-cocycle of Γ in K^*. From Hilbert's Theorem 90 it follows that there exists an element $\mu \in K^*$ such that $e_{\alpha_i,\sigma} = \mu^\sigma \mu^{-1}$. Since $\bar{\Delta}_{\Gamma_\theta}$ is a basis of the root system $\bar{\Phi}_{\Gamma_\theta}$ there exists $t_{\lambda_r} \in A_0$ such that $\alpha(t_{\lambda_r}) = \lambda_r(t_{\lambda_r}) = \mu^{-1}$ and $\lambda_j(t_{\lambda_r}) = 1$ for $\lambda_j \in \bar{\Delta}_{\Gamma_\theta}$ with $\lambda_j \neq \lambda_r$. Replacing φ by $\varphi \operatorname{Int}(t_{\lambda_r})$ then $c_{\alpha_i,\varphi}$ is replaced by $\mu^{-1} c_{\alpha_i,\varphi}$, so (10.45.2) implies $e_{\alpha_i,\sigma} = 1$ for all $\sigma \in \Gamma$. Now for each $\lambda_j \in \bar{\Delta}_{\Gamma_\theta}$ choose an element $t_{\lambda_j} \in A_0$ as above and let $t_0 = \prod_{\lambda_j \in \bar{\Delta}_{\Gamma_\theta}} t_{\lambda_j}$. Replacing φ by $\varphi \operatorname{Int}(t_0)$ we get $e_{\alpha,\sigma} = 1$ for all $\alpha \in \Delta$ and $\sigma \in \Gamma$, hence $\varphi \operatorname{Int}(t_0)$ is a k-involution.

Finally from Corollary 10.5 it follows that A_0 is a maximal (θ, k)-split torus of G. This proves the result. □

Remark 10.46. If $k = \mathbb{R}$ and $G_{\mathbb{R}}$ is compact, then an involution $\theta \in \operatorname{Aut}(X, \Phi)$ can be lifted to an admissible k-involution of $G_{\mathbb{R}}$ if and only if it can be lifted to an admissible involution of G. From this it follows that for $k = \mathbb{R}$ a restriction index $\mathcal{D}_0$ is admissible if and only if it is an admissible θ-index. So in the case of real groups one can drop condition (4) of Theorem 10.45.

Remark 10.47. For the classification of admissible θ-indices and Γ-indices a reduction to the restricted rank 1 indices was needed and these were classified. For the admissible (Γ, θ)-indices this reduction is not needed, as follows from the above result.

11. Classification of the admissible (Γ, θ)-indices

It follows from Theorem 10.45 that in order to classify the admissible (Γ, θ)-indices one needs to have a classification of 4 different indices. Two of these are already known. The admissible Γ-indices were classified by Tits in [Tit66] for a number of base fields k and the admissible θ-indices were classified in [Hel88]. It remains to classify the basic Γ_θ-indices and the restriction indices of k-anisotropic groups. In this section we discuss a classification of these for a number of base fields and combine these classifications to obtain a classification of the admissible (Γ, θ)-indices for k the real numbers, a $\mathfrak{p}$-adic field $\mathbb{Q}_p$, a number field $\mathfrak{n}$ and a finite field $\mathbb{F}_q$.

11.1. **Basic Γ_θ-indices.** To classify the basic Γ_θ-indices we use Proposition 10.38 and Corollary 10.39. For a number of base fields one can sharpen those conditions. For example in the case that $k = \mathbb{R}$ we have that $\Phi_0(\Gamma) = \{\alpha \in \Phi \mid \alpha^\sigma + \alpha = 0\} = \{\alpha \in \Phi \mid \alpha^\sigma = -\alpha\}$. Let Δ be a Γ-basis of Φ and $w_\sigma \in W_0(\Gamma)$ such that $\sigma(\Delta_0(\Gamma) = w_\sigma(\Delta_0(\Gamma)$. Since $\sigma(\Delta_0(\Gamma)) = -\Delta_0(\Gamma)$ it follows that w_σ is the opposition involution in $\in W_0(\Gamma)$ with respect to $\Delta_0(\Gamma)$ (see also

[Hel88, 2.9]). But then $[\sigma]$ and w_σ commute. From Proposition 10.38 it follows now that in this case θ^*, $w_0(\theta)$, $[\sigma]$ and w_σ commute. Combined with Corollary 10.39 we get the following characterization of the basic Γ_θ-indices in the case that $k = \mathbb{R}$:

Corollary 11.2. *Let $k = \mathbb{R}$, Ψ a semisimple root datum and assume Γ, θ act on (X, Φ) as in 5.21. A sixtuple $\mathcal{D} = (X,\ \Delta,\ \Delta_0(\Gamma),\ \Delta_0(\theta),\ [\sigma],\ \theta^*)$ is a basic Γ_θ-index if and only if it satisfies the following conditions:*

(1) *$[\sigma]$ and θ^* commute.*
(2) *$\Delta_0(\theta)$ is $[\sigma]$-stable, $\sigma \in \Gamma$, and $\Delta_0(\Gamma)$ is θ^*-stable.*
(3) *for every connected component Δ_1 of $\Delta_0(\theta) \cup \Delta_0(\Gamma)$ we have $\Delta_1 \subset \Delta_0(\Gamma)$ or $\Delta_1 \subset \Delta_0(\theta)$.*

11.3. Similarly as in the case of Γ-indices (see 10.19) the classification of the (Γ, θ)-indices can be reduced to a classification of absolutely irreducible (Γ, θ)-indices. That it suffices to consider absolutely irreducible indices can be seen as follows. Suppose $\mathcal{D} = (X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta), \theta^*, [\sigma])$ and X is simply connected. If $\mathcal{D}$ is irreducible, but not absolutely irreducible, then $\Delta = \Delta_1 \cup \ldots \cup \Delta_s$, where the Δ_i are mutually disjoint connected components of Δ and correspondingly one has $X = X_1 + \cdots + X_s$ and $\Phi = \Phi_1 \cup \cdots \cup \Phi_s$. Define $\mathcal{E}_1 = \{\sigma \in \Gamma_\theta \mid \Delta_1^{[\sigma]} = \Delta_1\}$ and $\Gamma_1 = \{\sigma \in \Gamma \mid \Delta_1^{[\sigma]} = \Delta_1\}$. Then $\Gamma_\theta = \bigcup_{i=1}^s \mathcal{E}_1\sigma_i$, where $\Delta_i = \Delta_1^{[\sigma_i]}$. Let $\mathcal{D}_1 = (X_1, \Delta_1, \Delta_1 \cap \Delta_0(\Gamma), \Delta_1 \cap \Delta_0(\theta), \theta^*, [\sigma])$, where $\sigma \in \Gamma_1$ and let k_1 be the fixed field of Γ_1. Denote the underlying Γ_1-index of $\mathcal{D}_1$ by $\mathcal{D}_1(k_1)$. We note that $\mathcal{E}_1$ is the subgroup of Γ_θ spanned by Γ_1 and $\{\mathrm{id}, -\theta\}$ if $\theta(\Delta_1) = \Delta_1$ and $\mathcal{E}_1 = \Gamma_1$ if $\theta(\Delta_1) \neq \Delta_1$.

We have the following cases:

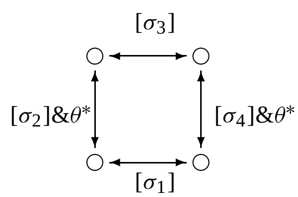
11.3.1. *$\mathcal{E}_1 = \Gamma_\theta$.* We note first that the (Γ, θ)-index $\mathcal{D}$ is absolutely irreducible if and only if $\mathcal{E}_1 = \Gamma_\theta$. In Table 1 we list the absolutely irreducible (Γ, θ)-indices together with the corresponding reduced root system of the symmetric k-variety for k the real numbers, a $\mathfrak{p}$-adic field $\mathbb{Q}_p$, a number field $\mathfrak{n}$ and a finite field $\mathbb{F}_q$. In this table we use the diagrammatic representation of the (Γ, θ)-indices as in 5.28. For a more detailed discussion of the notion used, see 11.4.

11.3.2. *$\mathcal{E}_1 \neq \Gamma_\theta$ and $\mathcal{E}_1 \supsetneq \Gamma_1$.* The condition $\mathcal{E}_1 \supsetneq \Gamma_1$ implies that $-\theta \in \mathcal{E}_1$, hence $\theta(\Delta_1) = \Delta_1$. In this case we get s copies of the (Γ, θ)-index $\mathcal{D}_1$.

11.3.3. *$\mathcal{E}_1 \neq \Gamma_\theta$ and $-\theta \notin \mathcal{E}_1 = \Gamma_1 = \Gamma$.* Since $-\theta \notin \mathcal{E}_1$, it follows that $\theta^*(\Delta_1) \neq \Delta_1$. So in this case $s = 2$ and θ exchanges two copies of the absolutely irreducible Γ-index $\mathcal{D}_1(k)$.

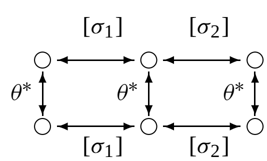
11.3.4. *$\mathcal{E}_1 \neq \Gamma_\theta$, $\Gamma_1 \subsetneq \Gamma$, $-\theta \notin \mathcal{E}_1 = \Gamma_1$ and $\mathcal{D}_\theta$ is irreducible.* Since $-\theta \notin \mathcal{E}_1$ we have $\theta^*(\Delta_1) \neq \Delta_1$ and since $\mathcal{D}_\theta$ is irreducible it follows that $s = 2$. Finally since $\Gamma_1 \subsetneq \Gamma$ it follows that Γ also exchanges two copies of the diagram $\mathcal{D}_1(k)$.

11.3.5. *$\mathcal{E}_1 \neq \Gamma_\theta$, $\Gamma_1 \subsetneq \Gamma$, $-\theta \notin \mathcal{E}_1 = \Gamma_1$, $\mathcal{D}_\theta$ is not irreducible and $\mathcal{D}_k$ is irreducible.* In this case $s = 2r$ is even and $\theta^*(\Delta_i) \neq \Delta_i$ for all $i = 1, \ldots, s$. So θ^* maps one half of the irreducible components into the other half. Since θ^* and $[\sigma]$ commute for all $\sigma \in \Gamma$ this restricts the number of possible cases considerably. An example of this is the following index where $\Phi_1 \subset X_1$ is of type A_1 and $\mathcal{D}_k$ consists of 4 copies of A_1 exchanged by $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \Gamma$:



In this case the restricted root system $\Phi(A_\theta^-)$ is isomorphic to Φ_1. We note that if $|\Gamma| = 2$ (for example if $k = \mathbb{R}$) then this case does not occur.

11.3.6. *$\mathcal{E}_1 \neq \Gamma_\theta$, $\Gamma_1 \subsetneq \Gamma$, $-\theta \notin \mathcal{E}_1 = \Gamma_1$ and $\mathcal{D}_\theta$, $\mathcal{D}_k$ are not irreducible.* Since $\mathcal{D}$ is irreducible, $\mathcal{D}_k$ not irreducible and θ^* commutes with the action of Γ the Γ-index $\mathcal{D}_k$ consists of two irreducible components $\mathcal{D}_k^1$ and $\mathcal{D}_k^2$. Similarly since $\mathcal{D}$ is irreducible and θ^* and $[\sigma]$ commute for all $\sigma \in \Gamma$ it follows that $\theta^*(\mathcal{D}_k^1) = \mathcal{D}_k^2$. Finally since $\Gamma_1 \subsetneq \Gamma$ the Γ-indices $\mathcal{D}_k^1$ and $\mathcal{D}_k^2$ are irreducible, but not absolutely irreducible. An example of this is the following index where $\Phi_1 \subset X_1$ is of type A_1 and $\mathcal{D}_k^1$ consists of 3 copies of A_1 exchanged by $\sigma_1, \sigma_2 \in \Gamma$:



Note that in this case the restricted root system $\Phi(A_\theta^-)$ is isomorphic to Φ_1.

11.4. In order to be able to refer to the absolutely irreducible (Γ, θ)-indices in Table 1, we will use the following notation, which combines the notation of Tits [Tit66] for Γ-indices and the notation in [Hel88] for θ-indices. In particular let $\mathcal{D} = (X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta), [\sigma], \theta^*)$ be a basic Γ_θ-index as in 10.40 and let $\mathcal{D}_k$ and $\mathcal{D}_\theta$ be the corresponding Γ-index and the θ-index as in 10.41. For the

Γ-indices we use the notation ${}^g X^t_{n,r}$. Here X denotes the type of Φ, i.e. one of $A,\ B, \dots,\ G$, n the rank of Φ, r the rank of $\bar{\Delta}_\Gamma$ and g the order of the action of Γ on the Dynkin diagram. In the case that $g = 1$ (i.e. the Dynkin diagram has no nontrivial automorphism) we will omit it in the notation. Finally t denotes either the degree of the division algebra, which occurs in the definition of the considered form or the dimension of the anisotropic kernel. To differentiate between these two cases we put t between parentheses when it stands for the degree of the division algebra. In fact the degree of the division algebra is only used if X is of classical type.

As for the θ-indices they can be described by the type in the Cartan notation together with the rank of the restricted root system $\bar{\Delta}_\theta$, see [Hel88, Table II]. We will use a superindex to indicate the rank of $\bar{\Delta}_\theta$. Similar as in [Hel88] we omit the action of θ^* on $\Delta_0(\theta)$ in the θ-index, because $\theta^*|\Delta_0(\theta) = -w_0(\theta)$ is completely determined by the type of the root system $\Phi_0(\theta)$. So combining these two we will denote a (Γ, θ)-index by ${}^g X^t_{n,r}(\text{type}\,\theta^p)$, where ${}^g X^t_{n,r}$ is as above, type θ is the Cartan notation of the involution and p denotes the rank of $\bar{\Delta}_\theta$. For example ${}^2A^{(1)}_{2n+1,n+1}(III^p_b)$ means that Φ is of type A_{2n+1}, θ is of type $AIII_b$, the action of Γ on the Dynkin diagram is the diagram automorphism, the degree of the division algebra is 1 and rank $\bar{\Delta}_\Gamma = n+1$, rank $\bar{\Delta}_\theta = p$.

In the next column of this table we list the Dynkin diagram of the restricted root system of the corresponding symmetric k-variety, which is the root system of a maximal (θ, k)-split torus of G. The multiplicities easily follow from this restricted root system and the corresponding (Γ, θ)-index.

In the last 4 columns of this table we indicate if a particular (Γ, θ)-index is admissible or not for the 4 different types of fields we consider. Here a "$+$" means that this (Γ, θ)-index is admissible for at least one field of that type (say for example number fields). Similarly a "$-$" means that this (Γ, θ)-index is not admissible for all fields of that type.

In the Table we will also write Γ^* instead of $[\sigma]$ if the action $[\cdot]$ of Γ on Δ is non-trivial, i.e. a Γ^* in the index means that for each the diagram automorphism which is indicated with an arrow, there exists a $\sigma \in \Gamma$, such that the action of $[\sigma]$ is precisely this diagram automorphism. If Φ is irreducible then Δ has more then one non-trivial diagram automorphism if and only if Φ is of type D_4. So only in this case Γ^* can stand for the action of more then one element $[\sigma]$, with $\sigma \in \Gamma$.

Table 1: Absolutely irreducible (Γ, θ)-indices

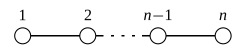
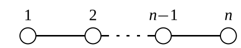
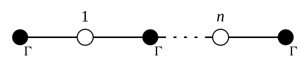
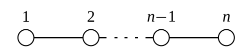
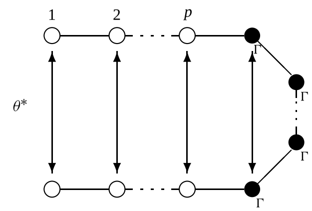
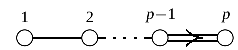
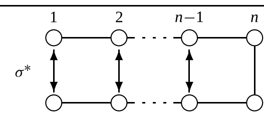
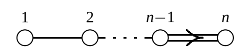
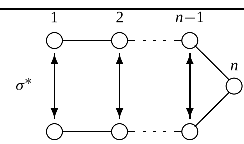
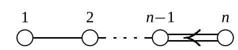
Type G	(Γ, θ)-index	$\Phi(A_\theta^-)$	admissible $\mathbb{R}$	admissible $\mathbb{Q}_p$	admissible $\mathbb{F}_q$	admissible $\mathfrak{n}$
$A_{n,n}^{(1)}(I)$			+	+	+	+
$A_{2n+1,2n+1}^{(1)}(II)$			+	+	+	+
$A_{n,n}^{(1)}(III_a^p)$ $n-2p \geq 1$			+	+	+	+
$A_{2n,2n}^{(1)}(III_b)$, $\sigma=\theta$ ${}^2A_{2n,n}^{(1)}(III_b)$, $\sigma=\Gamma\&\theta$ ${}^2A_{2n,n}^{(1)}(I)$, $\sigma=\Gamma$			+	+	+	+
$A_{2n-1,2n-1}^{(1)}(III_b)$, $\sigma=\theta$ ${}^2A_{2n-1,n}^{(1)}(III_b)$, $\sigma=\Gamma\&\theta$ ${}^2A_{2n-1,n}^{(1)}(I)$, $\sigma=\Gamma$			+	+	+	+

Table 1: (continued)

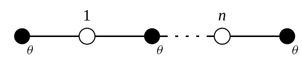
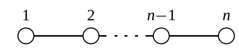
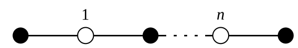
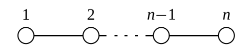
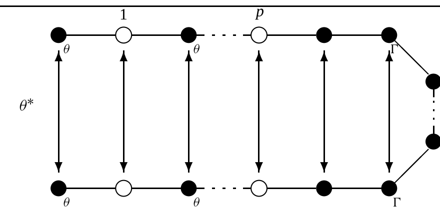
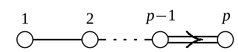
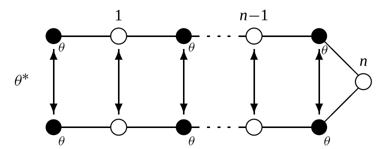
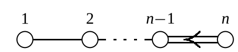
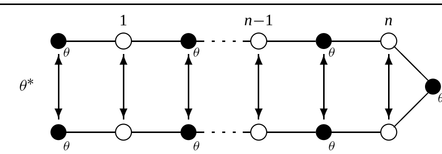
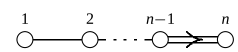
Type (G, θ)	(Γ, θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
$A^{(2)}_{2n+1,n}(I)$			+	+	−	+
$A^{(2)}_{2n+1,n}(II)$			+	+	−	+
$A^{(2)}_{2n+1,n}(III_a^{2p})$ $1 \le 2p < n$			+	+	−	+
$A^{(2)}_{4n-1,2n-1}(III_b)$			+	+	−	+
$A^{(2)}_{4n+1,2n}(III_b)$			+	+	−	+

Table 1: (continued)

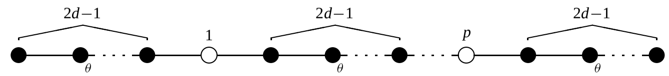
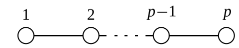
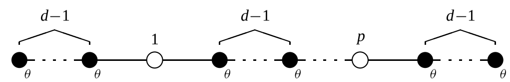
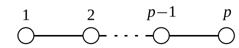
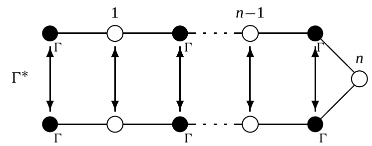
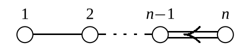
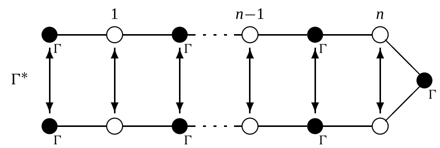
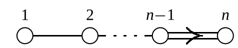
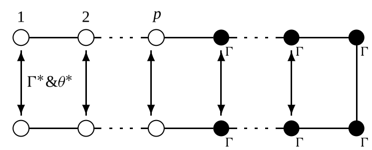
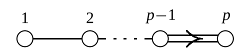
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
$A_{n,p}^{(2d)}(II)$ $2d = \frac{n+1}{p+1} > 2$			−	+	−	+
$A_{n,p}^{(d)}(I)$ $d = \frac{n+1}{p+1} > 2$			−	+	−	+
${}^2A_{4n-1,2n}^{(1)}(II)$			+	+	+	+
${}^2A_{4n+1,2n+1}^{(1)}(II)$			+	+	+	+
${}^2A_{2n,n}^{(1)}(III^p)$ $1 \leq p < n$			+	+	+	+

Table 1: (continued)

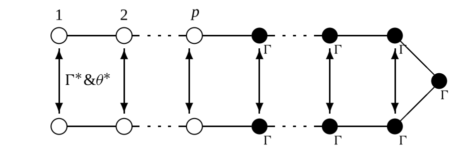
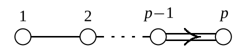
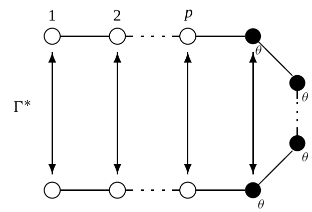
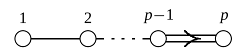
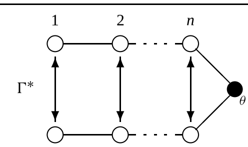
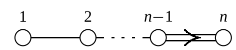
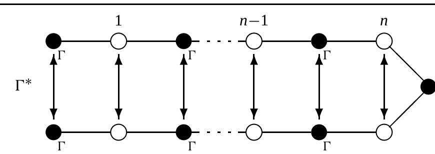
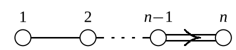
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
${}^2A^{(1)}_{2n+1,n+1}(III^p)$ $1 \le p \le n$			+	+	+	+
${}^2A^{(1)}_{n,p}(I)$ $n-2p>1$			+	−	−	+
${}^2A^{(1)}_{2n+1,n}(I)$			+	+	−	+
${}^2A^{(1)}_{4n+1,2n}(II)$			+	+	−	+

Table 1: (continued)

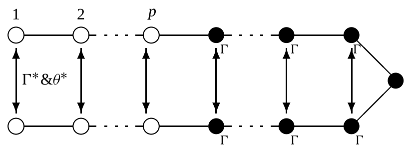
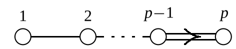
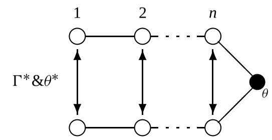
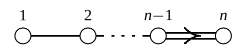
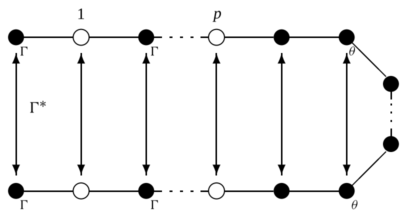
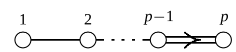
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
${}^2A^{(1)}_{2n+1,n}(III^p)$ $1 \le p \le n$			+	+	−	+
${}^2A^{(1)}_{2n+1,n}(III_b)$			+	+	−	+
${}^2A^{(1)}_{2n+1,2p}(II)$ $1 \le 2p < n$			+	−	−	+

Table 1: (continued)

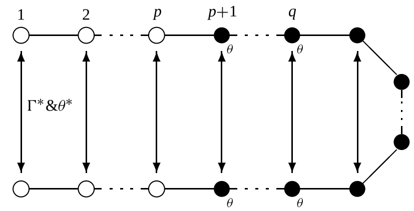
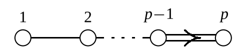
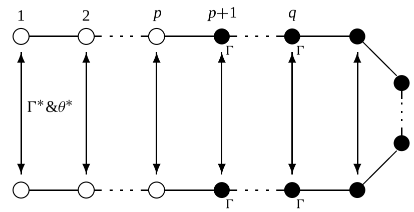
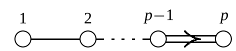
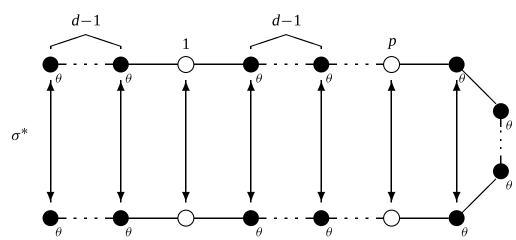
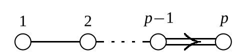
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
${}^2A_{n,p}^{(1)}(III^q)$ $1 \le p < q \le \frac{1}{2}(n+1)$ $p < \frac{1}{2}(n-1)$			+	−	−	+
${}^2A_{n,q}^{(1)}(III^p)$ $1 \le p \le q \le \frac{1}{2}(n-1)$			+	−	−	+
${}^2A_{n,p}^{(d)}(I)$ $\sigma^*=\Gamma^*$ ${}^2A_{n,p}^{(d)}(III_b)$ $\sigma^*=\Gamma^*\&\theta^*$ $d \ge 2$ $2pd \le n+1$ $d \mid n+1$			−	−	−	+

Table 1: (continued)

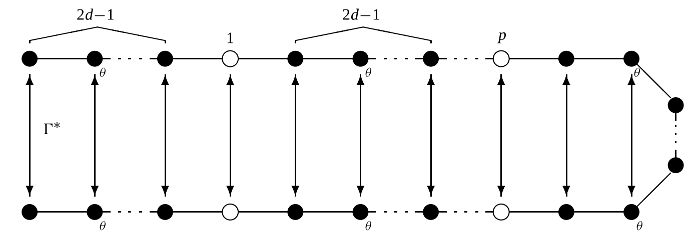
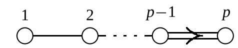
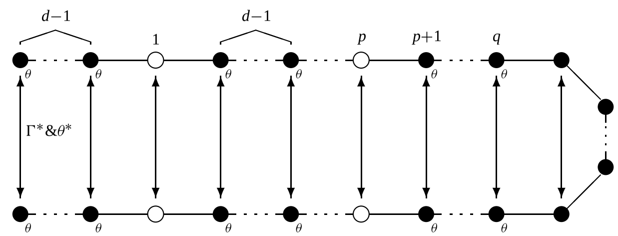
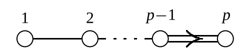
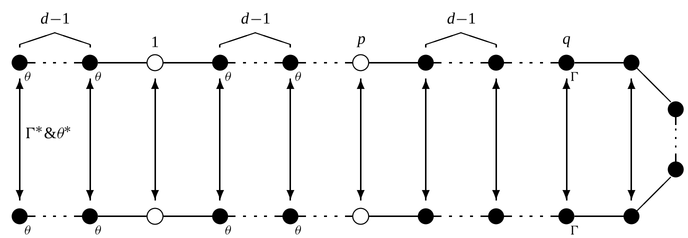
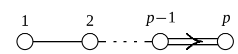
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
${}^2A^{(2d)}_{2n-1,p}(II)$ $d \geq 1$ $2pd \leq n$ $d \mid n$			−	−	−	+
${}^2A^{(d)}_{n,p}(III^q_a)$ $d \geq 2$ $2pd \leq n+1$ $d \mid n+1$ $q \geq pd$			−	−	−	+
${}^2A^{(d)}_{n,q}(III^p_a)$ $d \geq 2$ $2qd \leq n+1$ $d \mid n+1$ $p = rd,\ r < q$			−	−	−	+

Table 1: (continued)

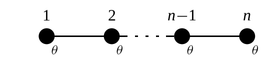
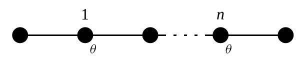
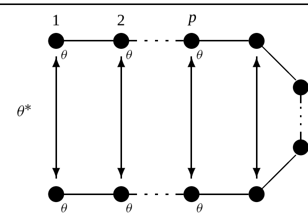
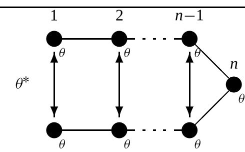
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
$A_{n,0}^{(1)}(I)$		$\emptyset$	n=1	−	−	+
$A_{2n+1,0}^{(1)}(II)$		$\emptyset$	−	+	−	+
$A_{n,0}^{(1)}(III_a^p)$ $n-2p\geq 1$		$\emptyset$	−	+	−	+
$A_{2n-1,0}^{(1)}(III_b)$		$\emptyset$	−	+	−	+

Table 1: (continued)

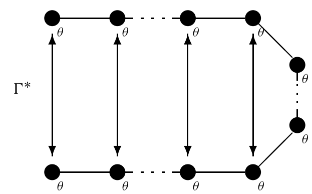
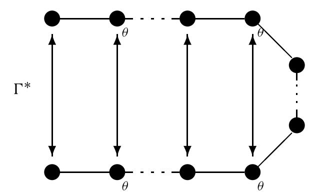
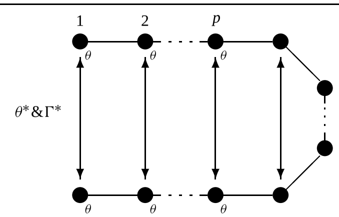
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	n
$A_{n,0}^{(1)}(I)$		$\emptyset$	+	−	−	+
$A_{2n+1,0}^{(1)}(II)$		$\emptyset$	+	−	−	+
$A_{n,0}^{(1)}(III_a^p)$ $n-2p\geq 1$		$\emptyset$	+	−	−	+

Table 1: (continued)

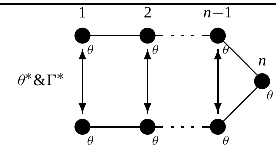
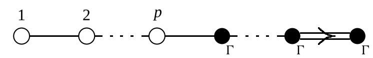
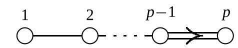
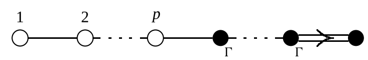
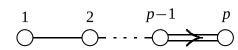
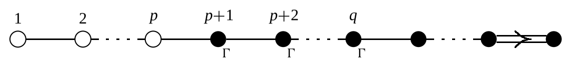
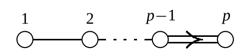
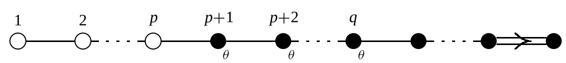
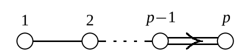
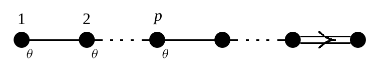
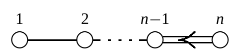
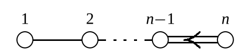
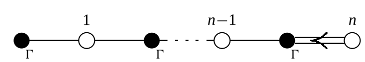
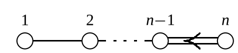
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
$A^{(1)}_{2n-1,0}(III_b)$		$\emptyset$	+	−	−	+
$B_{n,n}(I^p)$ $1 \le p \le n$			+	+	+	+
$B_{n,n-1}(I^p)$ $p{\le}n{-}1$			+	+	−	+
$B_{n,q}(I^p)$ $1{\le}p{\le}q{\le}n{-}2$			+	−	−	+
$B_{n,p}(I^q)$ $1{\le}p{\le}q{<}n$			+	−	−	+
$B_{n,0}(I^p)$ $1 \le p \le n$		$\emptyset$	+	−	−	+
$C^{(1)}_{n,n}(I)$			+	+	+	+
$C^{(1)}_{2n,2n}(II_b)$			+	+	+	+

Table 1: (continued)

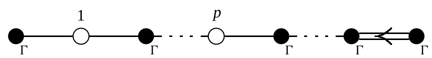
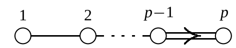
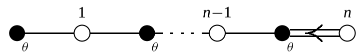
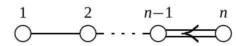
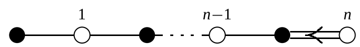
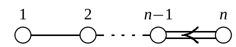
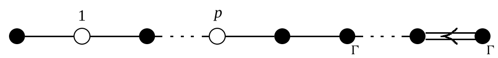
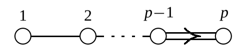
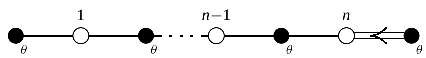
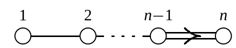
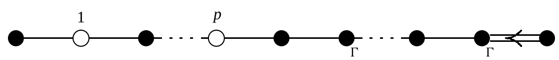
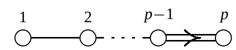
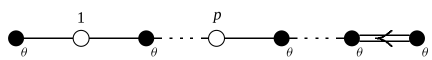
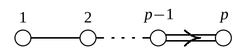
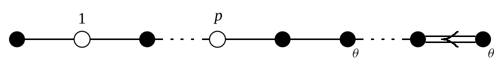
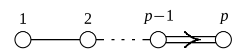
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
$C_{n,n}^{(1)}(II^p)$ $1\le p\le\frac{1}{2}(n-1)$			+	+	+	+
$C_{2n,n}^{(2)}(I)$			+	+	−	+
$C_{2n,n}^{(2)}(II_b)$			+	+	−	+
$C_{2n,n}^{(2)}(II^p)$ $1\le p\le(n-1)$			+	+	−	+
$C_{2n+1,n}^{(2)}(I)$			+	+	−	+
$C_{2n+1,n}^{(2)}(II^p)$ $1\le p\le n$			+	+	−	+
$C_{n,p}^{(2)}(I)$ $p<[\frac{n-1}{2}]$			+	−	−	+
$C_{2n,p}^{(2)}(II_b)$ $p<n-1$			+	−	−	+

Table 1: (continued)

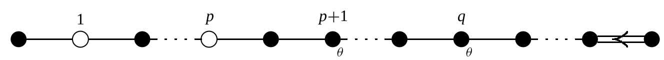
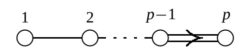
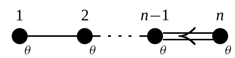
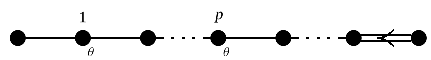
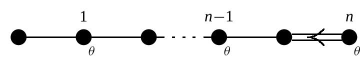
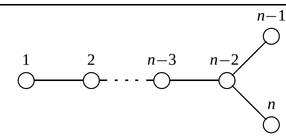
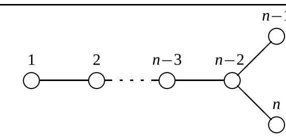
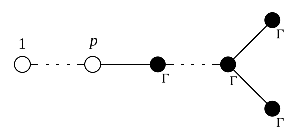
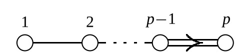
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
$C_{n,p}^{(2)}(II^q)$, $\sigma=\theta$ $C_{n,q}^{(2)}(II^p)$, $\sigma=\Gamma$ $1\le p\le q<\frac{1}{2}n$			+	−	−	+
$C_{n,0}^{(1)}(I)$		$\emptyset$	+	−	−	+
$C_{n,0}^{(1)}(II^p)$ $1\le p\le \frac{1}{2}(n-1)$		$\emptyset$	+	−	−	+
$C_{2n,0}^{(1)}(II_b)$		$\emptyset$	+	−	−	+
$D_{n,n}^{(1)}(I_b)$			+	+	+	+
$D_{n,n}^{(1)}(I^p)$ $p<n-1$			+	+	+	+

Table 1: (continued)

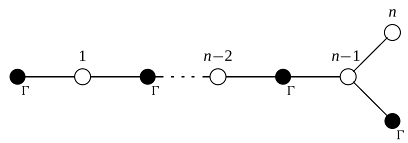
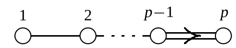
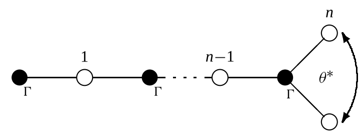
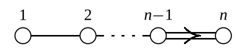
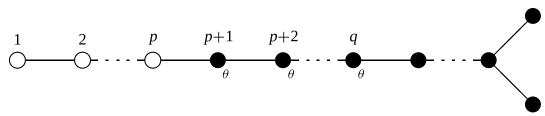
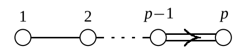
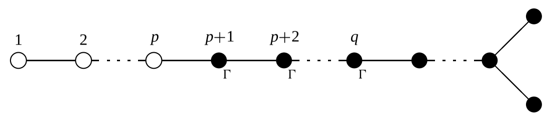
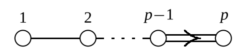
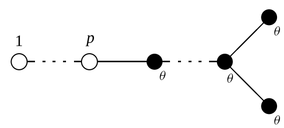
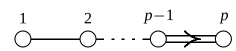
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
$D^{(1)}_{2n,2n}(III_a)$			+	+	+	+
$D^{(1)}_{2n+1,2n+1}(III_b)$			+	+	+	+
$D^{(1)}_{n,p}(I_a^q)$ $1\leq p\leq q\leq n-2$ $n-p=2m\geq 2$			+	$p=n-2$	−	+
$D^{(1)}_{n,q}(I_a^p)$ $1\leq p< q\leq n-2$ $n-p=2m\geq 2$			+	$q=n-2$	−	+
$D^{(1)}_{n,p}(I_b)$ $n-p=2m\geq 2$			+	$p=n-2$	−	+

Table 1: (continued)

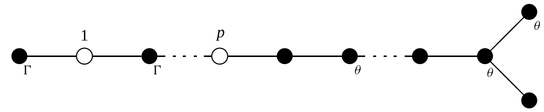
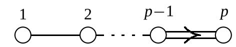
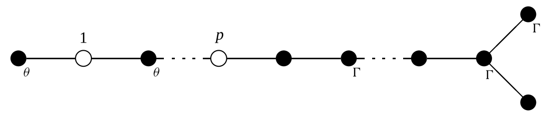
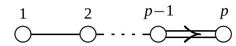
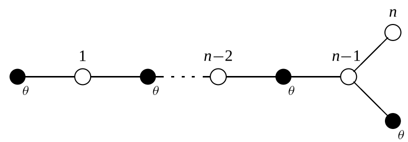
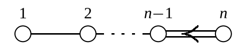
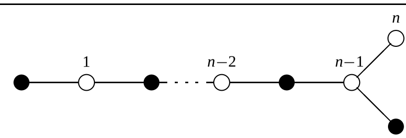
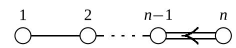
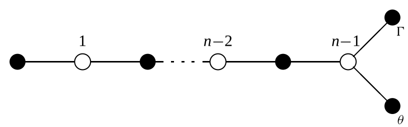
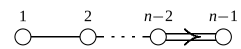
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
$D^{(1)}_{2n,2p}(III_a)$ $p<n$			+	$p=n-1$	−	+
$D^{(2)}_{2n,n}(I_a^{2p})$ $p\le n-1$			+	+	−	+
$D^{(2)}_{2n,n}(I_b)$			+	+	−	+
$D^{(2)}_{2n,n}(III_a)$			+	+	−	+
$D^{(2)}_{2n,n}(III_a)'$			+	+	−	+

Table 1: (continued)

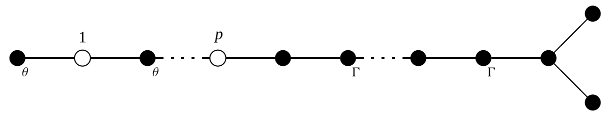
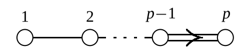
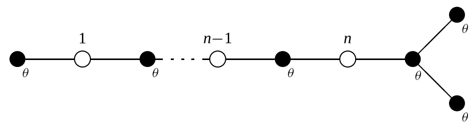
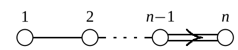
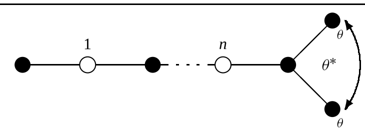
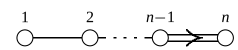
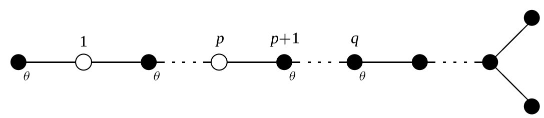
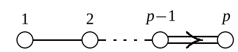
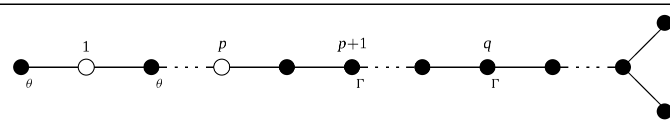
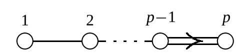
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
$D^{(2)}_{2n+3,n}(I_a^{2p})$ $1\le p\le n$			−	+	−	+
$D^{(2)}_{2n+3,n}(I_b)$			−	+	−	+
$D^{(2)}_{2n+3,n}(III_b)$			−	+	−	+
$D^{(2)}_{2n,p}(I_a^{q})$ $1\le 2p\le q<2n-1$			−	−	−	+
$D^{(2)}_{2n,q}(I_a^{2p})$ $1\le 2p<2q\le 2n-1$			−	−	−	+

Table 1: (continued)

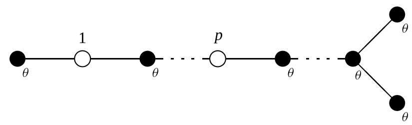
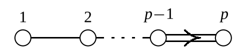
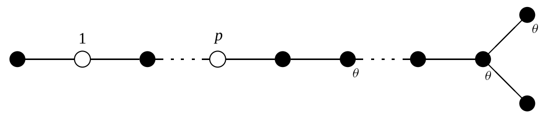
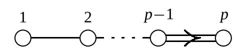
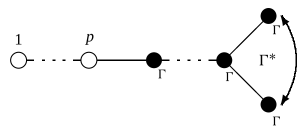
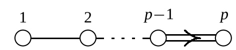
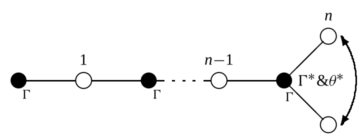
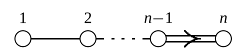
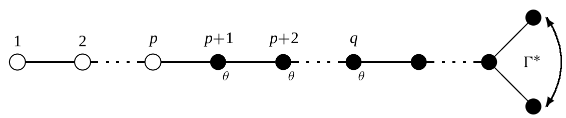
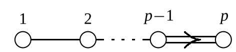
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
$D_{2n,p}^{(2)}(I_b)$ $n-p\geq 1$			$-$	$-$	$-$	$+$
$D_{2n,p}^{(2)}(III_a)$ $n-p\geq 1$			$-$	$-$	$-$	$+$
${}^2D_{n+1,n}^{(1)}(I^p)$ $p\neq n$			$+$	$+$	$+$	$+$
${}^2D_{2n+1,2n}^{(1)}(III_b)$			$+$	$+$	$+$	$+$
${}^2D_{n,p}^{(1)}(I_a^q)$ $p\leq q\leq n-2$			$n-p=2l+1$	$-$	$-$	$+$

Table 1: (continued)

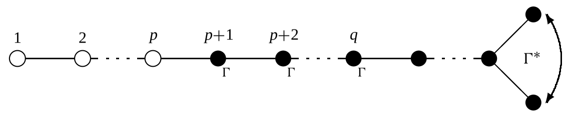
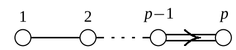
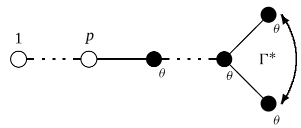
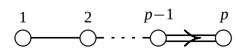
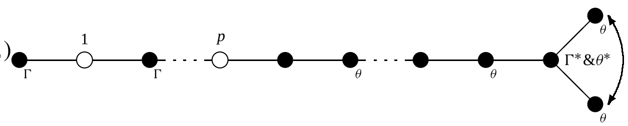
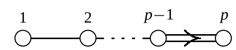
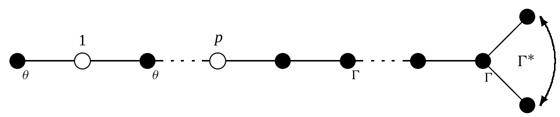
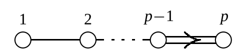
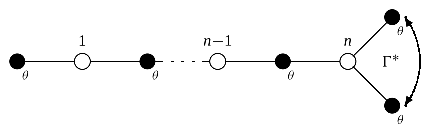
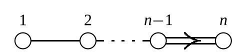
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
${}^2D_{n,q}^{(1)}(I_a^p)$ $1\leq p<q\leq n-2$ $n-p=2m\geq 2$			$n-q=$ $2l+1$	$-$	$-$	$+$
${}^2D_{n,p}^{(1)}(I_b)$ $n-p>1$			$n-p=$ $2l+1$	$-$	$-$	$+$
${}^2D_{2n+1,2p}^{(1)}(III_b)$ $p<n$			$+$	$-$	$-$	$+$
${}^2D_{2n+2,n}^{(2)}(I_a^{2p})$ $p\leq n$			$-$	$+$	$-$	$+$
${}^2D_{2n+2,n}^{(2)}(I_b)$			$-$	$+$	$-$	$+$

Table 1: (continued)

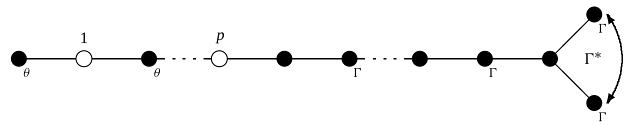
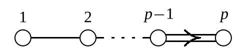
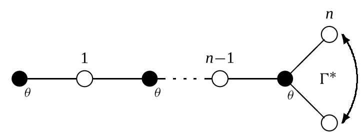
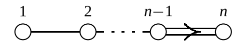
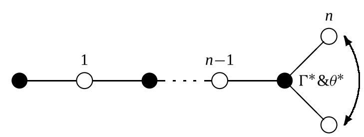
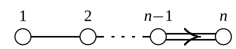
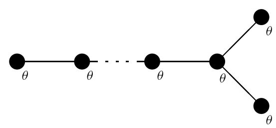
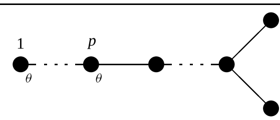
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
${}^2D^{(2)}_{2n+1,n}(I_a^{2p})$ $1\le p<n$			+	+	−	+
${}^2D^{(2)}_{2n+1,n}(I_b)$			+	+	−	+
${}^2D^{(2)}_{2n+1,n}(III_b)$			+	+	−	+
$D^{(1)}_{n,0}(I_b)$		$\emptyset$	n even	−	−	+
$D^{(1)}_{n,0}(I^p)$ $p<n-1$		$\emptyset$	n even	−	−	+

Table 1: (continued)

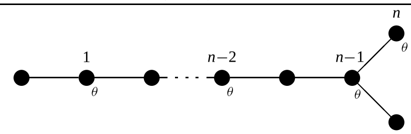
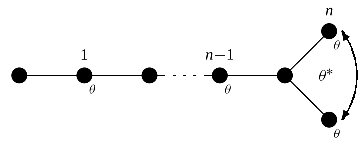
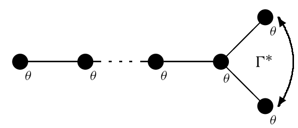
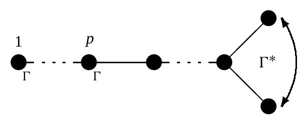
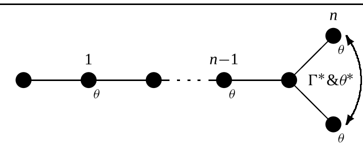
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
$D_{2n,0}^{(1)}(III_a)$		$\emptyset$	$+$	$-$	$-$	$+$
$D_{2n+1,0}^{(1)}(III_b)$		$\emptyset$	$-$	$-$	$-$	$+$
$D_{n,0}^{(1)}(I_b)$		$\emptyset$	n odd	$-$	$-$	$+$
$D_{n,0}^{(1)}(I^p)$ $p<n-1$		$\emptyset$	n odd	$-$	$-$	$+$
$D_{2n+1,0}^{(1)}(III_b)$		$\emptyset$	$+$	$-$	$-$	$+$

Table 1: (continued)

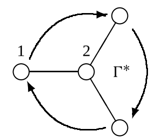
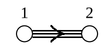
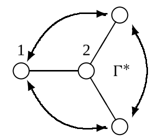
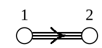
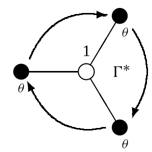
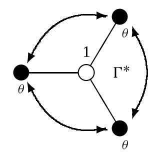
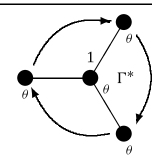
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
${}^3D^2_{4,2}(I_b)$			−	+	+	+
${}^6D^2_{4,2}(I_b)$			−	+	−	+
${}^3D^9_{4,1}(I_b)$			−	−	−	+
${}^6D^9_{4,1}(I_b)$			−	−	−	+
${}^3D^{28}_{4,0}(I_b)$		$\emptyset$	−	−	−	+

Table 1: (continued)

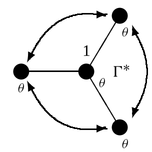
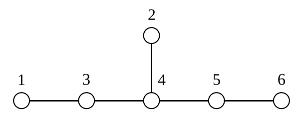
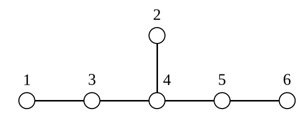
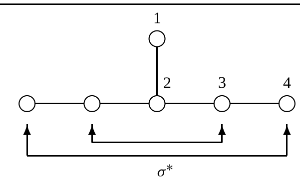
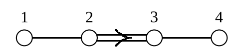
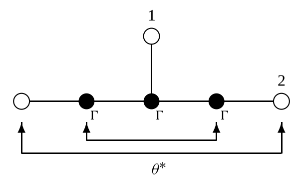
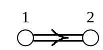
Type (G, θ)	(Γ, θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
${}^6D_{4,0}^{28}(I_b)$		$\emptyset$	−	−	−	+
${}^1E_{6,6}^0(I)$			+	+	+	+
${}^1E_{6,6}^0(II)$, $\sigma^*=\theta^*$ ${}^2E_{6,4}^{16}(I)$ $\sigma^*=\Gamma^*$ ${}^2E_{6,4}^{16}(II)$ $\sigma^*=\Gamma^*\&\theta^*$			+	+	+	+
${}^1E_{6,6}^0(III)$			+	+	+	+

Table 1: (continued)

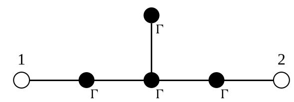
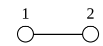
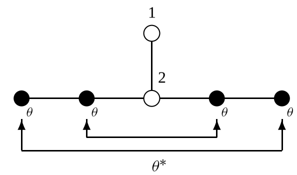
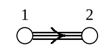
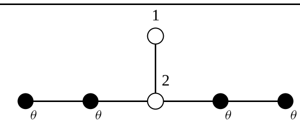
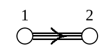
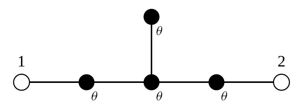
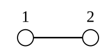
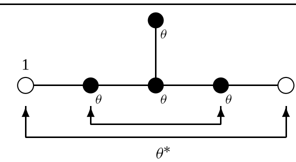
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
${}^1E_{6,6}^0(IV)$			+	+	+	+
${}^1E_{6,2}^{16}(II)$			−	+	−	+
${}^1E_{6,2}^{16}(I)$			−	+	−	+
${}^1E_{6,2}^{28}(I)$			+	−	−	+
${}^1E_{6,2}^{28}(II)$			+	−	−	+

Table 1: (continued)

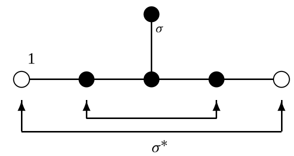
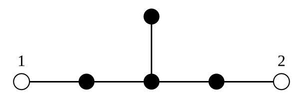
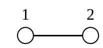
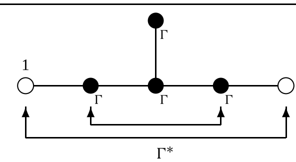
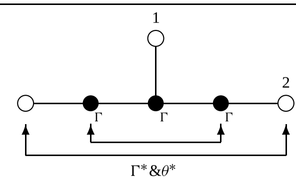
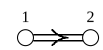
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
${}^1E_{6,2}^{28}(III)$, $\sigma^*{=}\theta^*$ ${}^2E_{6,2}^{16'}(IV)$, $\sigma^*{=}\Gamma^*$			+	−	−	+
${}^1E_{6,2}^{28}(IV)$			+	−	−	+
${}^2E_{6,4}^{16}(IV)$			+	+	+	+
${}^2E_{6,4}^{16}(III)$			+	+	+	+

Table 1: (continued)

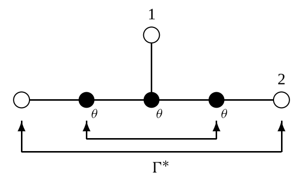
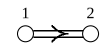
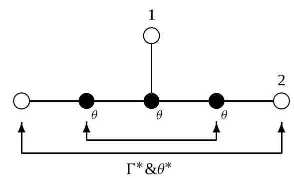
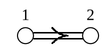
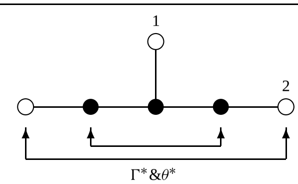
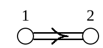
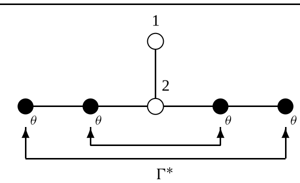
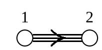
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
${}^2E_{6,2}^{16'}(I)$			+	−	−	+
${}^2E_{6,2}^{16'}(II)$			+	−	−	+
${}^2E_{6,2}^{16'}(III)$			+	−	−	+
${}^2E_{6,2}^{16''}(I)$			−	−	−	+

Table 1: (continued)

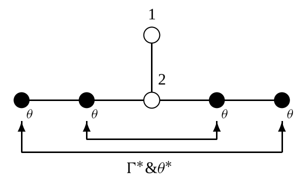
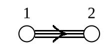
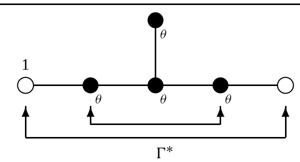
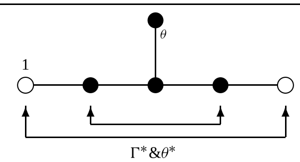
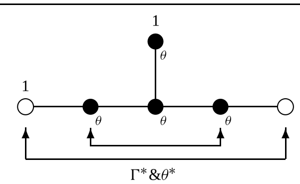
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
${}^2E_{6,2}^{16''}(II)$			−	−	−	+
${}^2E_{6,1}^{29}(I)$			−	−	−	+
${}^2E_{6,1}^{29}(III)$			−	−	−	+
${}^2E_{6,1}^{29}(II)$			−	−	−	+

Table 1: (continued)

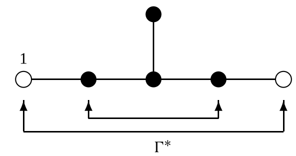
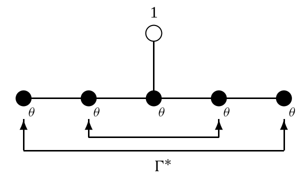
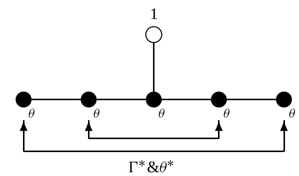
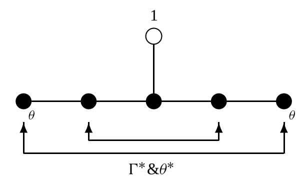
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
${}^2E_{6,1}^{29}(IV)$			−	−	−	+
${}^2E_{6,1}^{35}(I)$			−	−	−	+
${}^2E_{6,1}^{35}(II)$			−	−	−	+
${}^2E_{6,1}^{35}(III)$			−	−	−	+

Table 1: (continued)

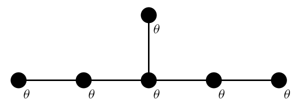
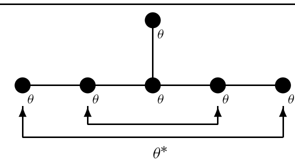
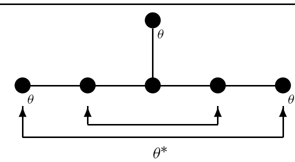
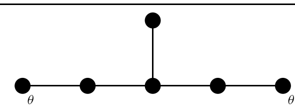
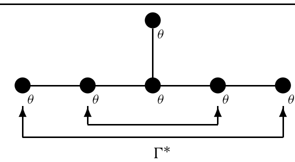
Type (G, θ)	(Γ, θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
${}^1E_{6,0}^{78}(I)$		$\emptyset$	−	−	−	+
${}^1E_{6,0}^{78}(II)$		$\emptyset$	−	−	−	+
${}^1E_{6,0}^{78}(III)$		$\emptyset$	−	−	−	+
${}^1E_{6,0}^{78}(IV)$		$\emptyset$	−	−	−	+
${}^2E_{6,0}^{78}(I)$		$\emptyset$	+	−	−	+

Table 1: (continued)

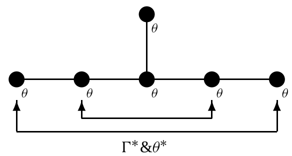
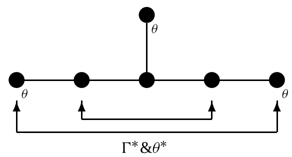
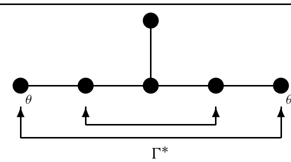
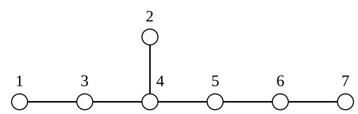
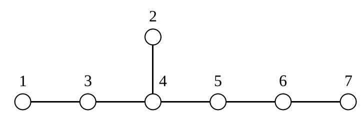
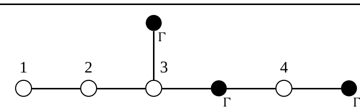
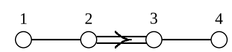
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
${}^2E_{6,0}^{78}(II)$		$\emptyset$	+	−	−	+
${}^2E_{6,0}^{78}(III)$		$\emptyset$	+	−	−	+
${}^2E_{6,0}^{78}(IV)$		$\emptyset$	+	−	−	+
$E_{7,7}^{0}(V)$			+	+	+	+
$E_{7,7}^{0}(VI)$			+	+	+	+

Table 1: (continued)

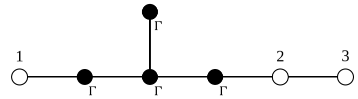
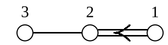
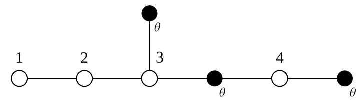
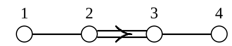
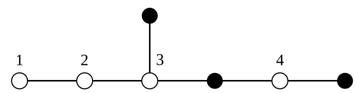
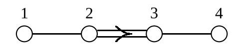
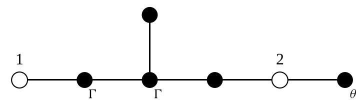
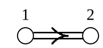
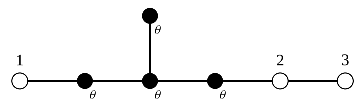
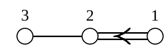
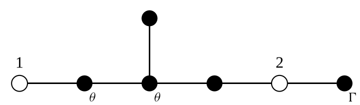
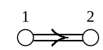
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
$E_{7,7}^{0}(VII)$			+	+	+	+
$E_{7,4}^{9}(V)$			+	+	−	+
$E_{7,4}^{9}(VI)$			+	+	−	+
$E_{7,4}^{9}(VII)$			+	+	−	+
$E_{7,3}^{28}(V)$			+	−	−	+
$E_{7,3}^{28}(VI)$			+	−	−	+

Table 1: (continued)

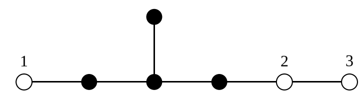
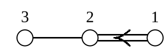
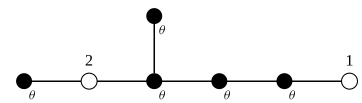
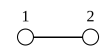
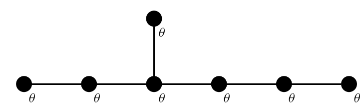
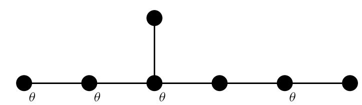
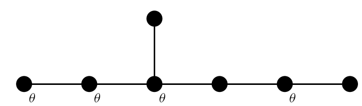
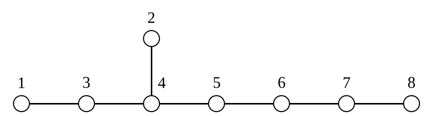
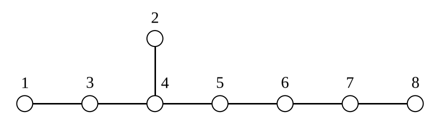
Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
$E_{7,3}^{28}(VII)$			+	−	−	+
$E_{7,2}^{31}(V)$			−	−	−	+
$E_{7,0}^{133}(V)$		∅	+	−	−	+
$E_{7,0}^{133}(VI)$		∅	+	−	−	+
$E_{7,0}^{133}(VII)$		∅	+	−	−	+
$E_{8,8}^{0}(VIII)$			+	+	+	+

Table 1: (continued)

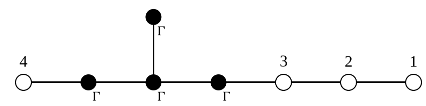
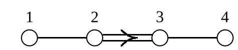
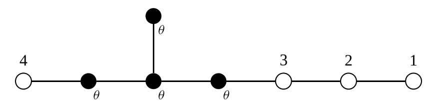
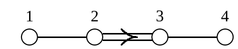
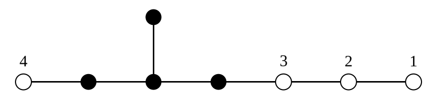
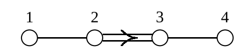
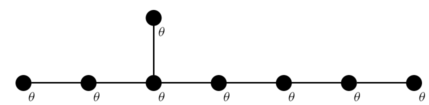
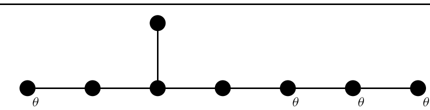
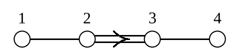
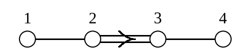
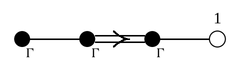
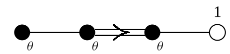
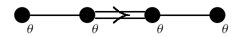
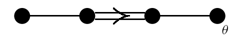
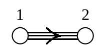
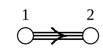
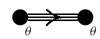
Type (G, θ)	(Γ, θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
$E_{8,8}^{0}(IX)$			+	+	+	+
$E_{8,4}^{28}(VIII)$			+	−	−	+
$E_{8,4}^{28}(IX)$			+	−	−	+
$E_{8,0}^{248}(VIII)$		$\emptyset$	+	−	−	+
$E_{8,0}^{248}(IX)$		$\emptyset$	+	−	−	+
$F_{4,4}^{0}(I)$			+	+	+	+
$F_{4,4}^{0}(II)$			+	+	+	+

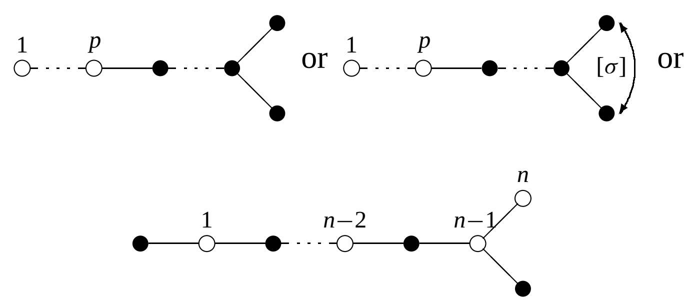
Table 1: (continued)

Type (G,θ)	(Γ,θ)-index	$\Phi(A_\theta^-)$	$\mathbb{R}$	$\mathbb{Q}_p$	$\mathbb{F}_q$	$\mathfrak{n}$
$F_{4,1}^{21}(I)$			+	−	−	+
$F_{4,1}^{21}(II)$			+	−	−	+
$F_{4,0}^{52}(I)$		∅	+	−	−	+
$F_{4,0}^{52}(II)$		∅	+	−	−	+
$G_{2,2}^{0}(I)$			+	+	+	+
$G_{2,0}^{14}(I)$		∅	+	−	−	+

For the isomorphy of the k-involutions of G we will use a notation similar to that of the (Γ,θ)-indices. Since a (Γ,θ)-index $\mathcal{D}$ determines only the isomorphy class of an k-involution under $N_G(A)$ (see Theorem 8.33) we have to add some notation to represent these k-involutions. If σ is a k-involution of G, normally related to (T, A) as in 8.2 with (Γ,θ)-index $\mathcal{D}$, then the other isomorphy classes in $\mathcal{C}_A(\sigma)$ differ at most a k-inner element. Therefor we will denote the k-involutions in a class $\mathcal{C}_A(\sigma)$ by: ${}^gX^t_{n,r}(\text{type}\,\theta^p)(\sigma,\epsilon_i)$, where ${}^gX^t_{n,r}(\text{type}\,\theta^p)$ represents the (Γ,θ)-index $\mathcal{D}$ and $\{\epsilon_i \mid i \in I\}$ is a set of k-inner elements in A representing the different isomorphy classes in $\mathcal{C}_A(\sigma)$. All these involutions have the same (Γ,θ)-index $\mathcal{D}$.

Remark 11.5. If the Dynkin diagram of the restricted root system $\Phi(A^-_\theta)$ of the maximal (θ, k)-split torus A^-_θ of G is of type B_p, then $\Phi(A^-_\theta)$ is of type B_p or BC_p. In fact $\Phi(A^-_\theta)$ is always of type BC_p except in the following cases:

(1) Φ is of type B_n.
(2) Φ is of type D_n and the Γ_θ-index is one of the following:



(3) rank $\Phi(A^-_\theta) = 1$ and Φ is of exceptional type (i.e not a restricted rank 1 case of one of the infinite families). In these case $\Phi(A^-_\theta)$ is of type BC_1.

We conclude this section with the following result about the irreducibility of the restricted root system $\Phi(A^-_\theta)$:

Proposition 11.6. *Let A be a maximal k-split torus of G, $T \supset A$ a maximal k-torus of G, $\mathcal{D}$ an admissible (Γ,θ)-index and $\theta \in \mathrm{Aut}(G, A)$ the corresponding k-involution. Then $\mathcal{D}$ is irreducible if and only if the restricted root system $\Phi(A^-_\theta)$ is irreducible.*

Proof. This result is immediate from the classification of the admissible (Γ,θ)-indices in Table 1. A proof independent of the above classification can be obtained using an argument similar to the one used in [Hel91, 2.15]. ☐

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