# COMBINATORICS RELATED TO ORBIT CLOSURES OF SYMMETRIC SUBGROUPS IN FLAG VARIETIES

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ABSTRACT. Let $G$ be a connected reductive group over an algebraically closed field $k$ of characteristic not 2; let $\theta \in \mathrm{Aut}(G)$ be an involution and $K = G^{\theta} \subseteq G$ the fixed point group of $\theta$ and let $P \subseteq G$ be a parabolic subgroup. The set $K\backslash G/P$ of $(K, P)$-double cosets in $G$ plays an important role in the study of Harish Chandra modules.

In [BH00] we gave a description of the orbits of symmetric subgroups in a flag variety $G/P$ mainly using geometric arguments. For general $P$, it is difficult to describe the combinatorics of the decomposition of the closure of a double coset in terms of $K \times P$ double cosets. However, in some special cases one can describe the combinatorics of the closures of the double cosets in more detail. This paper discusses the special case that $P$ contains a $\theta$-stable Levi factor $L$ and the set of roots of the connected center $S$ of $L$ is a root system with Weyl group $W(S) = N_G(S)/Z_G(S)$. Here $N_G(S)$ (resp. $Z_G(S)$) is the normalizer (resp. centralizer) of $S$ in $G$. In this case the combinatorics of the Weyl Group can be used to describe the closures of the double cosets of a part of the double coset space which includes the open and closed orbits and we get a number of results similar to the case that $P = B$ a Borel subgroup. This root system condition on $P$ is satisfied in many cases. For example in the case that $P$ is a minimal parabolic $k_0$-subgroup of $G$ or a minimal $\theta$-split parabolic subgroup of $G$ or a minimal $(\theta, k_0)$-split parabolic $k_0$-subgroup of $G$. Here $k_0 \subseteq k$ is a subfield of $k$ and $G, \theta$ are defined over $k_0$.

## INTRODUCTION

Let $G$ be a connected reductive group defined over an algebraically closed field $k$ of characteristic not 2 and let $B \subseteq G$ be a Borel subgroup. A closed subgroup $K \subseteq G$ is called *spherical* if the number of $K$-orbits in the flag variety $G/B$ is finite; equivalently, the set of $(K, B)$-double cosets $K\backslash G/B$ is finite. If $H$ and $K$ are spherical subgroups, we may also consider the double coset space $H\backslash G/K$. Results on double coset decompositions are important in representation theory (see, e.g., [vdBS97, BD92, Del98, FJ80, ŌS80]), for the orbit method of Kirillov and Kostant [Kir93] and in the study of Harish Chandra modules (see [Vog83]). In the later case the group $K$ is a *symmetric* subgroup of $G$, that is, $K$ consists of all fixed points of an involutive automorphism $\theta$ of $G$, and $H = P$ a parabolic subgroup of $G$.

These double cosets have been studied for several types of spherical subgroups, but best known are the case that both groups are parabolic subgroups or the case that one or both of the subgroups is a symmetric subgroup. One usually distinguishes the finite from the infinite case. Results in the infinite case include a detailed description of these double cosets and their invariants in the case that both $H$ and $K$ are symmetric subgroups (see [HS01, HS02b]), as well as some results for general spherical subgroups [HS02a].

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Several of the finite cases have been studied in detail as well. In the finite case a lot of the geometry related to these double cosets, for example their closures, can be described by combinatorial properties of the set of double cosets. The best example of this is the case when $H = P$ is a parabolic subgroup and $K = B$ is a Borel subgroup. Then by the Bruhat decomposition, each $(B, P)$-double coset intersects the Weyl group $W$ of a maximal torus into a unique coset of $W_P$, the Weyl group of $P$. This decomposition is one of the fundamental tools in the study of algebraic groups and their representations and in this case the double cosets and their closures can be completely described in terms of the Bruhat order of the Weyl group. Another important case is the case that $K$ is a symmetric subgroup and $H = B$ is a Borel subgroup. In [Spr84] Springer gave several characterizations of these double cosets, including one similar to the Bruhat decomposition of the group. In this case also much is know about the closures of the $(K, B)$-double cosets in $G$. For example, Richardson and Springer [RS90] gave a combinatorial description of a Bruhat type order on, which enables one to compute the double cosets contained in a closure of the $(K, B)$-double coset in $G$ using combinatorial methods (see [Hel96]).

When $H = P$ a parabolic subgroup and $K$ a symmetric subgroup the set $K\backslash G/P$ of $(K, P)$-double cosets in $G$ was studied extensively in [BH00]. Again there are several descriptions of the double cosets, similar in nature to the previous case where $H = B$ is a Borel subgroup. However, for general parabolic subgroups $P$ the combinatorics of the decomposition of the closure of a double coset in terms of $K \times P$ double cosets is much more complicated than in the case that $P = B$ a Borel subgroup. In some special cases one can give a more detailed description of these double cosets. In this paper we consider the special case that $P$ contains a $\theta$-stable Levi factor $L$ and the set of roots of the connected center $S$ of $L$ is a root system with Weyl group $W(S) = N_G(S)/Z_G(S)$. This root system structure can be used to obtain a detailed description of the combinatorics of the closures of the double cosets, which can then be used to describe a short resolution of singularities of closures of double cosets (see Remark 4). In this paper we will give some of these new results on the combinatorics of these orbits. There are many classes of parabolic subgroups for which there is a natural root system and Weyl group associated with the Levi subgroups of a parabolic subgroup. Examples are the case that $P$ is a minimal parabolic $k_0$-subgroup of $G$ or a minimal $\theta$-split parabolic subgroup of $G$ or a minimal $(\theta, k_0)$-split parabolic $k_0$-subgroup of $G$ (see [HW93, HH98, Ric82]). Here $k_0 \subseteq k$ is a subfield of $k$ and $G, \theta$ are defined over $k_0$.

A brief summary of the contents is as follows. In section 1 we first introduce the notation and briefly review the description of the double cosets $K\backslash G/P$ from [BH00]. After that we restrict ourselves to the case that $P$ contains a $\theta$-stable Levi factor $L$ and give a characterization of the double cosets $V^S := K\backslash G^P/P$, where $G^P$ is the set of all $g \in G$ such that $gPg^{-1}$ contains a $\theta$-stable Levi subgroup. The set $V^S$ contains a significant part of the $K \times P$ double cosets in $G$, including the open and closed orbits. Moreover in the case that $G, K$ are defined over a subfield $k_0 \subseteq k$ of $k$ and $P$ is a minimal parabolic $k_0$-subgroup of $G$ the set $V^S$ contains all the double cosets $KgP$ with $g \in G(k_0)$. Here $G(k_0)$ is the set of $k_0$-rational points of $G$. Let $S = Z(L)^0$ be the connected center of $L$; let $\Phi(S) = \Phi(G, S)$ be the set of the roots of $S$ in $G$, let $N_G(S)$ (resp. $Z_G(S)$) be the normalizer (resp. centralizer) of $S$ in $G$ and let $W(S) = W(G, S) = N_G(S)/Z_G(S)$ the Weyl group of $S$. In the case that $\Phi(S)$ is a root system with Weyl group $W(S)$ one has a natural map from $V^S$ into the set

$\mathcal{I} = \mathcal{I}_\theta = \{w \in W(S) \mid \theta(w) = w^{-1}\}$ of twisted involutions of the Weyl group $W(S)$. In the remainder of section 1 we analyze this map and in section 2 we analyze the combinatorics of the root system related to the set of orbits $V^S$. Let $\Phi^+ = \Phi(P, S)$ be the set of positive roots of $\Phi(S)$ related to $P$ and $\Delta$ the corresponding basis of $\Phi(S)$. By a result of Steinberg [Ste68] any Borel subgroup admits a $\theta$- stable conjugate, hence if $P = B$ one may assume that $\Delta$ is $\theta$-stable. In [BH00] it was shown that this does not extend to arbitrary parabolic subgroups. In particular $G$ does not need to contain a $\theta$-stable conjugate of $P$ and so we cannot assume that $\Delta$ is $\theta$-stable. This condition was used by Springer [Spr84] in the case that $P = B$ is a Borel subgroup and this enables one to give a detailed description of the twisted involutions in the Weyl group. To enable us to use this characterization of the twisted involutions we switch to an involution $\theta' = \theta w_0$, which stabilizes $\Delta$. Here $w_0 \in W(S)$. This involution is conjugate to $\theta$ if and only if the subset $G^P/P$ of $G/P$ contains a $\theta$-stable conjugate of $P$. In Proposition 9 it is shown that the set $G^P/P$ actually contains a $\theta$-stable conjugate of $P$ if $\Phi(S)$ is a root system with Weyl group $W(S)$. In the remainder of this section we analyze the relation between the orbit decompositions for the involutions $\theta'$ and $\theta$ and use these results to obtain some dimension formulas.

In section 3 we show that the open and closed orbits are contained in $V^S$ and give several characterizations of these orbits using the combinatorial structure developed in section 2. Finally in section 4 we study the decomposition of the closures of the double cosets $KgP$ with $g \in G^P$ in terms of $K \times P$ double cosets and give a combinatorial description of these orbit closures using the combinatorial description of the related twisted involution in the Weyl group.

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## 1. Preliminaries and recollection of results on double cosets

In this section, we recall some results about the double cosets and discuss some properties of the parabolic subgroups containing a $\theta$-stable Levi factor.

1.1. **Notation and recollection.** Throughout the paper, the ground field $k$ is algebraically closed of characteristic $\neq 2$. We denote by $G$ a connected reductive group, by $B$ a Borel subgroup of $G$, and by $T$ a maximal torus of $B$. The unipotent part of $B$ is denoted by $U$. We denote by $P$ a parabolic subgroup of $G$ containing $B$, and by $L$ the Levi subgroup of $P$ which contains $T$. If $S$ is a subset of $G$ and $H$ a closed subgroup of $G$, then we write $N_H(S)$ (resp. $Z_H(S)$) for the normalizer (resp. centralizer) of $S$ in $H$ and we write $\mathrm{Aut}(G, S)$ for the set of automorphisms $\phi \in \mathrm{Aut}(G)$ satisfying $\phi(S) \subset S$. We write $Z(G)$ for the center of $G$. The commutator subgroup of $G$ is denoted by $D(G)$ or $[G, G]$.

For a torus $A$ of $G$ we denote by $X^*(A)$ (resp. $X_*(A)$) the group of characters of $A$ (resp. one-parameter subgroups of $A$) and by $\Phi(A) = \Phi(G, A)$ the set of the roots of $A$ in $G$. If $\Phi(A)$ is a root system, then we let $\Delta(A)$ denote a set of simple roots of $\Phi(A)$. For a closed subgroup $H$ of $G$ we denote the Weyl group of $H$ relative to $A$ by $W_H(A) = N_H(A)/Z_H(A)$. If $H = G$, then we will also write $W(A) = W(G, A) = N_G(A)/Z_G(A)$.

For each $\alpha \in \Phi(T)$, let $U_\alpha \subset G$ be the corresponding root subgroup. If $\Delta(T)$ is the basis of $\Phi(T)$ corresponding to $B$, then each simple root $\alpha \in \Delta(T)$ defines a parabolic subgroup $P_\alpha$ of semisimple rank one, generated by $B$ and $U_{-\alpha}$. We denote by $L_\alpha$ the Levi subgroup of $P_\alpha$ which contains $T$, and by $G_\alpha$ the quotient of $L_\alpha$ by its center; then $G_\alpha$ is isomorphic to $\mathrm{PSL}(2)$.

We shall identify $U_\alpha$ and $U_{-\alpha}$ with their images in $G_\alpha$, and we denote by $T_\alpha$ the image of $T$. Any parabolic subgroup $P \supseteq B$ is generated by the $P_\alpha$'s that it contains. We write $P = P_\Pi$ where $\Pi$ is the set of all $\alpha \in \Delta$ such that $P_\alpha \subseteq P$. We denote by $\Phi_\Pi$ the sub-root system of $\Phi$ generated by $\Pi$, and by $W_\Pi$ its Weyl group.

Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, ... be the Lie algebras of $G$, $B$, $T$, .... For a subgroup $H$ of $G$ will also write $L(H)$ for the Lie algebra of $H$. We have the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(T)} \mathfrak{g}_\alpha$; for each $\alpha \in \Phi(T)$, we choose a non-zero root vector $X_\alpha \in \mathfrak{g}_\alpha$.

Let $\theta$ be an automorphism of order 2 of $G$; let $K \subseteq G$ be the fixed point subgroup. Then $K$ is reductive by [Ste68, §8]; let $K^0 = G^{\theta,0}$ be its connected component containing 1. For the $\theta$-action on $\mathfrak{g}$, the fixed point subspace $\mathfrak{g}^\theta$ is the Lie algebra of $K$ by [Bor91, Corollary 9.2]. Given $g, x \in G$, the *twisted action* associated to $\theta$ is given by $(g, x) \mapsto g{*}x = g^{-1}x\,\theta(g)$. This is also called the twisted $\theta$-action. Let $Q$ denote the set given by $Q = \{g \in G \mid \theta(g) = g^{-1}\}$. Then $Q$ is a finite union of orbits under the twisted $\theta$-action and $Q^0 = \{g^{-1}\theta(g) \mid g \in G\}$ is the connected component of $Q$ containing the identity. Let $\tau : G \to G$ be the map $g \mapsto g^{-1}\theta(g)$; observe that $\theta(x) = x^{-1}$ for all $x \in \tau(G)$ and $\tau(G) = Q^0$.

If $A \subset G$ is a torus and $\phi \in \mathrm{Aut}(G, A)$ an involution, then we write $A^+_\phi = (A \cap G^\phi)^0$ and $A^-_\phi = \{x \in A \mid \phi(x) = x^{-1}\}^0$. It is easy to verify that the product map

$$\mu : A^+_\phi \times A^-_\phi \to A, \quad \mu(t_1, t_2) = t_1 t_2$$

is a separable isogeny. In particular $A = A^+_\phi A^-_\phi$ and $A^+_\phi \cap A^-_\phi$ is a finite group. (In fact it is an elementary abelian 2-group.) If $\phi = \theta$ we reserve the notation $A^+$ and $A^-$ for $A^+_\theta$ and $A^-_\theta$ respectively. For other involutions of $A$, we shall keep the subscript. The involution $\phi$ of $(G, A)$ induces an automorphism of $W(A)$, also denoted by $\phi$, which is given by

$$\phi(w) = \phi \circ w \circ \phi, \quad w \in W(A).$$

The action of $\Phi(A)$ induced by $\phi$ will also be denoted by $\phi$ or $\phi|\Phi(A)$.

1.1.1. *Characterization of double cosets.* For $g \in G$, define an automorphism $\zeta_g$ of $G$ by

$$\zeta_g := \mathrm{Int}(g^{-1}) \circ \theta \tag{1}$$

and similar as in [BH00] define the automorphism

$$\psi_g := \zeta_{g^{-1}\theta(g)} = \mathrm{Int}(g^{-1}) \circ \theta \circ \mathrm{Int}(g). \tag{2}$$

This is an involution of $G$. If $x \in G$, then $\theta(gxg^{-1}) = g\zeta_{g^{-1}\theta(g)}(x)g^{-1} = g\psi_g(x)g^{-1}$. Hence for a subgroup $M$ of $G$ we have the condition that

$$(gMg^{-1})^\theta = gM^{\zeta_{g^{-1}\theta(g)}}g^{-1} = gM^{\psi_g}g^{-1}.$$

So in particular, when $g$ normalizes $M$ we have

$$M^{\psi_g} = g^{-1}M^\theta g. \tag{3}$$

From now on we assume that $T$ is a $\theta$-stable maximal torus of $G$; then its normalizer $N$ is $\theta$-stable, too. Set $\mathcal{V} = \{g \in G \mid T \text{ is } \psi_g\text{-stable}\} = \{g \in G \mid g^{-1}\theta(g) \in N\}$. Then $\mathcal{V}$ is stable under left multiplication by $K$ and right multiplication by $N$ and by [Spr84] and [HW93], any $(K, B)$-double coset in $G$ meets $\mathcal{V}$, along a unique $(K, T)$-double coset. For parabolic subgroups we obtained in [BH00] a similar parametrization of the $(K, P)$-double cosets in $G$.

Set $\mathcal{V}^P := \{g \in \mathcal{V} \mid KgB$ is open in $KgP\}$. Then by [BH00, Proposition 2] any $(K, P)$-double coset in $G$ meets $\mathcal{V}^P$, along a unique $(K, T)$-double coset.

1.1.2. *$\theta$-stable Levi subgroups.* In the following, we assume that $P$ contains a $\theta$-stable Levi subgroup $L$. By [BH00, Lemma 5] any $\theta$-stable Levi subgroup of $P$ is conjugate to $L$ in $R_u(P)^\theta$. Let $G^P$ be the set of all $g \in G$ such that $gPg^{-1}$ contains a $\theta$-stable Levi subgroup. Clearly, $G^P$ is a union of $(K, P)$-double cosets, but it can happen that $G^P \subsetneq G$. In the case that $P = B$ we have $G^B = G$ (see [Ste68]). Let $S = Z(L)^0$ denote the connected center of $L$, and $N_G(S)$ resp. $Z_G(S)$ the normalizer, resp. centralizer of $S$ in $G$. Then $L = Z_G(S)$, $N_G(L) = N_G(S)$, and these groups are $\theta$-stable. Let $\mathcal{V}^S = \{g \in G \mid g^{-1}\theta(g) \in N_G(S)\}$, a union of $(K, N_G(S))$-double cosets contained in $G^P$. Finally, let $\mathcal{V}^{S,P} = \mathcal{V}^S \cap \mathcal{V}^P$. Then by [BH00, Proposition 3] any $(K, P)$-double coset in $G^P$ meets $\mathcal{V}^S$ along a unique $(K, L)$-double coset, which itself meets $\mathcal{V}^{S,P}$ along a unique $(K, T)$-double coset.

Set $V^S := K\backslash\mathcal{V}^S/L$; then we have $V^S = K\backslash G^P/P = K\backslash\mathcal{V}^{S,P}/T$. The action of $N_G(S)$ on $\mathcal{V}^S$ by right multiplication induces an action of the Weyl group $W(S) := N_G(S)/Z_G(S)$ on $V^S$. We denote this action of $W(S)$ on $V^S$ by $(w, v) \to w \cdot v$ $(w \in W(S),\ v \in V^S)$. We interpret the orbit set $V^S/W(S)$ in terms of certain conjugacy classes of $\theta$-stable tori, as follows.

Let $\mathcal{S}$ be the set of all conjugates of $S$ by elements of $G$. This is an affine variety, isomorphic to $G/N_G(S)$, on which $\theta$ acts. Let $\mathcal{S}^\theta$ be the fixed point set of $\theta$, then $\mathcal{S}^\theta$ is the set of conjugates of $S$ by elements of $\mathcal{V}^S$. It is an affine variety, on which $K$ acts by conjugation. The bijective map $\mathcal{V}^S/N_G(S) \to \mathcal{S}^\theta : gN_G(S) \mapsto gSg^{-1}$ is $K$-equivariant; thus, the induced map $\eta : V^S/W(S) \to \mathcal{S}^\theta/K$ is bijective. In the case that $P = B$ this was observed in [RS90, Proposition 2.7].

1.2. **Orbits and twisted involutions.** In the remainder of this section we assume that $\Phi(S)$ is a root system with Weyl group $W(S) = N_G(S)/Z_G(S)$. Similar as in the case of $K\backslash G/B$-double cosets there exists a natural map from the orbit set into the set of twisted involutions in the Weyl group $W(S)$. First we give a characterization of this Weyl group.

1.2.1. *The group $W(S)$.* Since $S$ is $\theta$-stable its centralizer $Z_G(S)$ is $\theta$-stable as well. By [Ste68, 7.5] $Z_G(S)$ contains a $\theta$-stable maximal torus $T \supset S$. Let $X_0(S) = \{\chi \in X(T) \mid \chi(S) = e\}$ and $\Phi_0(S) = \Phi(T) \cap X_0(S)$. This is a closed subsystem of $\Phi(T)$. Denote the Weyl group of $\Phi_0(S)$ by $W_0(S)$ and identify it with the subgroup of $W(T)$ generated by the reflections $s_\alpha$, $\alpha \in \Phi_0(S)$. Let $W_1 = \{w \in W(T) \mid w(X_0(S)) = X_0(S)\}$. We note that $W_0(S)$ is a normal subgroup of $W_1$. Put $\bar{X} = X(T)/X_0(S)$, $\pi$ the natural projection from $X(T)$ to $\bar{X}$ and let $\bar{\Phi} = \pi(\Phi(T) - \Phi_0(S))$ denote the set of *restricted roots of* $\Phi$ *relative to* $X_0(S)$. Using similar arguments as in [Hel00, Proposition 4.11] we easily get the following characterization of $\Phi(S)$ and $W(S)$:

**Lemma 1.** $\bar{\Phi} = \Phi(S)$ *and* $W(S) \cong W_1/W_0(S)$.

In this case we also have the following characterization of $\mathcal{V}^{S,P}$:

**Lemma 2.** *Let* $v \in \mathcal{V}^P$. *Then the following are equivalent:*

(i) $v \in \mathcal{V}^{S,P}$.
(ii) $S$ *is* $\psi_v$*-stable.*

(iii) $\psi_v(X_0(S)) = X_0(S)$.

1.2.2. *Twisted involutions.* Recall that an element $w \in W(S)$ is a *twisted involution* if $\theta(w) = w^{-1}$ (see [Spr84, §3] or [HW93, §7]). Let $\mathcal{I} = \mathcal{I}_\theta = \{w \in W(S) \mid \theta(w) = w^{-1}\}$ be the set of twisted involutions in $W(S)$. If $v \in V^S$ with representative $x(v)$, then $\varphi(v) := \tau(x(v))Z_G(S) \in W(S)$ is a twisted involution. The element $\varphi(v) \in \mathcal{I}_\theta$ is independent of the choice of representative $x(v) \in \mathcal{V}^S$ for $v$. So this defines a map $\varphi : V^S \to \mathcal{I}_\theta$. In the case that $P = B$ a Borel subgroup the map $\varphi$ plays an important role in the study of the Bruhat order on $V$. For more details, see [RS90].

The Weyl group $W(S)$ acts also on $\mathcal{I}_\theta$. This action comes from the *twisted action* of $W(S)$ on (the set) $W(S)$, which is defined as follows: if $w,\ w_1 \in W(S)$, then $w * w_1 = ww_1\theta(w)^{-1}$. If $w_1 \in W(S)$, then $W(S) * w_1 = \{w * w_1 \mid w \in W(S)\}$ is the *twisted $W(S)$-orbit* of $w_1$. Now $\mathcal{I}_\theta$ is stable under the twisted action, so that we get a twisted action of $W(S)$ on $\mathcal{I}_\theta$. The image of $\varphi$ in $\mathcal{I}_\theta$ is a union of twisted $W(S)$-orbits, as follows from the following result:

**Lemma 3.** *Let $w \in W(S)$ and $v \in V^S$. Then $\varphi(w \cdot v) = w * \varphi(v)$.*

From this result it follows now that the map $\varphi : V^S \to \mathcal{I}_\theta$ is equivariant with respect to the action of $W(S)$ on $V^S$ and the twisted action of $W(S)$ on $\mathcal{I}_\theta$. So there is a natural orbit map $\phi : V^S/W(S) \to \mathcal{I}_\theta/W(S)$. In the case that $P = B$ a Borel subgroup Richardson and Springer [RS90] showed that this map is in fact injective. Moreover in this case one can easily proves then the following properties of the maps $\varphi$ and $\phi$ (see [HW93] and [RS90]).

**Proposition 1.** *Let $G$, $\varphi$ and $\phi$ be as above and assume $S$ is a maximal torus. Then we have the following.*

(i) *$\phi : V^S/W(S) \to \mathcal{I}_\theta/W(S)$ is injective.*
(ii) *There is a bijection from $\varphi(V^S)/W(S)$ onto $\mathcal{S}^\theta/K$.*

*Remark* 1. It is an open question if the above result holds when $S$ not a maximal torus but still satisfies the condition that $\Phi(S)$ is a root system with Weyl group $W(S)$. The statement (ii) is immediate from (i). To prove (i) it suffices to show that if $\varphi(v_1) = \varphi(v_2) = \mathrm{id} \in W(S)$, then $v_1 = w \cdot v_2$ for some $w \in W(S)$.

## 2. Twisted Involutions and dimension formulas

In this section we first recall and establish some results on twisted involutions in a Weyl group. In the last subsection we apply these results to obtain some dimension formulas.

2.1. **Preliminaries about twisted involutions.** For the remainder of this paper we assume that $P$ is a parabolic subgroup of $G$ containing a $\theta$-stable Levi factor $L$, $S = Z(L)^0$ and that $\Phi(S)$ is a root system with Weyl group $W(S)$. To simplify notation we will write $X = X^*(S)$, $\Phi = \Phi(S)$ and $W = W(S)$. For $\alpha \in \Phi$ denote the corresponding reflection in $W$ by $s_\alpha$. Let $\Phi^+ = \Phi(P, S)$ be the set of positive roots of $\Phi$ related to $P$, $\Delta$ the corresponding basis of $\Phi$ and $\Sigma = \{s_\alpha \mid \alpha \in \Delta\}$. The Weyl group $W$ is generated by $\Sigma$. In the sequel, the length function $\ell$ and Bruhat order $\leq$ on $W$ are defined relative to $\Sigma$.

For a subset $\Lambda$ of $\Delta$ denote the subset of $\Phi$ consisting of the integral combinations of $\Lambda$ by $\Phi_\Lambda$. Then $\Phi_\Lambda$ is a closed subsystem of $\Phi$ with Weyl group $W_\Lambda$. We identify $W_\Lambda$ with the finite subgroup $W(\Phi_\Lambda)$ of $W$ generated by the $s_\alpha$ for $\alpha \in \Phi_\Lambda$. Let $w^0_\Lambda$ denote the longest element of $W_\Lambda$ with respect to $\Lambda$.

2.1.1. The roots of $\Phi$ can be divided into three subsets, according to the action of $\theta$, as follows.

(a) $\theta(\alpha) \neq \pm\alpha$. Then $\alpha$ is called *complex* (relative to $\theta$).
(b) $\theta(\alpha) = -\alpha$. Then $\alpha$ is called *real* (relative to $\theta$).
(c) $\theta(\alpha) = \alpha$. Then $\alpha$ is called *imaginary* (relative to $\theta$).

These definitions of real, complex and imaginary roots carry over to the Weyl group as follows. Let $\mathcal{I} = \mathcal{I}_\theta = \{w \in W \mid \theta(w) = w^{-1}\}$ denote the set of twisted involutions. Given $w \in \mathcal{I}_\theta$, an element $\alpha \in \Phi$ is called *complex* (resp. *real*, *imaginary*) relative to $w$ if $w\theta\alpha \neq \pm\alpha$ (resp. $w\theta\alpha = -\alpha$, $w\theta\alpha = \alpha$). We use the following notation:

$$\begin{array}{ll} C'(w,\theta) = \{\alpha \in \Phi^+ \mid -\alpha \neq w\theta\alpha < 0\}, & R(w,\theta) = \{\alpha \in \Phi^+ \mid -\alpha = w\theta\alpha\}, \\ C''(w,\theta) = \{\alpha \in \Phi^+ \mid \alpha \neq w\theta\alpha > 0\}, & I(w,\theta) = \{\alpha \in \Phi^+ \mid \alpha = w\theta\alpha\}. \end{array}$$

We will omit $\theta$ from this notation if there is no ambiguity as to which involution we consider. These sets are related to the following subsets of $\Phi^+$. For $\sigma \in \mathrm{Aut}(X, \Phi)$ let $\Psi(\sigma)$ (resp. $\Psi^-(\sigma)$) denote the set given by $\Psi(\sigma) = \{\alpha \in \Phi^+ \mid \sigma^{-1}\alpha > 0\}$ (resp. $\Psi^-(\sigma) = \Phi^+ - \Psi(\sigma)$).

**Lemma 4.** *Let $w \in \mathcal{I}_\theta$. Then we have the following conditions:*

(i) $\Psi(w\theta) = \{\alpha \in \Phi^+ \mid w\theta\alpha > 0\}$ *and* $\Psi^-(w\theta) = \{\alpha \in \Phi^+ \mid w\theta\alpha < 0\}$.
(ii) $\Psi^-(w\theta) = C'(w) \cup R(w)$ *and* $\Psi(w\theta) = C''(w) \cup I(w)$ *are disjoint unions.*
(iii) $\Psi^-(w\theta)$, $C'(w)$ *are* $-w\theta$ *stable and* $-w\theta|R(w) = 1$.
(iv) $\Psi(w\theta)$, $C''(w)$ *are* $w\theta$ *stable and* $w\theta|I(w) = 1$.

*Proof.* Since $\theta(w) = w^{-1}$, $(w\theta)^2 = 1$ the first statement is clear. Since $\Psi(w\theta)$ is $w\theta$ stable and $\Psi^-(w\theta))$ is $-w\theta$ stable, the other statements are clear. □

2.2. **Switch to an involution fixing $\Phi^+$.** The discussion on twisted involutions in [Spr84] in the case that $P = B$ depends on the fact that $\theta(\Phi^+) = \Phi^+$. This is equivalent to the condition that $\theta(B) = B$. This in its turn is equivalent to the condition that the orbit $KB$ is closed (see [Spr84] or [HW93]). In general $\theta$ does not need to fix $\Phi^+$, but we can modify the involution $\theta$ so that this is the case. Let $w_0 \in W$ be the unique element such that

$$\theta(\Phi^+) = w_0(\Phi^+) \tag{4}$$

and let $\theta' = \theta w_0$. Then $w_0$ and $\theta'$ satisfy the following conditions:

**Proposition 2.** *Let $\Phi$, $\Phi^+$, $\theta$, $w_0$ and $\theta'$ be as above, $w \in \mathcal{I}_\theta$ and $w' = ww_0$. Then we have the following properties:*

(i) $w_0 \in \mathcal{I}_\theta$ *and* $\theta w_0 = w_0^{-1}\theta$.
(ii) $\theta'(\Phi^+) = \Phi^+$.
(iii) $\theta'$ *is an involution of* $\Phi$.
(iv) $\mathcal{I}_{\theta'} = \mathcal{I}_\theta \cdot w_0$.
(v) $w'\theta' = w\theta$.
(vi)

$$\begin{array}{rclrcl} I(w,\theta) & = & I(w',\theta'), & R(w,\theta) & = & R(w',\theta'), \\ C'(w,\theta) & = & C'(w',\theta'), & C''(w,\theta) & = & C''(w',\theta'). \end{array}$$

*Proof.* (i). Since $\theta(\Phi^+) = w_0(\Phi^+)$ it follows that $w_0^{-1}\theta(\Phi^+) = \Phi^+$ and consequently $\theta(w_0^{-1})(\Phi^+) = \theta w_0^{-1}\theta(\Phi^+) = \theta(\Phi^+) = w_0(\Phi^+)$. So $\theta(w_0^{-1}) = w_0$, which proves (i).

(ii) follows from the fact that $\theta w_0 = w_0^{-1}\theta$.

(iii) From (i) it follows that $\theta(w_0) = \theta w_0 \theta = w_0^{-1}$, hence $(\theta')^2 = \theta w_0 \theta w_0 = \text{id}$.

(iv). Let $w \in \mathcal{I}_{\theta'}$ and consider $ww_0^{-1}$. Since $\theta w_0 = w_0^{-1}\theta$ we have

$$\theta(ww_0^{-1}) = \theta w w_0^{-1}\theta = \theta w \theta w_0 = w_0 w_0^{-1}\theta w \theta w_0 = w_0\theta' w \theta' = w_0 w^{-1}.$$

So $\mathcal{I}_{\theta'} \subset \mathcal{I}_\theta \cdot w_0$. A similar argument shows the opposite inclusion, which proves the result.

(v). Let $w \in \mathcal{I}_\theta$ and $w' = ww_0$. Then $w'\theta' = ww_0\theta w_0 = ww_0\theta w_0\theta\theta = ww_0\theta(w_0)\theta = ww_0w_0^{-1}\theta = w\theta$, what proves the result.

Finally (vi) is immediate from (v). □

2.2.1. *Relation between $\theta$ and $\theta'$.* On the level of the root system and Weyl group we can now work with $\theta'$ and $w'$ instead of $\theta$ and $w$. The question remains if $\theta'$ can be lifted to an involution of $G$, and if so, whether that involution is conjugate to $\theta$ and what is the relation between the orbits for $\theta$ and $\theta'$. For $P = B$ these questions were studied in [Hel97]. In the following we generalize this to the parabolic subgroups as in this section. First we address the question of lifting.

Let $w \in W(S)$ and $\xi = w\theta$. Then $\xi$ is an involution of $\Phi$ if and only if $w \in \mathcal{I} = \mathcal{I}_\theta$. Moreover $\xi$ can be lifted to an involution of $G$ if and only if $w$ has a representative $n \in N_G(S)$ satisfying $\theta(n)n \in Z(G)$. The question remains which twisted involutions in $w \in \mathcal{I}$ have such a representative and when this lifted involution is conjugate to $\theta$. This comes down to twisted involutions in $\varphi(V^S)$:

**Lemma 5.** *Let $\mathcal{V}^S$, $V^S$ and $\varphi : V^S \to \mathcal{I}_\theta$ be as in 1.2.2 and let $w \in \mathcal{I}$. Then the following are equivalent.*

(i) *There exists a representative $n \in N_G(S)$ for $w$, such that $\xi = \text{Int}(n)\theta$ is an involution of $G$ conjugate to $\theta$.*
(ii) *$w \in \varphi(V^S) \subset \mathcal{I}$.*

*Proof.* Assume first that $\xi = \text{Int}(n)\theta$ is an involution of $G$ conjugate to $\theta$. Let $g \in G$ such that $\text{Int}(g)^{-1}\theta\,\text{Int}(g) = \text{Int}(n)\theta$. Then $g^{-1}\theta(g) = nz$ for some $z \in Z(G)$. It follows that $g \in \mathcal{V}^S$ and $w \in \varphi(V^S)$.

Conversely if $w \in \varphi(V^S)$, take $v \in V^S$ with $w = \varphi(v)$, let $x = x(v) \in \mathcal{V}^S$ be a representative and set $n = x^{-1}\theta(x) \in N_G(S)$; then $n$ represents $w$, and $\xi = \text{Int}(n)\theta = \text{Int}(x)\theta\,\text{Int}(x^{-1}) = \psi_x$. So the involution $\xi$ is conjugate to $\theta$. □

The question whether $\theta'$ as in (4) can be lifted to an involution conjugate to $\theta$ reduces now to the question if $w_0 \in \varphi(V^S)$. The latter condition is equivalent to the following:

**Proposition 3.** *Let $\mathcal{V}^S$, $V^S$, $\varphi : V^S \to \mathcal{I}$ and $w_0$ be as above. Then $w_0 \in \varphi(V^S)$ if and only if there exists $g \in G^P$ such that $gPg^{-1}$ is $\theta$-stable.*

*Proof.* Assume first $w_0 \in \varphi(V^S)$. Let $v_0 \in V^S$ be such that $\varphi(v_0) = w_0$ and let $x_0 = x(v_0) \in \mathcal{V}^S$ be a representative of $v_0$. Then $\tau(x_0)$ is a representative of $w_0$ in $N_G(S)$. Since $\theta(\Phi^+) = w_0(\Phi^+)$, we have $\theta(P) = \tau(x_0)P\tau(x_0^{-1})$. But then $P_1 = \theta(x_0^{-1})P\theta(x_0)$ is a $\theta$-stable parabolic subgroup of $G$.

Conversely, assume $P_0 \subset G$ is a $\theta$-stable conjugate of $P$. By [BH00, Proposition 3] there exists $x \in \mathcal{V}^S$ such that $P_0 = xPx^{-1}$. Since $\theta(P_0) = P_0$ it follows that

$$\theta(P) = \theta(x)^{-1}xPx^{-1}\theta(x) = \tau(\theta(x)^{-1})P\tau(\theta(x))$$

Now $\tau(\theta(x)^{-1}) \in N_G(S)$. Let $w \in W$ be the corresponding Weyl group element. Then $\theta(\Phi^+) = w(\Phi^+)$, so $w = w_0 \in \varphi(V^S)$. □

*Remark* 2. For arbitrary parabolic subgroups $G/P$ does not need to contain a $\theta$-stable parabolic subgroup (see [BH00]).

In Proposition 9 we will show that $G^P/P \subset G/P$ contains a $\theta$-stable conjugate of $P$. Since this result does not depend on the results in this section we can combine it with the above result to obtain the following:

**Corollary 1.** *Let $w_0$, $\theta'$ be as above. There exists a representative $n \in N_G(S)$ of $w_0$, such that $\xi = \mathrm{Int}(n)\theta$ is an involution of G conjugate to $\theta$ satisfying $\xi|\Phi = \theta'$.*

2.2.2. Since by Corollary 1 the involutions $\theta$ and $\theta'$ are conjugate, the orbit decompositions of the corresponding symmetric varieties under the action of a parabolic subgroup are similar. In the remainder of this section we discuss the relation between these double coset decompositions.

Let $P$, $S$, $\mathcal{V}^S$, $V^S$, $\mathcal{I}$ and $\varphi : V^S \to \mathcal{I}$ be as above. Take $w_0 \in W(S)$ such that $\theta(\Phi(S)^+) = w_0(\Phi(S)^+)$, $n_0 = x_0^{-1}\theta(x_0) \in N_G(S) \cap \tau(G)$ a representative of $w_0^{-1}$, $\theta' = \mathrm{Int}(n_0)\theta$ and $K' = G^{\theta'} = x_0G^\theta x_0^{-1}$. Denote the actions of $\theta$ and $\theta'$ on $\Phi(S)$ also by $\theta$ and $\theta'$. Then $\theta' = \theta w_0 = w_0^{-1}\theta$. As for $\theta$ let $\tau' : G \to G$ be the map defined by $\tau'(x) = x^{-1}\theta'(x)$, $\mathcal{V}^{S'} = \{x \in G \mid \tau'(x) \in N_G(S)\}$, $V^{S'}$ the set of $(Z_G(S) \times K')$-orbits in $\mathcal{V}^{S'}$, $\mathcal{I}_{\theta'}$ the set of twisted involutions of $W(S)$ with respect to $\theta'$ and $\varphi' : V^{S'} \to \mathcal{I}_{\theta'}$ as in 1.2.2. We have the following relations between the sets $\mathcal{V}^S$, $\mathcal{V}^{S'}$, $\mathcal{I}$ and $\mathcal{I}_{\theta'}$.

**Lemma 6.** *Let $\mathcal{V}^S$, $\mathcal{V}^{S'}$, $\mathcal{I}$, $\mathcal{I}_{\theta'}$, $x_0$, $n_0$ and $w_0$ be as above. Then we have the following.*

(i) $\mathcal{V}^{S'} = \mathcal{V}^S \cdot x_0^{-1}$.
(ii) $\tau'(\mathcal{V}^{S'}) = \tau(\mathcal{V}^S) \cdot n_0^{-1}$.

As in 1.2.2 there is also an action of the Weyl group $W(S)$ on $\mathcal{I}_{\theta'}$. Namely if $w \in W(S)$ and $a' \in \mathcal{I}_{\theta'}$, then define an action $w *' a' = wa'\theta'(w)^{-1}$. Since $\theta' = \theta w_0$ and $a' = aw_0$ for some $a \in \mathcal{I}_\theta$, we get

$$\begin{aligned} w *' a' &= waw_0\theta' w^{-1}\theta' = waw_0w_0^{-1}\theta w^{-1}\theta w_0 \\ &= wa\theta w^{-1}\theta w_0 = (w * a)w_0. \end{aligned} \tag{5}$$

This means that right translation by $w_0$ gives an isomorphism $\iota_{w_0} : \mathcal{I} \to \mathcal{I}_{\theta'}$, which is equivariant with respect to the actions $*$ resp. $*'$ of $W(S)$ on $\mathcal{I}$ resp. $\mathcal{I}_{\theta'}$. On the other hand, the map $g \to gx_0^{-1}$ from $\mathcal{V}^S$ to $\mathcal{V}^{S'}$ induces a map $\delta : V^S \to V^{S'}$. Then $\varphi' \circ \delta = \iota_{w_0} \circ \varphi$. We summarize the above analysis in the following result.

**Proposition 4.** *Let $\mathcal{V}^S$, $\mathcal{V}^{S'}$, $\mathcal{I}$, $\mathcal{I}_{\theta'}$, $n_0$ and $w_0$ be as above. Then we have the following.*

(i) *The map $\iota_{w_0} : \mathcal{I} \to \mathcal{I}_{\theta'}$ induces an isomorphism between $\mathcal{I}/W(S)$ and $\mathcal{I}_{\theta'}/W(S)$.*
(ii) $V^{S'}/W(S') \simeq V^S/W(S) \simeq \varphi'(V^{S'})/W(S) \simeq \varphi(V^S)/W(S)$.

2.3. **Dimension formulas.** Using the description of the twisted involutions as in the previous subsections we can refine the description of the group $P^\theta = P \cap K$ as given in [BH00]. This leads to some dimension formulas which are needed for our discussion on closures of double cosets. First we need some notation. Given a quasi-closed subset $\psi$ of $\Phi(S)$, we write $U_\psi$ for the unipotent subgroup generated by the product of the $U_\alpha$ with $\alpha \in \psi$.

**Proposition 5.** *Let $g \in G^P$, $M = gPg^{-1}$ and $v \in \mathcal{V}^S$ such that $M = vPv^{-1}$ (see 1.1.2). Let $n = v^{-1}\theta(v) \in N_G(S)$, $w$ the image of $n$ in $W(S)$, $\Psi(w\theta) = \Phi(S)^+ \cap w\theta\Phi(S)^+$ as in 2.1 and $\mathfrak{g}_w = \bigoplus_{\alpha \in C''(w)} \mathfrak{g}_\alpha$. Then we have the following:*

(i) $M^\theta \simeq Z_G(S)^{\psi_v} \ltimes U^{\psi_v}_{\Psi(w\theta)}$.

(ii) $\dim(M^\theta) = \dim(Z_G(S)^{\psi_v}) + \dim(U^{\psi_v}_{I(w)}) + \frac{1}{2}\dim(\mathfrak{g}_w)$.

*Proof.* (i). Since $M^\theta = v(P \cap n\theta(P)n^{-1})v^{-1}$ it follows that $M^\theta = v(Z_G(S) \ltimes U_{\Psi(w\theta)})v^{-1}$. The groups $vZ_G(S)v^{-1}$ and $vU_{\Psi(w\theta)}v^{-1}$ are $\theta$-stable. Hence by (3), $M^\theta = v(Z_G(S)^{\psi_v} \ltimes U^{\psi_v}_{\Psi(w\theta)})v^{-1}$. Now (i) is obvious.

(ii). By Lemma 4(ii), $\Psi(w\theta) = I(w) \cup C''(w)$ is a disjoint union. The set $I(w)$ is a closed subset of $\Phi(S)^+$ and $L(U_{I(w)}) = \bigoplus_{\alpha \in I(w)} \mathfrak{g}_\alpha$. It follows that $L(U_{\Psi(w\theta)}) = L(U_{I(w)}) \oplus \mathfrak{g}_w$. Note that $\psi_v\alpha = w\theta\alpha$, $\alpha \in \Phi(S)$. Since $I(w)$ and $C''(w)$ are $w\theta$-stable by Lemma 4, $L(U_{\Psi(w\theta)})^{\psi_v} = L(U_{I(w)})^{\psi_v} \oplus \mathfrak{g}_w^{\psi_v}$. Then we have that $\dim(U^{\psi_v}_{\Psi(w\theta)}) = \dim(U^{\psi_v}_{I(w)}) + \dim(\mathfrak{g}_w^{\psi_v})$. It remains to show that $\dim(\mathfrak{g}_w^{\psi_v}) = \frac{1}{2}\dim(\mathfrak{g}_w)$. From the definition of $C''(w)$ and the fact that $\psi_v = w\theta$ on $\Phi(S)$, $\psi_v\alpha \neq \alpha$ for $\alpha \in C''(w)$. Hence there exists a subset $J$ of $C''(w)$ such that $C''(w) = J \cup \psi_v(J)$ is a disjoint union. Set $V = \bigoplus_{\alpha \in J} \mathfrak{g}_\alpha$. Then $\mathfrak{g}_w = V \oplus \psi_v(V)$. It is now easy to see that $\mathfrak{g}_w^{\psi_v} \simeq V$ and $\dim(\mathfrak{g}_w^{\psi_v}) = \frac{1}{2}\dim(\mathfrak{g}_w)$. □

**Corollary 2.** *Let $v \in \mathcal{V}^S$ and $v_1 = vm$ with $m \in N_G(S)$. Let $n = v^{-1}\theta(v)$, $n_1 = v_1^{-1}\theta(v_1) = m^{-1}n\theta(m)$ and $w$, $w_1$, $w_2$ the images of $n$, $n_1$, $m$ in $W(S)$ respectively. If $w_2(I(w)) = I(w_1)$, then*

$$\dim\left((v_1Pv_1^{-1})^\theta\right) - \dim\left((vPv^{-1})^\theta\right) = \frac{1}{2}(\dim(\mathfrak{g}_{w_1}) - \dim(\mathfrak{g}_w)).$$

*Proof.* From Proposition 5, we have the identities

$$\dim((v_1Pv_1^{-1})^\theta) = \dim(Z_G(S)^{\psi_{v_1}}) + \dim(U^{\psi_{v_1}}_{I(w_1)}) + \frac{1}{2}\dim(\mathfrak{g}_{w_1}),$$

$$\dim((vPv^{-1})^\theta) = \dim(Z_G(S)^{\psi_v}) + \dim(U^{\psi_v}_{I(w)}) + \frac{1}{2}\dim(\mathfrak{g}_w).$$

On the other hand from (3) it follows that $Z_G(S)^{\psi_{v_1}} = m^{-1}v^{-1}(Z_G(S)^\theta)vm = m^{-1}(Z_G(S)^{\psi_v})m$ and $U^{\psi_{v_1}}_{I(w_1)} = m^{-1}(U^{\psi_v}_{I(w)})m$. Since $w_2(I(w)) = I(w_1)$ we have $m^{-1}U_{I(w)}m = U_{I(w_1)}$, what proves the result. □

2.3.1. The group $\theta(P)$ is also a parabolic subgroup of $G$ with $Z_G(S)$ as a Levi factor. Since $\Phi(S)$ is a root system with Weyl group $W(S)$, there exists $n_0 \in N_G(S)$ with $n_0 P n_0^{-1} = \theta(P)$. Let $w_0$ denote the image of $n_0$ in $W(S)$. By our choice, $\theta(\Phi(S)^+) = w_0(\Phi(S)^+)$. Similarly as in 2.2 set $\theta' = \theta w_0$ and for $w \in W(S)$ set $w' = w w_0$. For $\alpha \in \Phi(S)$ let $P_\alpha$ as in 3.2.1 be the parabolic subgroup of $G$ containing $P$ with $\Phi(P_\alpha, S) = (\mathbb{Z}\alpha \cap \Phi(S)) \cup \Phi(S)^+$. Then we have:

**Proposition 6.** *Let $v \in \mathcal{V}^S$, $n = v^{-1}\theta(v)$, $w$ the image of $n$ in $W(S)$ and $w' = w w_0$. Let $\alpha \in \Delta(S)$, $s_\alpha$ in $W(S)$ the reflection defined by $\alpha$, $n_\alpha \in N_G(S)$ a representative of $s_\alpha$, $v_1 = v n_\alpha$ and $n_1 = v_1^{-1}\theta(v_1) = n_\alpha^{-1} n \theta(n_\alpha)$. If $\ell(s_\alpha w' \theta'(s_\alpha)) = \ell(w') + 2$, then we have the following:*

(i) $\dim((v_1 P v_1^{-1})^\theta) = \dim((v P v^{-1})^\theta) + \dim P - \dim P_\alpha$.
(ii) $\dim(P * n_1) = \dim(P * n) + \dim P_\alpha - \dim P$.

*Proof.* (i). By Proposition 2(vi) $w'\theta' = w\theta$, $I(w,\theta) = I(w',\theta')$ and $C''(w,\theta) = C''(w',\theta')$. Since $s_\alpha w' \theta'(s_\alpha)\theta' = s_\alpha w \theta(s_\alpha)\theta$ we also have $I(s_\alpha w\theta(s_\alpha), \theta) = I(s_\alpha w'\theta'(s_\alpha), \theta')$, and $C''(s_\alpha w\theta(s_\alpha), \theta) = C''(s_\alpha w'\theta'(s_\alpha), \theta')$. From [HW93, Lemma 7.8] it follows that

$$I(s_\alpha w\theta(s_\alpha)) = s_\alpha I(w), \tag{6}$$

$$C''(s_\alpha w\theta(s_\alpha)) = s_\alpha(C''(w) - Y), \tag{7}$$

where $Y = \Phi(S)^+ \cap (\mathbb{Z}\alpha \cup \mathbb{Z}w\theta\alpha)$. But then by (6) and Corollary 2,

$$\dim\left((v_1 P v_1^{-1})^\theta\right) - \dim\left((v P v^{-1})^\theta\right) = \frac{1}{2}\left(\dim(\mathfrak{g}_{s_\alpha w\theta(s_\alpha)}) - \dim(\mathfrak{g}_w)\right).$$

By (7) this number coincides with $\dim\big(\bigoplus_{\gamma \in \Phi(S)^+ \cap \mathbb{Z}\alpha} \mathfrak{g}_\alpha\big)$. Now (i) follows from (8).

(ii). By (i), we have that $\dim((v_1 P v_1^{-1})^\theta) = \dim((v P v^{-1})^\theta) + \dim P - \dim P_\alpha$. Since $\dim(P * n) = \dim(P) - \dim((v P v^{-1})^\theta)$ and $\dim(P * n_1) = \dim(P) - \dim((v_1 P v_1^{-1})^\theta)$, the result follows. ☐

## 3. Open and closed orbits

In this subsection we give a characterization of the open and closed orbits in $V$ and show that they are actually contained in $V^S$. We first set the notation and prove some preliminary results.

3.1. **Preliminaries and facets.** Let $P$, $L$, $S$, etc. be as in section 2. By chambers, facets of $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$, we mean those with respect to the hyperplanes $\ker(\alpha)$, $\alpha \in \Phi(G,S)$. In the case that $\Phi(S)$ is a root system with Weyl group $W(S) = N_G(S)/Z_G(S)$ the parabolic subgroups of $G$ which contain a parabolic subgroup with $Z_G(S)$ as a Levi factor are in bijective correspondence with the facets of $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. Given a facet $F$, the corresponding parabolic subgroup $P(F)$ of $G$ containing $S$ is determined by

$$\Phi(P(F), S) = \{\alpha \in \Phi(G,S) \mid \langle x, \alpha \rangle \geq 0, \quad x \in F\}.$$

For $\lambda \in X_*(S)$, let $F(\lambda)$ denote the facet containing $\lambda$. For simplicity, we write $P(\lambda)$ for the parabolic subgroup $P(F(\lambda))$ of $G$ containing $S$.

**Lemma 7.** *Assume $S$ is as above, $\Phi(S)$ is a root system with Weyl group $W(S)$, $P$ a parabolic subgroup containing $Z_G(S)$, $F$ the facet with $P = P(F)$ and $L = Z_G(F)$ the corresponding Levi factor of $P$. If $P$ is $\theta$-stable, then we have the following:*

(i) $\theta(F) = F$.
(ii) *There is $\lambda \in X_*(S^+)$ such that $P = P(\lambda)$ and $L = Z_G(\lambda)$.*

*Proof.* (i). If $P$ is $\theta$-stable, then $P(F) = \theta(P(F)) = P(\theta(F))$. It follows that $\theta(F) = F$.

(ii). There exists $\mu \in X_*(S) \cap F$. By (i), $\theta(\mu) \in F$ and $\lambda = \mu + \theta(\mu) \in X_*(S^+) \cap F$. Then $\lambda$ has the desired properties. □

For $\theta$-split parabolic subgroups of $G$ we have a similar characterization. First we define the following:

**Definition 1.** A parabolic subgroup $P_1$ of $G$ will be called $(\theta, \mathcal{S}^\theta)$-split if $P_1$ is $\theta$-split and there exists $S_1 \in \mathcal{S}^\theta$ such that $S_1 \subset P_1$ and $P_1 \cap \theta(P_1) = Z_G(S_1^-)$.

Similar as in [HW93, §4] we have the following result.

**Lemma 8.** *Let $P_1$ be a $(\theta, \mathcal{S}^\theta)$-split parabolic subgroup of $G$ and $S_1 \in \mathcal{S}^\theta$ such that $S_1 \subset P_1$. Then there exists $\lambda \in X_*(S_1^-)$ such that $P_1 = P(\lambda)$ and $P_1 \cap \theta(P_1) = Z_G(\lambda)$.*

*Proof.* Since $S_1$ is $\theta$-stable, $S_1 \subset \theta(P_1)$. Let $F$ be the facet of $X_*(S_1) \otimes_{\mathbb{Z}} \mathbb{R}$ such that $P_1 = P(F)$. Note that $\theta(P_1) = P(-F)$. Hence $\theta(F) = -F$. Now choose $\tau \in X_*(S_1) \cap F$. Since $\tau$, $-\theta(\tau) \in F$ and $F$ is a convex cone, $\lambda = \tau - \theta(\tau) \in F$. Then $\lambda$ has the desired property. □

3.2. ***θ*-singular roots.** In this subsection we prove some results about $\theta$-singular roots which are needed for the description of the open and closed orbits.

3.2.1. Similarly as for Borel subgroups we can characterize the parabolic subgroups containing $P$ by subsets of the roots in $\Phi(S)$. Let $\Phi(S) = \Phi(G, S)$ denote the set of roots of $S$ in $G$, $\Phi(S)^+ = \Phi(P, S)$ and $\Delta(S)$ the corresponding basis. For $\lambda \in \Phi(S)$, let $\mathfrak{g}_\lambda$ be the root subspace of the Lie algebra $\mathfrak{g}$ of $G$ corresponding to $\lambda$. We have the decomposition $\mathfrak{g} = L(Z_G(S)) \bigoplus_{\lambda \in \Phi(S)} \mathfrak{g}_\lambda$. Given $\lambda \in \Delta$, let $P_\lambda$ denote the parabolic subgroup of $G$ containing $P$ such that $\Phi(P_\lambda, S) = (\mathbb{Z}\lambda \cap \Phi(S)) \cup \Phi(S)^+$. It is easy to see that

$$\dim(P_\lambda) = \dim(P) + \dim\Big(\bigoplus_{\beta \in \mathbb{Z}\lambda \cap \Phi(S)^+} \mathfrak{g}_\beta\Big). \tag{8}$$

If $\Lambda \subset \Delta(S)$, then similar as in 2.1 we denote by $\Phi(S)_\Lambda$ the subsystem of $\Phi(S)$ consisting of integral combinations of $\Lambda$ and by $P_\Lambda$ the parabolic subgroup of $G$ containing $P$ generated by all the $P_\lambda$ with $\lambda \in \Lambda$. Note that $\Phi(P_\Lambda, S) = \Phi(S)_\Lambda \cup \Phi(S)^+$. We denote by $L_\Lambda$ the Levi subgroup of $P_\Lambda$ which contains $S$, by $G_\Lambda$ the quotient of $L_\Lambda$ by its center, and by $S_\Lambda$ the image of $S$. In the case that $\Lambda = \{\lambda\}$ we will also write $L_\lambda$ for $L_\Lambda$, $G_\lambda$ for $G_\Lambda$ and $S_\lambda$ for $S_\Lambda$. We shall identify $U_\lambda$ and $U_{-\lambda}$ with their images in $G_\lambda$.

**Definition 2.** A root $\lambda \in \Phi(S)$ with $\theta(\lambda) = \pm\lambda$ is called *$\theta$-singular* (resp. *$\theta$-compact*) if $G_\lambda \not\subset K$ (resp. $G_\lambda \subset K$). A root $\lambda \in \Phi$ with $\theta(\lambda) = \lambda$ is called *$\theta$-compact imaginary* if $G_\lambda \subset K$. A root $\lambda \in \Phi$ with $\theta(\lambda) = \lambda$ and $G_\lambda \not\subset K$ is called *imaginary $\theta$-singular*. This definition is a refinement of 2.1.1.

In the case that $S$ is a maximal torus of $G$ all real roots are $\theta$-singular (see [Hel88] or [Spr84]). For $S$ not a maximal torus we can show the following:

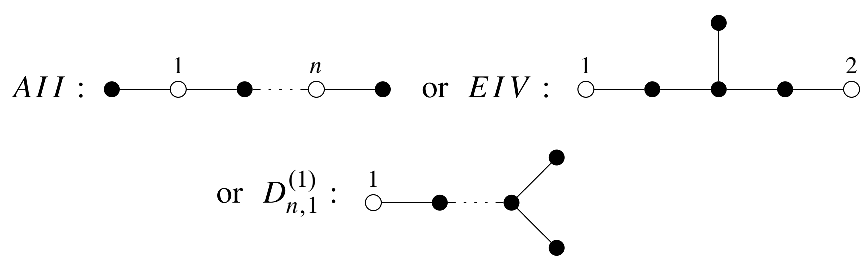
**Proposition 7.** *Let $S \in \mathcal{S}^\theta$. Then we have the following.*

(i) *If $\lambda \in \Phi(S)$ is real, then there exist a $x \in G_\lambda$ such that $S_1 = xSx^{-1}$ is $\theta$-stable and* $\dim S^- > \dim S_1^-$.

(ii) *If $\lambda \in \Phi(S)$ is $\theta$-singular imaginary, then there exist a $x \in G_\lambda$ such that $S_1 = xSx^{-1}$ is $\theta$-stable and* $\dim S^- < \dim S_1^-$.

*Proof.* We may assume that $G = G_\lambda$ and $S$ is $\theta$-split. Let $T \supset S$ be a $\theta$-stable maximal torus with $T^+$ a maximal torus of $Z_G(S) \cap K$. If $T^+$ is not a maximal torus of $K$, then there exists $\alpha \in \Phi(T)$, $\theta(\alpha) = -\alpha$. If $\alpha \in \Phi(S) \cap \Phi(T)$, then by [Hel91, 4.6] $\alpha$ is $\theta$-singular and there exists $g \in Z_G((\ker \alpha)^0)$ such that $T_1 = gTg^{-1}$ is $\theta$-stable with $\dim T_1^+ = \dim T^+ + 1$. Then also $S_1 = gSg^{-1}$ is $\theta$-stable and $\dim S_1^+ = \dim S^+ + 1$, what proves the result. So assume $\alpha \notin \Phi(S)$. Since $T^+$ is maximal in $Z_G(S) \cap K$, we get $\alpha|S \neq 0$, so $\pi(\alpha) \in \Phi(S)$. Moreover reducing to $Z_G(T^+)$ we may assume $T = T^-$ is $\theta$-split. Since $\alpha \notin \Phi(S)$ there exists $\beta \in \Phi(T)$ with $\beta \neq \pm\alpha$. Then also $\beta|S \neq 0$, so $\pi(\alpha) \in \Phi(S)$. Since $\Phi(S) = \{\pm\lambda, \pm 2\lambda\}$ or $\Phi(S) = \{\pm\lambda, \pm\frac{1}{2}\lambda\}$ it follows that $\alpha$ and $\beta$ are perpendicular and span a root system of type $A_1 + A_1$. Repeating the same argument it follows that $\Phi(T)$ is of type $A_1 + \cdots + A_1$ ($n$ copies of $A_1$). Let $\alpha_1, \dots, \alpha_n$ be a basis of $\Phi(T)$. Since $\alpha_1, \dots, \alpha_n$ are real roots, it follows from [Hel91, 4.6] that they are $\theta$-singular strongly orthogonal roots and there exists $g \in G$ such that $T_1 = gTg^{-1}$ is $\theta$-stable with $\dim T_1^+ = \dim T^+ + n$. Since $gSg^{-1} \subset T_1^+$ the result follows.

Assume next that $T^+$ is a maximal torus of $K$. Let $w_0 \in \Phi(T)$ be a maximal $\theta$-singular involution of $W(T)$ (see [Hel91]). Then $T_{w_0}^- \subset T^+$ and $T_{w_0}^+ \supset S$. We may restrict to $Z_G(T_{w_0}^-)$. So assume that $\Phi(T)$ has no $\theta$-singular roots. Then the $\theta$-index of $(G, \theta)$ must be one of the following:




In each of these cases one easily shows that there exists $w_1 \in W(T_1)$ such that $w_1(T_1^-) \subset T_1^+$. Let $n \in N_G(T_1)$ be a representative of $w_1$. Then, since $S \subset T_1^-$, we get $nSn^{-1} \subset nT_1^- n^{-1} \subset T_1^+$, what proves (i).

(ii) follows with a similar argument as in (i). □

**Definition 3.** Let $S \in \mathcal{S}^\theta$. We will say that $S^+$ is maximal if $\Phi(S)$ has no real roots. Similarly we will say that $S^-$ is maximal if $\Phi(S)$ has no $\theta$-singular imaginary roots.

3.3. **Closed orbits.** In [BH00, Proposition 8] we gave a description of the closed $(K, P)$-orbits in $G$. Whether an orbit $KgP$ is closed in $G$ can also be seen from the order induced by $P$ on the restricted root system $\Phi(S)$. For this we need to recall the notion of $\theta$-order from [Hel91]. In the following let $\theta$ be an involution in $\mathrm{Aut}(X, \Phi)$. Let $X_0(-\theta) = \{\chi \in X \mid \theta(\chi) = -\chi\}$,

$\Phi_0(-\theta) = \Phi \cap X_0(-\theta)$ and $\pi^-$ the natural projection from $X$ to $X/X_0(-\theta)$. As in [Hel91] we define a $\theta^-$-order on $\Phi$ by choosing orders on $X_0(-\theta)$ and $X/X_0(-\theta)$:

**Definition 4.** Let $\succ$ be a linear order on $X$. The order $\succ$ is called a $\theta^-$-order if it has the following property:

$$\text{if } \chi \in X,\ \chi \succ 0, \text{ and } \chi \notin X_0(-\theta), \text{ then } \theta(\chi) \succ 0. \tag{9}$$

A basis $\Delta$ of $\Phi$ with respect to a $\theta^-$-order on $X$ will be called a $\theta^-$-basis of $\Phi$. If $\Delta$ is a basis of $\Phi$ with respect to a $\theta^-$-order on $X$, then we write $\Delta_0(-\theta) = \Delta \cap \Phi_0(-\theta)$ and $\bar{\Delta}_{-\theta} = \pi^-(\Delta - \Delta_0(-\theta))$. Clearly $\Delta_0(-\theta)$ is a basis of $\Phi_0(-\theta)$. A similar property holds for $\bar{\Delta}_{-\theta}$ (see [Hel88, 2.4]).

3.3.1. *A characterization of $\theta$ on a $\theta^-$-basis of* $\Phi$. Let $\Delta$ be a $\theta^-$-basis of $\Phi$. Then $\theta = \theta^* \cdot w_0(-\theta)$, where $w_0(-\theta)$ is the longest element of $W(\Phi_0(-\theta))$ with respect to $\Delta_0(-\theta)$ and $\theta^* \in \mathrm{Aut}(X, \Phi, \Delta, \Delta_0(-\theta)) = \{\phi \in \mathrm{Aut}(X, \Phi) \mid \phi(\Delta) = \Delta \text{ and } \phi(\Delta_0(-\theta)) = \Delta_0(-\theta)\}$, $(\theta^*)^2 = \mathrm{id}$. This is called a characterization of $\theta$ on its $(-1)$-eigenspace.

For closed orbits we have now the following:

**Proposition 8.** *Let $P$ be a parabolic subgroup containing a $\theta$-stable Levi subgroup $L$ of $P$. Let $S = Z(L)^0$ and assume $\Phi(S)$ is a root system with Weyl group $W(S)$. Then $P$ is $\theta$-stable if and only if*

(i) *$\Phi(S)$ has no real roots.*
(ii) *$P$ induces a $\theta^-$-order on $\Phi(S)$.*

*Proof.* Assume $P$ is $\theta$-stable. Let $\Delta$ be the basis of $\Phi$ corresponding to the order induced by $P$. If $\alpha \in \Phi(S)^+$ is a real root then $\theta(\alpha) = -\alpha \in \Phi(S)^-$, what contradicts the fact that $P$ is $\theta$-stable. Similarly, it follows from the fact that $P$ is $\theta$-stable that $\theta(\Phi^+) = \Phi^+$ and consequently also $\theta(\Delta) = \Delta$. The result now follows from 3.3.1.

Conversely assume (i) and (ii) hold. Let $\Delta$ be the basis of $\Phi(S)$ related to the order induced by $P$. Since $\Phi(S)$ has no real roots, it follows from 3.3.1 that $\theta(\Delta) = \Delta$, what proves the result. □

**Proposition 9.** *Let $P$ be a parabolic subgroup containing a $\theta$-stable Levi subgroup $L$ of $P$. Let $S = Z(L)^0$ and assume $\Phi(S)$ is a root system with Weyl group $W(S)$. Then $G^P/P \subset G/P$ contains a $\theta$-stable conjugate of $P$.*

*Proof.* let $L$ be a $\theta$-stable Levi factor of $P$ and $S = Z(L)^0$. Let $\Phi_1 = \{\alpha \in \Phi(S) \mid \theta(\alpha) = -\alpha\}$ be the set of real roots, $\Delta_1 \subset \Phi_1$ a basis and $w_1$ the longest involution of $W(\Phi_1)$ with respect to $\Delta_1$. By [Hel91, 2.7] there exist $\alpha_1, \dots, \alpha_n \in \Phi_1$ strongly orthogonal roots, such that $w_1 = s_{\alpha_1} \dots s_{\alpha_n}$. Since each of the $\alpha_i$, $(i = 1, \dots, n)$ is $\theta$-singular, it follows from Proposition 7, with an easy induction, that there exists $x \in Z_G(S^-_{w_1})$ such that $S_1 = xSx^{-1}$ is $\theta$-stable, $\dim S_1^- = \dim S - n$ and $S_1$ has no real roots. From Proposition 8 and Lemma 7 it follows that there exists $\lambda \in X_*(S_1)$ such that $P(\lambda)$ is $\theta$-stable. □

*Remark* 3. If $P$ is $\theta$-stable, then $KP$ is closed in $G$ and any closed orbit is of the form $KxP$ with $xPx^{-1}$ a $\theta$-stable conjugate of $P$.

3.4. **Open orbit.** For $g \in \mathcal{V}^S$ let $\Delta_{c,g}(S)$ be the set of all $\psi_g$-compact simple roots. From Lemma 2 it follows that

$$\mathcal{V}^{S,P} = \{v \in G \mid S \text{ and } T \text{ are } \psi_v\text{-stable and } \Pi \subset \Delta_{c,v} \cup \psi_v(\Phi^-)\}.$$

Combined with [BH00, Proposition 5] we get:

**Corollary 3.** *Let $\mathcal{O}$ be the unique open orbit of $(K, P)$ in $G$. Then the following are equivalent:*

(i) $\mathcal{O} \subset G^P$.
(ii) *There exists $v \in \mathcal{V}^{S,P}$ such that $\mathcal{O} = KvP$.*
(iii) *There exists $v \in \mathcal{V}^{S,P}$ such that $\Delta \subset \Delta_{c,v} \cup \psi_v(\Phi^-)$.*
(iv) *There exists $v \in \mathcal{O}$ such that $S$ and $T$ are $\psi_v$-stable and $\Delta \subset \Delta_{c,v} \cup \psi_v(\Phi^-)$.*

Using the restricted root system $\Phi(S)$ we can actually show that the open orbit is contained in $G^P$. For this we first characterize the conjugates of $P$ containing a $S \in \mathcal{S}^\theta$ with $S^-$ maximal. These are related to $(\theta, \mathcal{S}^\theta)$-split parabolic subgroups.

**Proposition 10.** *Let $P_1$ be a $(\theta, \mathcal{S}^\theta)$-split parabolic subgroup of $G$ and $S \in \mathcal{S}^\theta$ such that $S \subset P_1$ and $P_1 \cap \theta(P_1) = Z_G(S^-)$. Then the following are equivalent:*

(i) *$P_1$ is a minimal $(\theta, \mathcal{S}^\theta)$-split parabolic subgroup of $G$.*
(ii) *$S^-$ is maximal.*

*Proof.* (i) $\Rightarrow$ (ii). Assume $S^-$ is not maximal. Let $\lambda \in \Phi(S)$ be a $\theta$-singular imaginary root. Then $S^- \subset \ker(\lambda)^0$. Let $T \supset S$ be a $\theta$-stable maximal torus with $T^-$ maximal $\theta$-split in $P_1 \cap \theta(P_1) = Z_G(S^-)$. From Proposition 7 it follows that there exist $x \in Z_G(T^-.\ker(\lambda)^0)$ such that $S_1 = xSx^{-1}$ is $\theta$-stable with $\dim S_1^- = \dim S^- + 1$. Since $Z_G(T^-.\ker(\lambda)^0) \subset Z_G(S^-)$ it follows that $S_1 \subset P_1 \cap \theta(P_1) = Z_G(S^-)$. Let $\delta$ be a nontrivial element of $X_*(S_1^-)$ and $P(\delta)$ the parabolic subgroup of $G$ containing $S_1$ given by $\Phi(P(\delta), S_1) = \{\alpha \in \Phi(S_1) \mid \langle \delta, \alpha \rangle \geq 0\}$. Then $P(\delta)$ is a proper parabolic subgroup of $P_1$ and since $\theta(\delta) = -\delta$ it follows that $P(\delta)$ is $(\theta, \mathcal{S}^\theta)$-split. Clearly this is a contradiction. So $S^-$ is maximal.
(ii) $\Rightarrow$ (i). Assume $P_0$ a $(\theta, \mathcal{S}^\theta)$-split parabolic subgroup with $P_0 \subset P_1$. Then $P_0 \cap \theta(P_0) \subset P_1 \cap \theta(P_1) = Z_G(S^-)$. By Lemma 8 there exists $\delta \in X_*(S^-)$ such that $P_0 = P(\delta)$ and $P_0 \cap \theta(P_0) = Z_G(\delta)$. Since clearly $Z_G(S^-) \subset Z_G(\delta)$ it follows that $P_0 \cap \theta(P_0) = P_1 \cap \theta(P_1) = Z_G(S^-)$. So $P_0 = P_1$. ☐

**Proposition 11.** *Let $P_1$ be a minimal $(\theta, \mathcal{S}^\theta)$-split parabolic subgroup of $G$ containing a conjugate $P_0$ of $P$. Then $K^0 P_1 = K^0 P_0$ is open in $G$.*

*Proof.* Since $\Phi(Z_G(S^-), S)$ has no $\theta$-singular roots it follows that $K^0 Z_G(S^-) = K^0 Z_G(S)$. Since $P_1 = Z_G(S^-) P_0$ we get $K^0 P_1 = K^0 P_0$, which by [Vus74, Theorem 1.3.] is open. ☐

**Proposition 12.** *All minimal $(\theta, \mathcal{S}^\theta)$-split parabolic subgroups of $G$ are conjugate under $K^0$.*

*Proof.* Let $P_1, P_2$ be minimal $(\theta, \mathcal{S}^\theta)$-split parabolic subgroups of $G$ and let $P_0 \subset P_1$, $P_0' \subset P_2$ be parabolic subgroups of $G$ conjugate to $P$. Let $g \in G$ such that $gP_0'g^{-1} = P_0$. By Proposition 11 both $K^0 P_0'$ and $K^0 P_0 = K^0 g P_0' g^{-1}$ are open in $G$. This yields that $K^0 P_0'$ and $K^0 g P_0' g^{-1}$ are the same open orbit of $P_0'$ in $K^0 \backslash G$. Hence $g \in K^0 P_0'$. Write $g = h.p$, $h \in K^0$, $p \in P_0'$. Then $hP_0'h^{-1} = P_0$. So we may assume that $P_0 \subset P_1 \cap hP_2h^{-1}$.

Let $S_0 \in \mathcal{S}^\theta$ such that $S_0 \subset P_0$ and $\delta_1, \delta_2 \in X_*(S_0^-)$ such that $P_1 = P(\lambda_1)$, $P(\lambda_2) = hP_2h^{-1}$. Since $P(\lambda_1) \cap P(\lambda_2) \supset P_0$ it follows with the same argument as in [Vus74, 1.2. Prop. 4] that $P_1 \cap hP_2h^{-1} = P(\lambda_1 + \lambda_2)$. Clearly $P(\lambda_1 + \lambda_2)$ is a $(\theta, \mathcal{S}^\theta)$-split parabolic subgroup of $G$. By the minimality condition of $P_1$ we get $P_1 = P(\lambda_1 + \lambda_2) \subset hP_2h^{-1}$. By symmetry, we also have that $h^{-1}P_1h \subset P_2$. Thus $hP_2h^{-1} = P_1$. □

For the tori $S \in \mathcal{S}^\theta$ with $S^-$ maximal we can prove a similar result:

**Corollary 4.** *Let $S_1, S_2 \in \mathcal{S}^\theta$ with $S_1^-$ and $S_2^-$ maximal. Then $S_1$ and $S_2$ are conjugate under $K$.*

*Proof.* Let $P_1 \supset S_1$ and $P_2 \supset S_2$ be minimal $(\theta, \mathcal{S}^\theta)$-split parabolic subgroups of $G$. By Proposition 12 There exists $h \in K^0$ such that $hP_2h^{-1} = P_1$. So we may assume $P_1 = P_2$. Let $P_0 \supset S_1$ and $P_0' \supset S_2$ be parabolic subgroups, conjugate to $P$, contained in $P_1$. Since $K^0P_0 = K^0P_0'$ open in $G$, it follows that $P_0$ and $P_0'$ are $K^0$-conjugate. So we may assume $P_0 = P_0'$ contains $S_1$ and $S_2$. But then by [BH00, Lemma 5] $S_1$ and $S_2$ are conjugate under $K \cap R_u(P_0)$. □

We conclude this subsection by giving a characterization of the open orbit in terms of $\Phi(S)$ and $W(S)$. Let $\mathcal{O}$ be the unique open orbit of $(K, P)$ in $G$ and let $v \in \mathcal{V}^S$ such that $\mathcal{O} = KvP$. If $n = v^{-1}\theta(v)$, then the orbit $P * n$ is the unique open orbit of $P$ in $Q^0$, which is also called the big cell. Similar as in 2.3.1 let $w_0$ be the element in the Weyl group $W(S)$ with $w_0(\Phi(S)^+) = \theta(\Phi(S)^+)$ and let $\theta' = \theta w_0$. For $w \in W(S)$ let $w' = ww_0$.

**Theorem 1.** *Let $v \in \mathcal{V}^S$, $n = v^{-1}\theta(v)$, $w$ the image of $n$ in $W(S)$, $w' = ww_0$ and $\Lambda = I(w) \cap \Delta(S)$. The following conditions are equivalent:*

(i) *$P * n$ is open in $\tau(G) = Q^0$.*
(ii) *$C''(w) \cap \Delta(S) = \emptyset$ and $\Lambda \subset \Delta_{c,v}(S)$.*
(iii) *$w' = w_\Lambda^0 w_{\Delta(S)}^0$ and $\Lambda \subset \Delta_{c,v}(S)$.*
(iv) *$vP_\Lambda v^{-1}$ is a minimal $(\theta, \mathcal{S}^\theta)$-split parabolic subgroup of $G$, containing $P$.*
(v) *There exists a minimal $(\theta, \mathcal{S}^\theta)$-split parabolic subgroup of $G$, containing $v^{-1}Pv$.*

*Proof.* (i) ⇒ (ii). By Proposition 6(ii), $C''(w) \cap \Delta(S) = \emptyset$. By Proposition 2(vi) $w'\theta' = w\theta$. Since $C''(w) \cap \Delta(S) = \emptyset$ it follows from [Spr84, Lemma 3.2(ii)] that $s_\alpha w'\theta'(s_\alpha) = w'$ for all $\alpha \in \Delta(S)$ with $s_\alpha w' > w'$. Then from [HW93, Proposition 7.11] it follows that $w' = w_\Lambda^0 w_{\Delta(S)}^0$. Observe that $vP_\Lambda v^{-1} \cap \theta(vP_\Lambda v^{-1}) = v(P_\Lambda \cap n\theta(P_\Lambda)n^{-1})v^{-1}$. From [HW93, Proposition 7.11] we get $(\Phi(S)_\Lambda \cup \Phi(S)^+) \cap w\theta(\Phi(S)_\Lambda \cup \Phi(S)^+) = \Phi(S)_\Lambda \cup \Phi(S)_\Lambda^+ = \Phi(S)_\Lambda$. It follows that $vP_\Lambda v^{-1} \cap \theta(vP_\Lambda v^{-1}) = vG_{\Phi(S)_\Lambda}v^{-1}$.

Now let $P_0 = P \cap G_{\Phi(S)_\Lambda}$. A simple dimension argument shows that $P_0 * n$ is open in $G_{\Phi(S)_\Lambda} * n$. It follows that $P_0 G_{\Phi(S)_\Lambda}^{\psi_v}$ is open in $G_{\Phi(S)_\Lambda}$. However $\psi_v|\Phi(S)_\Lambda = 1$ and $P_0$ is $\psi_v$-stable. Now by [BH00, Proposition 8], $G_{\Phi(S)_\Lambda} = P_0 G_{\Phi(S)_\Lambda}^{\psi_v}$ and by [HW93, Lemma 1.8], $\Lambda \subset \Delta_{c,v}(S)$.

(ii) ⇒ (iii). Using the same argument as in (i) ⇒ (ii), the condition $C''(w) \cap \Delta(S) = \emptyset$ yields that $w' = w_\Lambda^0 w_{\Delta(S)}^0$.

(iii) ⇒ (iv). Using the same argument as in (i) ⇒ (ii), the condition $w' = w_\Lambda^0 w_{\Delta(S)}^0$ yields that $vP_\Lambda v^{-1} \cap \theta(vP_\Lambda v^{-1}) = vG_{\Phi(S)_\Lambda}v^{-1}$. Consequently $vP_\Lambda v^{-1}$ is a $\theta$-split parabolic subgroup of $G$. Let $S_1 = vSv^{-1}$. Since $\Lambda \subset \Delta_{c,v}(S)$, $S^-_{\psi_v}$ is maximal with respect to $\psi_v$. But then $S^-_1$ is maximal with respect to $\theta$. By Proposition 10, $vP_\Lambda v^{-1}$ is a minimal $\theta$-split parabolic subgroup of $G$.

(iv) $\Rightarrow$ (v) is trivial.

(v) $\Rightarrow$ (i). By Proposition 11, $KvPv^{-1}$ is open in $G$. Hence $KvP$ is open in $G$ and (i) is immediate. □

## 4. Orbit closures

In this section we study the decomposition of the closures in $G$ of double cosets in $G^P$ in terms of $K \times P$ double cosets. These results are very similar to those obtained in [HW93] for minimal parabolic $k_0$-subgroups acting on the symmetric $k_0$-variety $G_{k_0}/H_{k_0}$, where $k_0 \subseteq k$ is a subfield of $k$ and $G$, $\theta$ are defined over $k_0$.

For $g \in G^P$, consider the double coset $KgP$. Let $v \in \mathcal{V}^S$ such that $KvP = KgP$, $n = v^{-1}\theta(v)$, $w$ the image of $n$ in $W(S)$, $\theta' = \theta w_0$ and $w' = ww_0$. Let $\Phi^+ = \Phi(P, S)$ be the set of positive roots related to $P$ and let $\Delta(S)$ be the corresponding basis of $\Phi$. By [HW93, Proposition 7.9] we can write

$$w' = s_{\alpha_1} \dots s_{\alpha_h} w^0_\Lambda \theta'(s_{\alpha_h}) \dots \theta'(s_{\alpha_1}),$$

with $\Lambda$ a $\theta'$-stable subset of $\Delta(S)$ satisfying $w^0_\Lambda \theta'\alpha = -\alpha$ for $\alpha \in \Phi(S)_\Lambda$, $\alpha_1, \dots, \alpha_h \in \Delta(S)$ and $\ell(w') = 2h + \ell(w^0_\Lambda)$. Choose $n_1, \dots, n_h \in N_G(S)_k$ with images $s_{\alpha_1}, \dots, s_{\alpha_h}$ in $W(S)$ respectively. Set $u = n_h^{-1} \dots n_1^{-1} v$ and $m = u\theta(u)^{-1}$. First we look at $P_\Lambda$:

**Lemma 9.** *Let $v \in \mathcal{V}^S$, $n = v^{-1}\theta(v)$, $w$ the image of $n$ in $W(S)$ and $w' = ww_0$. Let $\Lambda = \Delta(S) \cap R(w) = \{\alpha \in \Delta(S) \mid w\theta\alpha = -\alpha\}$, $P_\Lambda$ the parabolic subgroup of $G$ containing $P$ defined by $\Lambda$ and $P' = vP_\Lambda v^{-1}$. Assume $w' = w^0_\Lambda$. Then we have the following conditions:*

(i) *If $w' = w^0_\Lambda$, then the group $P' = vP_\Lambda v^{-1}$ is $\theta$-stable.*
(ii) *$\tau(P')$ is a closed irreducible and smooth subvariety of $\tau(G) = Q^0$.*
(iii) *$\tau(vPv^{-1})$ is dense in $\tau(P')$.*

*Proof.* (i). First note that $w\theta = w'\theta'$. From the condition $w' = w^0_\Lambda$, $w'\theta'(\Phi(S)_\Lambda \cup \Phi(S)^+) = \Phi(S)_\Lambda \cup \Phi(S)^+$. By [BT65, 3.22], we have that $P' \cap \theta(P') = vP_\Psi v^{-1}$ with $\Psi = (\Phi(S)_\Lambda \cup \Phi(S)^+) \cap w\theta(\Phi(S)_\Lambda \cup \Phi(S)^+) = \Phi(S)_\Lambda \cup \Phi(S)^+$. It follows that $P'$ is $\theta$-stable.

(ii). Since $P'$ is $\theta$-stable, by [BH00, Proposition 8], $KP'$ is a closed subvariety of $G$. It follows that $\tau(P')$ is a closed irreducible subvariety of $Q$. As $\tau(P')$ is homogeneous, it is smooth.

(iii). Note that $vG_{\Phi(S)_\Lambda}v^{-1}$ is $\theta$-stable. By the assumption on $\Lambda$, $vG_{\Phi(S)^+_\Lambda}v^{-1}$ is a $\theta$-split parabolic subgroup of $vG_{\Phi(S)_\Lambda}v^{-1}$. It follows from Proposition 11 that $KvG_{\Phi(S)^+_\Lambda}v^{-1}$ is dense in $KvG_{\Phi(S)_\Lambda}v^{-1}$. Hence $KvPv^{-1}$ is dense in $KP'$. Now the assertion is obvious. □

For a subset $Y$ of $G$ denote the Zariski closure of $Y$ in $G$ by $\mathrm{cl}(Y)$.

**Theorem 2.** *Let $v \in \mathcal{V}^S$, $n = v^{-1}\theta(v)$, $w$, etc. be as above. Then we have the following:*

(i) $\mathrm{cl}(P * n) = P_{\alpha_1} * \cdots * P_{\alpha_h} * P_\Lambda * m$.
(ii) $\dim(P * n) = \Sigma_{i=1}^h(\dim(P_{\alpha_i}) - \dim(P)) + \dim(P_\Lambda * m)$.

*Proof.* We prove the assertion by induction on $h$, starting $h = 0$. When $h = 0$, the result is immediate from Lemma 9. Set $v_1 = n_1^{-1} v = n_2 \dots n_h u$. We may assume that

$$\mathrm{cl}(P * v_1^{-1}\theta(v_1)) = P_{\alpha_2} * \cdots * P_{\alpha_h} * P_\Lambda * m.$$

By Proposition 6(ii), we have that

$$\dim(P * n) = \dim(P * v_1^{-1}\theta(v_1)) + \dim P_{\alpha_1} - \dim P. \tag{10}$$

Clearly $\dim(P_{\alpha_1} * P * v_1^{-1}\theta(v_1)) \leq \dim(P * v_1^{-1}\theta(v_1)) + \dim P_{\alpha_1} - \dim P$. Hence we have the relation that $\dim(P_{\alpha_1} * P * v_1^{-1}\theta(v_1)) \leq \dim(P * n)$. However $P_{\alpha_1} * P * v_1^{-1}\theta(v_1)$ contains $P * n$. This yields that

$$\dim(P_{\alpha_1} * P * v_1^{-1}\theta(v_1)) = \dim(P * n). \tag{11}$$

Since $P_{\alpha_1} * \mathrm{cl}(P * v_1^{-1}\theta(v_1))$ is closed and irreducible, we conclude that $\mathrm{cl}(P * n) = \mathrm{cl}(P_{\alpha_1} * P * v_1^{-1}\theta(v_1)) = P_{\alpha_1} * \mathrm{cl}(P * v_1^{-1}\theta(v_1)) = P_{\alpha_1} * \cdots * P_{\alpha_h} * P_\Lambda * m$. Finally (ii) follows from (10) by an easy induction. □

When $P = B$ a Borel subgroup, the above result is due to Springer [Spr84, Theorem 6.5]. In the case that $G, \theta$ are defined over a subfield $k_0$ of $k$ and $P$ is a minimal parabolic $k_0$-subgroup of $G$ this result can be found in [HW93].

*Remark* 4. Since $P_\Lambda * m$ closed in $Q^0$ as well it follows from Theorem 2 that the product map defines a proper morphism from $P_{\alpha_1} \times^P P_{\alpha_2} \times^P \dots \times^P P_\Lambda * m$ to the closure of $P * m$. From Theorem 2(ii) it follows that its general fibers are finite. One can show that this morphism is in fact birational and that we have a resolution of singularities of orbit closures. We intend to discuss this in more detail in future work.

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