Selected solutions

1 The coefficient matrix of the red-black ordering is

\[
A = \begin{pmatrix}
  D_1 & B \\
  B^T & D_2
\end{pmatrix}, \quad D_i = -\frac{4}{h^2} I, \quad i = 1, 2
\]

\(B\) is a sparse matrix in which each row has at most 4 non-zero entries. The size of the matrix is \((n - 1)^2\) by \((n - 1)^2\) with non-zero entries of \(O(n^2)\). The \(SOR(\omega)\) at \(k\)-th iteration can be written as

\[
v_{ij}^{k+1} = u_{ij}^k, \quad i, j = 1, 2, \ldots, n - 1.
\]

\[
v_{ij}^{k+1} = (1 - \omega)u_{ij}^k + \frac{\omega}{4} \left( u_{i-1,j}^{k+1} + u_{i+1,j}^{k+1} + u_{i,j-1}^{k+1} + u_{i,j+1}^{k+1} - h^2 f_{ij} \right), \quad i, j = 1, 2, \ldots, n - 1.
\]

The iterative method does not depend on the ordering of the equations and unknowns, but does depend on the index \(i\) and \(j\).

2 (a) The results of one iteration are

\[
\mathbf{x}_J = \begin{pmatrix}
  1/3 \\
  1/2 \\
  1/2
\end{pmatrix}, \quad \mathbf{x}_{GS} = \begin{pmatrix}
  1/3 \\
  1/2 \\
  5/4
\end{pmatrix}, \quad \mathbf{x}_{SOR(1,5)} = \begin{pmatrix}
  0 \\
  5/4 \\
  31/16
\end{pmatrix}
\]

(b) and (c), The iteration matrices are:

\[
R_J = \begin{bmatrix}
  0 & 1/3 & -1/3 \\
  0 & 0 & -1/2 \\
  0 & 1/2 & 0
\end{bmatrix}, \quad R_{GS} = \begin{bmatrix}
  0 & 1/3 & -1/3 \\
  0 & 0 & -1/2 \\
  0 & 0 & -1/4
\end{bmatrix}.
\]

Since \(\|R_J\|_\infty = 2/3 < 1\) and \(\|R_{GS}\|_\infty = 2/3 < 1\), both iterative methods converge.

3 For the first matrix, the eigenvalues of \(R\) are the diagonals. Notice that \(|a_{ii}| < 1\) for \(i = 1, 2, 3, 4\).

We just need to check \(|a_{55}| = 1 - \sin(\alpha \pi)|. Note that 0 < \sin x < 1 if 0 < x < \pi and \sin x is a periodic function of 2\pi. Thus, if 2k < x < 2k + 1, then the iterative method converges, where \(k\) is an integer.

For the second matrix we have \(\|R\|_1 = 0.9999 < 1\), the iterative method converges.

4 (a) The iteration matrices are:

\[
R_J = \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & -2 \\
  0 & 2 & 0
\end{bmatrix}, \quad R_{GS} = \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & -2 \\
  0 & 0 & -4
\end{bmatrix}.
\]

Since \(\rho(R_J) = 2 > 1\) and \(\rho(R_{GS}) = 4 > 1\), both iterative methods diverge.

(b) The matrix is weakly diagonally dominant and irreducible. Both Jacobi and Gauss-Seidel iterative methods converge.