1. Find $\|x\|_p$, $p = 1, 2, \infty$ for the following vectors
   (a) $x = (3, -4, 0, -3/2)^T$.
   (b) $x = (\sin k, \cos k, 2^k)^T$ for a fixed positive integer $k$.
   (c) $x = (4/(k+1), -2/k^2, k^2 e^{-k})^T$ for a fixed positive integer $k$.

2. Find $\|A\|_p$, $p = 1, 2, \infty$ for the following matrices:
   \[
   \begin{bmatrix}
     1 & 2 \\
     0 & -3
   \end{bmatrix} ;
   \begin{bmatrix}
     -2 & 1 \\
     1 & -2
   \end{bmatrix} ;
   \]

3. (a) Show that $\|x\|_\infty$ is equivalent to $\|x\|_2$. That is to find constants $C$ and $c$ such that $c \leq \|x\|_\infty \leq \|x\|_2 \leq C\|x\|_\infty$. Note that you need to determine such constants that the equalities are true for some particular $x$.
   (b) Show that $\|AB\| \leq \|A\|\|B\|$ for any natural matrix norm, and $\|QA\|_2 = \|A\|_2$ if $Q$ is an orthogonal matrix ($Q^HQ = I$, $QQ^H = I$).

4. Given
   \[
   A = \begin{bmatrix}
     0 & 1 & 2 & 3 \\
     3 & 0 & 1 & 2 \\
     2 & 3 & 0 & 1 \\
     1 & 2 & 3 & 0
   \end{bmatrix} ,
   b = \begin{bmatrix}
     6 \\
     6 \\
     6 \\
     6
   \end{bmatrix}
   \]
   (a) Use Gaussian elimination with the partial pivoting to find the matrix decomposition $PA = LU$.
   This is a paper problem and you are asked to use exact calculations (use fractions if necessary).
   (b) Find the determinant of the matrix $A$.
   (c) Use the factorization to solve $Ax = b$.

5. Consider solving $AX = B$ for $X$, $A \in \mathbb{R}^{m,n}$, $B \in \mathbb{R}^{n,m}$. There are two obvious algorithms. The first one is to get $A = PLU$ using Gaussian elimination, and then to solve for each column of $X$ by forward and backward substitution. The second algorithm is to compute $A^{-1}$ using Gaussian elimination and then to multiply $A^{-1}B$ to get $X$. Count the number of operations by each algorithm and determine which one is faster.

6. (Programming Part) Given a sequence of data
   \[(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m), (x_{m+1}, y_{m+1})\],
   write a program to interpolate the data using the following model
   \[y(x) = a_0 + a_1 x + \cdots + a_{m-1} x^{m-1} + a_m x^m.\]
(a) Derive the linear system of equations for the interpolation problem.

(b) Let \( x_i = (i-1)h, \, i = 1, 2, \ldots, m+1, \, h = 1/m, \, y_i = \sin \pi x_i \), write a computer code using the Gaussian elimination with column partial pivoting to solve the problem. Test your code with \( m = 4, 8, 16, 32, 64 \) and plot the error \( |y(x) - \sin \pi x| \) with 100 or more points between 0 and 1, that is, predict the function at more points in addition to the sample points. For example, you can set \( h1 = 1/100; \, x1 = 0: h1 : 1, \, y1(i) = a_0 + a_1 x1(i) + \cdots + a_{m-1} (x1(i))^{m-1} + a_m (x1(i))^m, \, \)
y2(i) = \( \sin(\pi x1(i)) \), plot \((x1, y1 - y2)\).

(c) Record the CPU time (in Matlab type help cputime) for \( m = 50, 100, 150, 200, \ldots, 350, 400 \). Plot the CPU time versus \( m \). Then use the Matlab function polyfit \( z = \text{polyfit}(m, \text{cputime}(m), 3) \) to find a cubic fitting of the CPU time versus \( m \). Write down the polynomial and analyze your result. Does it look like a cubic function?

7. **Extra Credit:** Choose one from the following (Note: please do not ask the instructor about the solution since it is extra credit):

(a) Let \( A \in \mathbb{R}^{m \times n} \). Show that \( \|A\|_2 = \max_{1 \leq i \leq n} \sqrt{\lambda_i(A^T A)} \), where \( \lambda_i(A^T A), \, i = 1, 2, \ldots, n \) are the eigenvalues of \( A^T A \).

(b) Show that if \( A \) is a symmetric positive definite matrix, then after one step of Gaussian elimination (without pivoting), then reduced matrix \( A_1 \) in

\[
A \iff \begin{bmatrix}
a_{11} & * \\
0 & A_1
\end{bmatrix}
\]

must be symmetric positive definite. Therefore no pivoting is necessary.