

1. Consider the Poisson equation

$$\begin{aligned} u_{xx} + u_{yy} &= xy, & (x, y) \in \Omega \\ u(x, y)|_{\partial\Omega} &= 0, \end{aligned}$$

where Ω is the unit square. Using the finite difference method, we can get a linear system of equations

$$\frac{U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{ij}}{h^2} = f(x_i, y_j), \quad 1 \leq i, j \leq 3, \quad (1)$$

where $h = 1/4$, $x_i = ih$, $y_j = jh$, $i, j = 0, 1, 2, 3, 4$, and U_{ij} is the approximation of $u(x_i, y_j)$. Write down the coefficient matrix and the right hand side using the *red-black* orderings given in the right diagram of Fig 1. What is the dimension of the coefficient matrix? How many nonzero entries and how many zeros? Generalize your results to general case when $0 \leq i, j \leq n$ and $h = 1/n$. Write down the component form of the $SOR(\omega)$ iterative method. Does the $SOR(\omega)$ iterative method depend on the ordering? From your analysis, explain whether you prefer to use Gaussian elimination method or an iterative method.

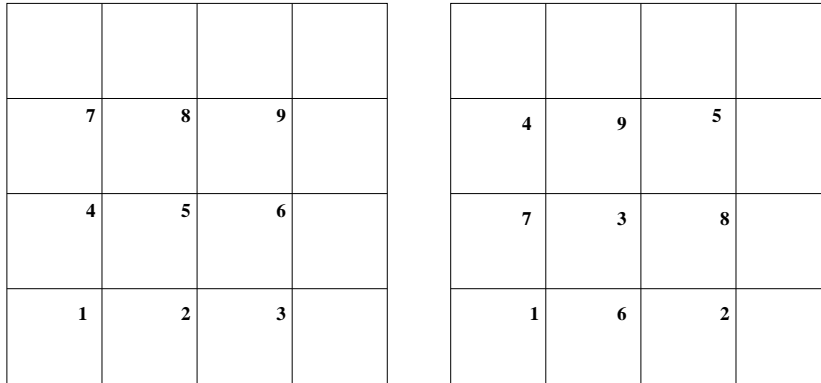


Figure 1: Digram for Exercise 1.

2. Given the following linear system of equations:

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 3 \\ 2x_2 + x_3 &= 2 \\ -x_2 + 2x_3 &= 2 \end{aligned}$$

- (a) With $x^{(0)} = [1, -1, 1]^T$, find the *first* iteration of the Jacobi, Gauss-Seidel, and SOR ($\omega = 1.5$) methods.
- (b) Write down the Jacobi and Gauss-Seidel iteration matrices R_J and R_{GS} .
- (c) Do the Jacobi and Gauss-Seidel iterative methods converge?

3. Judge whether the iterative method $x^{(k+1)} = R x^{(k)} + c$ converges or not.

$$(a) \quad R = \begin{bmatrix} e^{-1} & -e^1 & -1 & -1 & -10 \\ 0 & \sin \pi/4 & 10^4 & -1 & -1 \\ 0 & 0 & -0.1 & -1 & 1 \\ 0 & 0 & 0 & 1 - e^{-2} & -1 \\ 0 & 0 & 0 & 0 & 1 - \sin(\alpha\pi) \end{bmatrix}, \quad (b) \quad \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.3 & -0.7 \\ 0 & 0.69 & 0.2999 \end{bmatrix}$$

4. Determine the convergence of the Jacobi and Gauss-Seidel method applied to the system of equations $Ax = b$, where

$$(a) \quad A = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{bmatrix}, \quad (b) \quad A = \begin{bmatrix} 3 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ 2 & 3 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 2 & 3 & -1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 2 & 3 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & 2 & 3 \end{bmatrix}$$

5. Modify the Matlab code *poisson_drive.m* and *poisson_sor.m* to solve the following diffusion and convection equation:

$$u_{xx} + u_{yy} + au_x - bu_y = f(x, y), \quad 0 \leq x, y \leq 1,$$

Assume that solution at the boundary $x = 0$, $x = 1$, $y = 0$, $y = 1$ are given (Dirichlet boundary conditions). The central-upwinding finite difference scheme is

$$\frac{U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{ij}}{h^2} + a \frac{U_{i+1,j} - U_{ij}}{h} - b \frac{U_{i,j} - U_{i,j-1}}{h} = f_{ij}$$

- Assume the exact solution is $u(x, y) = e^{2y} \sin(\pi x)$, find $f(x, y)$.
- Use the $u(x, y)$ above for the boundary condition and the $f(x, y)$ above for the partial differential equation. Let $a = 1$, $b = 2$, and $a = 100$, $b = 2$, solve the problem with $n = 20$, 40, 80, and $n = 160$. Try $\omega = 1$, the best ω for the Poisson equation discussed in the class, the optimal ω by testing, for example $\omega = 1.9, 1.8, \dots, 1$.
- Tabulate the error, the number of iterations for $n = 20$, 40, 80, and $n = 160$ with your tested optimal ω , compare the number of iterations with the Gauss-Seidel method.
- Plot the solution and the error for $n = 40$ with your tested optimal ω . Label your plots as well.