MA 584
Homework #1
Due Thursday, September 9, 2004

1. When we deal with irregular boundaries or use adaptive grids, we need to use non-uniform grids. Derive the finite difference coefficients for the following:

$$u'(\bar{x}) \approx \alpha_1 u(\bar{x} - h_1) + \alpha_2 u(\bar{x}) + \alpha_3 u(\bar{x} + h_2), \tag{1}$$
$$u''(\bar{x}) \approx \alpha_1 u(\bar{x} - h_1) + \alpha_2 u(\bar{x}) + \alpha_3 u(\bar{x} + h_2), \tag{2}$$
$$u'''(\bar{x}) \approx \alpha_1 u(\bar{x} - h_1) + \alpha_2 u(\bar{x}) + \alpha_3 u(\bar{x} + h_2) \tag{3}$$

Are they consistent? In other words, as $h = \max\{h_1,\ h_2\}$ approaches zero, does the error also approach to zero? If so, what are the order of accuracy? Do you see any potential problems with the schemes you have derived?

2. Show that the finite difference method using the central formula for the BVP

$$u'' = f, \quad 0 < x < 1, \tag{4}$$
$$u(0) = 0, \qquad u'(1) = \sigma, \tag{5}$$

and the *ghost point* method at $x = 1$ is stable and $\|E\|_2 \le Ch^{3/2}$.

3. Program the central finite difference method for the self-adjoint BVP

$$(\beta(x)u')' - \gamma(x)u(x) = f(x), \quad 0 < x < 1,$$
$$u(0) = u_a, \qquad au(1) + bu'(1) = c.$$

using a uniform grid and the central difference scheme

$$\frac{\beta_{i+\frac{1}{2}}(U_{i+1} - U_i)/h - \beta_{i-\frac{1}{2}}(U_i - U_{i-1})/h}{h} - \gamma(x_i)U_i = f(x_i). \tag{6}$$

Test your code for the case where

$$\beta(x) = 1 + x^2, \quad \gamma(x) = x, \quad a = 2, \quad b = -3, \tag{7}$$

and the rest of functions or parameters are determined from the exact solution

$$u(x) = e^{-x}(x-1)^2. \tag{8}$$

Plot the computed solution and the exact solution, and the errors for a particular grid, say $n = 80$. Do the grid refinement analysis to determine the order of accuracy of the global solution. Also try to answer the following questions:

- Can your code handle the case when $a = 0$ or $b = 0$?
- If we use the central difference scheme for the equivalent differential equation

$$\beta u'' + \beta' u' - \gamma u = f, \tag{9}$$

what are the advantages or disadvantages?

4. Consider the finite difference scheme for the 1D steady state *convection-diffusion* equation

$$\epsilon u'' - u' = -1, \quad 0 < x < 1 \tag{10}$$

$$u(0) = 1, \qquad u(1) = 3. \tag{11}$$

(a) Verify the exact solution is

$$u(x) = 1 + x + \left(\frac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1} \right). \tag{12}$$

(b) Compare the following two finite difference methods for $\epsilon = 0.3, 0.1, 0.05$, and 0.0005,
(1): Central difference scheme:

$$\epsilon \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} - \frac{U_{i+1} - U_{i-1}}{2h} = -1. \tag{13}$$

(2): Central-upwind difference scheme:

$$\epsilon \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} - \frac{U_i - U_{i-1}}{h} = -1. \tag{14}$$

Do grid refinement analysis for each case to determine the order of accuracy. Plot the computed solution and the exact solution for $h = 0.1$, $h = 1/25$, and $h = 0.01$. You can use Matlab command *subplot* to put several graphs together.

(c) From you observation, give your opinion to see which method is better.