Computation of cerebral blood flow

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Support: NSF-DMS-0616597, NSF-DMS-0410561, SAMSI, American Diabetes Association 1-06-CR-25, NIH 2P60 AG08812, AG0043390, R01-NS045745

November 18, 2008
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Outline

1. Introduction
2. Model
3. BCs & Numerics
4. Data & calibration
5. Results
6. Conclusion
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Circle of Willis (CoW)
Circle of Willis (CoW) (2)
Medical issues

- CoW complete in only 50% of healthy brains
- redundant vessels ⇔ autoregulation
- network topology ⇒ blood flow
- stroke ⇒ perfusion
- stent location for stroke victims

Patient specific numerical simulation ⇒ flexible approach
Medical issues

- CoW complete in only 50% of healthy brains
- redundant vessels $\Leftrightarrow$ autoregulation
- network topology $\Rightarrow$ blood flow $\leftarrow$ THIS TALK
- stroke $\Rightarrow$ perfusion $\leftarrow$ THIS TALK
- stent location for stroke victims

Patient specific numerical simulation $\Rightarrow$ flexible approach
Modeling issues

- how detailed should the model be? **dimensionality**
- only **part** of vascular system can be considered
- how to link computed part to the rest? **boundary conditions**
- how to calibrate model to data? **reliability**
Relation to previous work

- **one-d**: Alastruey, Sherwin et al. (2007), Moore et al. (2006)
- **two-d**: Ferrandez et al. (2000), Kufahl et al. (1985)
- **three-d**: Alvaes et al. (2007), Cebral et al. (2003), Moore et al. (2006)
- **"other"**: Canic et al. (2006), Deparis, Quarteroni et al. (2007), Figueroa, Hughes et al. (2006), Nobile et al. (2007)

**what we bring**

- network + wall reaction
- fast and simple method
- extensive comparison with data
- study of role played by viscoelasticity
fluid equations in one vessel

blood density is assumed constant $\Rightarrow$ incompressible Navier-Stokes

$$\rho \left( \partial_t u + u \cdot \nabla u \right) - \nabla \cdot \sigma = \rho g$$
$$\nabla \cdot u = 0$$

- $\rho$: density
- $u$: velocity
- $\sigma$: stress tensor
- $g$: acceleration due to gravity
Simplifying assumptions (flow)

- Blood is **Newtonian** $\sigma = -\rho \mathbb{I} + \mu (\nabla \mathbf{u} + \nabla (\mathbf{u}^T))$
- Blood flow is **axisymmetric** and has no swirl
- $\mathbf{u}(r, \theta, x) = < u_r(r, x), 0, u_x(r, x) >$
the vessels are **tethered** in their longitudinal direction

**Kelvin model:**

\[
\rho - \rho_0 + \tau_\sigma \frac{\partial p}{\partial t} = \frac{Eh}{r_0} \left( s + \tau_\epsilon \frac{\partial s}{\partial t} \right)
\]

\[
s = 1 - \sqrt{\frac{A_0}{A}}, \quad A = \pi R^2, \quad \tau_\sigma \text{ and } \tau_\epsilon \text{ are relaxation times}
\]
Model: wrapping it up

- non-dimentionalization
- radial vel. $\ll$ axial vel.
- averaging on cross-section
- axial velocity

$$u_x(r, x, t) = \frac{\gamma + 2}{\gamma} U(x, t) \left(1 - \left(\frac{r}{R(x, t)}\right)^\gamma\right)$$

![Graph showing axial velocity profile for different nondimensional radii]
Model (finally)

unknowns:
- $A$ surface area
- $Q$ flow ($Q = AU$)
- $P$ pressure

\[
(\partial_t + B \partial_x) \begin{bmatrix} A \\ Q \\ P \end{bmatrix} = G,
\]

\[
B = \begin{bmatrix}
0 & \frac{1}{\gamma+1} \left( \frac{Q}{A} \right)^2 & 1 \\
\frac{\tau\epsilon}{\tau\sigma} & \frac{\gamma+2}{\gamma+1} \frac{Q}{A} & 0 \\
0 & \frac{1}{M^2} A^{-3/2} & 0
\end{bmatrix}
\quad\text{and}\quad
G = \begin{bmatrix}
-\frac{\gamma+2}{\mathcal{R}} \frac{Q}{A} - \frac{ex\cdot k}{\mathcal{F}} A \\
\frac{1}{\mathcal{W}} (1 - P) + \frac{2}{\mathcal{W}M^2} (1 - A^{-1/2})
\end{bmatrix}.
\]

Dimensionless numbers:
Reynolds $\mathcal{R}$, Froude $\mathcal{F}$, Mach $M$, Weissenberg $\mathcal{W}$
unknowns:

- $A$ surface area
- $Q$ flow ($Q = AU$)
- $P$ pressure

\[
(\partial_t + B \partial_x) \begin{bmatrix} A \\ Q \\ P \end{bmatrix} = G,
\]

- eigenvalues of $B$ are real $\Rightarrow$ hyperbolic balance law
- observations: $\lambda_1 < 0$, $\lambda_2 = 0$, $\lambda_3 > 0$ and solutions are smooth
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Junction conditions

$J$-th junction with $N_J$ vessels

- conservation of mass

\[ \sum_{i=1}^{N_J} Q_i = 0 \]

- continuity of pressure

\[ P_1 = P_2 = \cdots = P_{N_J} \]
Boundary conditions

- **INFLOW**: at Basilar, left and right Carotid
  \[ Q = U A, \]
  where \( U \) comes from data

- **OUTFLOW**: all other ends (Windkessel)
  \[
  R^s \partial_t Q + \frac{R^s + R^p}{R^p C} Q = \partial_t P + \frac{1}{R^p C} P,
  \]
  where resistances \( R^s, R^p \) and compliance \( C \) are adapted for each vessel from data (see below)
Spatial discretization

$v(\cdot, t)$ defined in $[-1, 1]$

**Chebyshev collocation** at the Chebyshev-Gauss-Lobatto nodes

$$x_j = \cos \left( \frac{\pi j}{N - 1} \right), \quad j = 0, \ldots, N - 1$$

$$v(x, t) \approx v_N(x, t) = \sum_{j=0}^{N-1} V_j(t) \psi_j(x),$$

$\{\psi_j\}_{j=0}^{N-1}$ Lagrange interpolating polynomials at $x_j$'s ($\psi_j(x_i) = \delta_{ij}$).

**advantage:** accuracy for low number of nodes
Time discretization

- explicit methods (TVD)
  - advantages: simple, nonlinear stability
  - disadvantages: conditionally stable, introduce bias in network

- implicit methods (Backward Euler)
  - advantages: no bias, larger time steps
  - disadvantage: low order (for BE)

Bottom line: 4 nodes per vessel, two Newton steps per time step OK
## Data (I), geometry

Length and area of vessels (magnetic resonance angiogram)

<table>
<thead>
<tr>
<th></th>
<th>name</th>
<th>diameter (mm)</th>
<th>length (mm)</th>
<th>$E$ ($10^6$ g/s²cm)</th>
<th>$c_0$ ($10^2$ cm/s)</th>
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<td>2.2</td>
<td>3.3*</td>
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<tr>
<td>4</td>
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<tr>
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<tr>
<td>6</td>
<td>R PCoA</td>
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<td>10*</td>
<td>16</td>
<td>14</td>
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<tr>
<td>7</td>
<td>L PCoA</td>
<td>2.0</td>
<td>10*</td>
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<td>R ICA</td>
<td>4.2</td>
<td>48</td>
<td>8.0</td>
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<td>L ICA</td>
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<td>48</td>
<td>8.0</td>
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<td>L ACA1</td>
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<td>11</td>
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<td>ACoA</td>
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<td>16</td>
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<tr>
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<td>R ACA 2</td>
<td>2.3</td>
<td>23</td>
<td>16</td>
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<tr>
<td>16</td>
<td>L ACA 2</td>
<td>2.3</td>
<td>23</td>
<td>16</td>
<td>14</td>
</tr>
</tbody>
</table>
Data (II)

Obtained from healthy volunteer in supine position
- velocity (digital transcranial Doppler)
- pressure (finger)
- electrocardiogram

![Graph showing ECG, pressure, and velocity over time]
Kalman filtering (I)

- \( x \): state of model with probability density \( p(x) \) (prior)
- \( d \): data; \( p(d|x) \) data likelihood
- Bayes: \( p(x|d) \propto p(d|x)p(x) \)
- observation operator \( H \): \( d \) would be \( Hx \) if no error but \( d \neq Hx \)

\[
p(x) \propto \exp \left( -\frac{1}{2}(x - \mu)^T Q^{-1} (x - \mu) \right) 
\]

\[
p(d|x) \propto \exp \left( -\frac{1}{2}(d - Hx)^T R^{-1} (d - Hx) \right) 
\]
Kalman filtering (II)

Algebra $\Rightarrow \hat{x} = x | d$ is Gaussian with

$$p(\hat{x}) \propto \exp \left( -\frac{1}{2}(\hat{x} - \hat{\mu})^T \hat{Q}^{-1}(\hat{x} - \hat{\mu}) \right)$$

- $\hat{\mu} = \mu + K(d - H\mu)$
- $\hat{Q} = (I - KH)Q$
- $K = QH^T(HQH^T + R)^{-1}$ Kalman gain

Posterior mean $\hat{\mu}$ tries matching observation $H\hat{\mu} \approx d$ and mean $\hat{\mu} \approx \mu$

**Lemma**

$\hat{\mu}$ minimizes

$$F(x) = (x - \mu)^T Q^{-1}(x - \mu) + (d - Hx)^T R^{-1}(d - Hx)$$
Ensemble Kalman filtering

Monte Carlo implementation of Kalman filter

- replace state covariance matrix \( Q \) by sampled covariance based on ensemble
- if state evolution is nonlinear, prior may be not be Gaussian...
- ensemble size = 100
- used to calibrate boundary condition parameters
Calibration

**blue line:** data, ×: model-original resistance parameters, o: model-EnKF optimized resistance parameters
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Velocities

20 realizations; blue line: $\mu$, green line: $\mu \pm \sigma$, dashed line: $\mu \pm 2\sigma$, $\times$: mean predicted outflow

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influence of the tricky parameters

LMCA

velocity (cm/s)
time (s)
data
$\lambda = 2, \gamma = 1$
$\lambda = 9, \gamma = 1$

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influence of the tricky parameters

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure}
\caption{LMCA}
\end{figure}

\begin{itemize}
\item data
\item $\gamma = 2, \tau = 1$
\item $\gamma = 2, \tau = 4$
\end{itemize}
influence of the tricky parameters

not much influence...
## Common anatomical variations

<table>
<thead>
<tr>
<th>Variation</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complete CoW</td>
<td>49%</td>
</tr>
<tr>
<td>Extra ACoA</td>
<td>10%</td>
</tr>
<tr>
<td>Missing ACA, seg 1</td>
<td>6%</td>
</tr>
<tr>
<td>Missing ACoA</td>
<td>1%</td>
</tr>
<tr>
<td>Missing One PCoA</td>
<td>9%</td>
</tr>
<tr>
<td>Missing Both PCoAs</td>
<td>9%</td>
</tr>
<tr>
<td>Missing All CoAs</td>
<td>*</td>
</tr>
<tr>
<td>Missing PCA, seg 1</td>
<td>9%</td>
</tr>
</tbody>
</table>
Perfusion (I)

```
1.61
1.61
2.91
2.91
1.93
1.93
```

```
1.61
1.61
2.91
2.91
1.93
1.93
```

normal
missing LPCoA
Stroke (II)

None

Vertebral

Basilar

LICA

perfusions (ml)

RPCA
LPCA
RMCA
LMCA
RACA
LACA
left: complete circle; right: missing LPCoA
left: complete circle; right: missing all CoAs
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Final remarks

- one-dimensional models work (for this problem)
- "cheap" discretization works (for this problem)
- elastic vs. visco-elastic, boundary conditions, flow profile not central (not as important as geometry)
- non Newtonian effects?
- better inclusion of uncertainties into the model
- comparison with perfusion data