TRANSONIC EVAPORATION WAVES
IN A SPHERICALLY SYMMETRIC NOZZLE

XIAOBIAO LIN AND MARTIN WECHSELBERGER

ABSTRACT. This paper studies the liquid to vapor phase transition in a cone shaped nozzle. Using the geometric method presented in [32, 34], we further develop results on subsonic and supersonic evaporation waves in [14] to transonic waves. It is known that transonic waves do not exist if restricted solely to the slow system on the slow manifolds [14]. Thus we consider the existence of transonic waves that include layer solutions of the fast system that cross or connect to the sonic surface. In particular, we are able to show the existence and uniqueness of evaporation waves that cross from supersonic to subsonic regions and evaporation waves that connect from the subsonic region to the sonic surface and then continue onto the supersonic branch via the slow flow.

1. INTRODUCTION

In this paper, we investigate the liquid to vapor phase transition in a spherically symmetric cone-shaped nozzle. Subsonic evaporation processes, such as fuel injection into a combustion engine, show up in many important engineering problems and are studied in many research laboratories around the world. The ring formation observed in a shock tube experiment [33] is an example of a supersonic process, as argued by Fan in [11]. Our focus is on transonic evaporation waves. While we are not aware of any experiments where transonic evaporation waves have been observed, we expect that our theoretical results might be useful in some physical and engineering process when the speed of the fluid in a nozzle approaches sonic speed.

The study of nozzle flows was pioneered by Courant & Friedrichs [6] and T.-P. Liu [26]. Transonic flow without liquid-gas phase transition has been studied by many authors, see recent articles [3, 4, 5, 23, 35, 36]. Among them [35] dealt with subsonic and subsonic to sonic flows through infinitely long nozzles. For transonic flows modeled by reaction-diffusion equations, W. Liu et al. [17, 18, 27] considered one dimensional standing waves for a simplified model of gas flows in a nozzle with general variable cross sectional area $a(x)$. To the best of our knowledge, transonic flows that include a reaction-diffusion equation describing evaporation or condensation have not been rigorously analysed mathematically.

Liquid-vapor phase transitions have been much studied using a van der Waals pressure function [15]. The van der Waals model requires detailed modeling of the evaporation of individual droplets [8, 30] or of the nucleation process by which vapor condenses. Figure 1(a) shows the graph of pressure $p$ as a function of specific volume $v$ (the inverse of density $\rho$) with the temperature held constant. The decreasing branch of the pressure function $p$ for $v < \alpha$ corresponds to the liquid state while the other decreasing branch for $v \geq \beta$ corresponds to the vapor state of the fluid. There is a ‘spinodal region’, $\alpha < v < \beta$, in which pressure...
Figure 1. Figure 1(a) shows the van der Waals pressure function $p(v)$ where the Maxwell line $p = p_{eq}$ connects $v = m$ to $v = M$ on $p = p(v)$. Figure 1(b) shows the functions $p(\lambda, v), \lambda = 0, 1$, where the Maxwell line $p = p_{eq}$ connects $v = m$ on $\lambda = 0$ to $v = M$ on $\lambda = 1$.

is, anomalously, an increasing function of $v$, i.e., a decreasing function of density $\rho$. This region is considered to be highly unstable and cannot be observed in experiments. In reality, this part of the pressure function $p$ is replaced by the Maxwell line $p = p_{eq}$ which is defined by the ‘equal area rule’; see Figure 1(a). The fluid can be in any liquid/vapor state for the equilibrium pressure $p_{eq}$, see [9].

An alternative model proposed by Fan [9] introduces a variable $0 \leq \lambda \leq 1$ to represent the mass fraction of vapor, where $\lambda = 0$ represents the pure liquid state, and $\lambda = 1$ represents the pure vapor state. Since Fan’s model allows intermediate values of $\lambda$, it can be used to study experiments in which mixtures of the liquid and vapor states occur [11]. In Fan’s model, pressure $p$ is a function of $(\lambda, v)$. The graphs of $p(0, v)$ and $p(1, v)$ are shown in Figure 1(b). This model was motivated by, among other works, [33, 20, 31].

An important feature of Fan’s model [9, 10, 11] is a reaction-diffusion equation that describes the phase transition between liquid and vapor. This model was proposed under the assumption that the heat capacity of the fluid under consideration is high (called a retrograde fluid), resulting in evaporation that is mainly caused by the pressure change, not the temperature change. Fan’s model consists of a viscose $p$-system that describes the motion of compressible liquid-vapor mixture and a reaction-diffusion equation that describes the liquid to vapor phase transition:

$$
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
(\rho u)_t + \nabla \cdot (\rho uu + p(\lambda, \rho)I) &= \eta_1 \nabla \cdot (\nabla u + \nabla u^T) + \eta_2 \nabla \cdot ((\nabla \cdot u)I), \\
(\lambda \rho)_t + \nabla \cdot (\lambda \rho u) &= \frac{w(\lambda, \rho)}{\gamma} + \mu \nabla \cdot (\rho \nabla \lambda),
\end{align*}
$$

where $\rho > 0$ is the density of the fluid, $u \in \mathbb{R}^3$ the velocity vector of the fluid, $\lambda \in [0, 1]$ is the mass fraction of vapor. All the constants in (1.1) are small parameters. Among them $\eta_1$ is the shear viscosity of the fluid, $\eta_2$ is a linear combination of the shear and the volume viscosity coefficients that is related to dilation of the fluid, and $\mu$ is the diffusion coefficient.
Assumption 1. The pressure $p(\lambda, \rho)$ is a function of the density $\rho$ and the mass fraction of vapor $\lambda$ which satisfies

\begin{align}
  p_\rho > 0, \quad p_\rho > 0, \quad p_\lambda > 0, \quad p(\lambda, 0) = 0, \quad p(\lambda, \infty) = \infty.
\end{align}

A typical pressure function which satisfies all the conditions in (1.2) is given by

\begin{align}
  p(\lambda, \rho) = C(1 + \lambda)\rho^k, \quad C > 0, \quad k > 1.
\end{align}

The last equation of system (1.1) comes from a simplified model proposed by Fan [9] where the vapor initiation term has been omitted and only the vapor growth term $w(\lambda, \rho)$ is included. The function $w$ is defined as

\begin{align}
  w(\lambda, \rho) := (p(\lambda, \rho) - p_{eq})\lambda(\lambda - 1)\rho,
\end{align}

and $w/\gamma$ represents the vapor production rate where $\gamma$ is the typical reaction time. If $p = p_{eq}$ then $w = 0$ and the mixture could be in any liquid/vapor configuration state $0 \leq \lambda \leq 1$.

Necessary and sufficient conditions for the existence of phase-changing traveling waves (liquid to vapor or vapor to liquid) in one-dimensional space were proved in [9, 10, 12]. Using dynamical systems methods, the proof of the existence of those one-dimensional waves was simplified in [13].

Let $r$ be the radial coordinate of the nozzle. We assume that the fluid is injected at the smaller end $r = r_0$ and discharged at the larger end $r = r_1$ (see, Figure 2). Assuming that the cone’s boundary is slippery and offering no resistance to tangential flows at the boundary, the boundary effect is therefore negligible. We consider the spherically symmetric solutions
which are functions of \((r, t)\) only and satisfy the following system:

\[
\begin{align*}
\rho_t + (\rho u)_r + \frac{2\rho u}{r} &= 0, \\
(\rho u)_t + (\rho u^2 + p)_r + \frac{2\rho u^2}{r} &= \epsilon (u_r + \frac{2u}{r}), \\
(\lambda \rho u)_t + (\lambda \rho u)_r + \frac{2\lambda \rho u}{r} &= \frac{1}{\gamma} (p - p_{eq}) \lambda (\lambda - 1) \rho + \mu ((\rho \lambda)_r + \frac{2\rho \lambda}{r}),
\end{align*}
\]

where \(\epsilon\) is the combined viscosity. A straightforward but rather lengthy calculation shows that \(\epsilon = \eta_1 + \eta_2\).

Fan and Lin in [14] studied the existence of non-transonic evaporation waves that are in either subsonic or supersonic regions for the entire evaporation process. In this paper we shall extend their results to evaporation waves that cross the sonic surface. That is, at \(r = r_0\) the liquid state fluid is subsonic (or supersonic) while at \(r = r_1\) the vapor state fluid becomes supersonic (or subsonic). In the sequel, we call such waves transonic evaporation waves. We look for standing waves that are stationary solutions of (1.5). The evaporation front of a stationary wave remains in the finite nozzle, making the process more useful. Assume that the fluid particles move from the smaller end to the larger end of the nozzle with the speed \(u > 0\). If, in the Eulerian coordinates, the speed of the wave is zero, then in the Lagrangian coordinates, the wave travels with the speed \(-u\), i.e., pointing towards the smaller end of the nozzle. Since the speed of sound is \(\sqrt{\frac{p}{\rho}}\) (see, e.g., [1]), we call the standing wave with \(u^2 < p_\rho\) (or \(u^2 = p_\rho\) or \(u^2 > p_\rho\)) the subsonic (or sonic, or supersonic) wave, which really means that the wave speed in the Lagrangian coordinates is subsonic (or sonic, or supersonic). The region of the phase space where \(u^2 < p_\rho\) (or \(u^2 = p_\rho\) or \(u^2 > p_\rho\)) will be called the subsonic (or sonic, or supersonic) region. The sonic region is a codimension one surface in the phase space.

The spherically symmetric standing waves are stationary solutions of (1.5). Based on the laws of fluid dynamics, the typical reaction time \(\gamma\), the diffusion coefficient of vapor \(\mu\) and the viscosity \(\epsilon\) are small parameters, and are proportional to the mean free path. Using this physical fact and to simplify the matter mathematically, we shall assume that

\[
\gamma = \epsilon/a, \quad \mu = \epsilon b, \quad a, b = O(1).
\]

Then the stationary solution in spherical coordinates satisfies

\[
\begin{align*}
(\rho u)_r + \frac{2\rho u}{r} &= 0, \\
(\rho u^2 + p)_r + \frac{2\rho u^2}{r} &= \epsilon (u_r + \frac{2u}{r}), \\
(\lambda \rho u)_r + \frac{2\lambda \rho u}{r} &= \frac{a}{\epsilon} (p - p_{eq}) \lambda (\lambda - 1) \rho + \epsilon b ((\rho \lambda)_r + \frac{2\rho \lambda}{r}).
\end{align*}
\]

In this paper we only consider an evaporation process where \(p < p_{eq}\) is a necessary condition for the vapor growth function \(w\) to be positive. We state this as an assumption for future reference:

**Assumption 2.** The fluid described by system (1.6) satisfies the condition \(p < p_{eq}\).

In the rest of the paper, we analyze this system with the emphasis on transonic evaporation waves. In particular, we look for standing waves with an internal layer at \(r = r_i, r_0 < r_i < r_1,\)
where the transition from the liquid state to the vapor state of the fluid happens (see, Figure 2). By formulating appropriate boundary conditions at the injection and discharge sites of the nozzle at \( r = r_0 \) and \( r = r_1 \), we prove the existence of unique super to subsonic as well as sub to supersonic evaporation waves. The precise statements are given in Theorem 5.1 and Theorem 5.2 for the super to subsonic case and in Theorem 5.3 for the sub to supersonic case.

In the following, we outline the structure of the paper that leads to the precise formulations and the proofs of these theorems. A key in our analysis is to identify system (1.6) as a singularly perturbed system. In §2, we introduce appropriate slow and fast variables so that system (1.6) can be converted into a slow-fast dynamical system and studied by means of geometric singular perturbation theory [14, 16, 19, 34].

In §3, we focus on the (internal) layer problem for the existence of evaporation layer solutions. In general, there may exist two kinds of evaporation transonic layer solutions connecting the pure liquid state \( \lambda = 0 \) to the pure vaporized state \( \lambda = 1 \): a supersonic to subsonic layer or a subsonic to supersonic layer. Each of these internal layers is a heteroclinic solution that crosses the sonic surface in fast radial ‘time’. We prove that supersonic to subsonic internal layer solutions at \( r = r_i \), \( r_0 < r_i < r_1 \) do exist but subsonic to supersonic layer solutions do not. However, we are still able to show the existence of internal subsonic (or supersonic) to sonic layer solutions.

In §4, we study the reduced outer problem on the critical manifold. We show that the flow on the critical manifold points away from the sonic surface. Thus transonic evaporation waves cannot be constructed solely by solutions on the slow manifold but must consist of internal layers. As a common practice in singular perturbation problems, we define in §5 singular evaporation waves \( \Gamma = \gamma_0 \cup \gamma_i \cup \gamma_1 \) which are concatenations of slow (outer) solutions \( \gamma_0, \gamma_1 \) and an internal layer solution \( \gamma_i \) (see Figure 3). As highlighted above, we are able to prove the existence and uniqueness of transonic evaporation waves \( \Gamma(\varepsilon) \). In the supersonic to subsonic case, \( \Gamma(\varepsilon) \) is \( O(\varepsilon) \) close to \( \Gamma \) while in the supersonic to subsonic case, \( \Gamma(\varepsilon) \) is \( O(\varepsilon^{2/3}) \) close to \( \Gamma \). The proof of the supersonic to subsonic case relies on a generalization of the Exchange Lemma that has been recently proven in [25]. The fractional power of the order of \( \varepsilon \) in the subsonic to supersonic case points to a turning point problem that stems from the sonic connection of the internal layer \( \gamma_i \). To prove this result we use a geometric desingularization known as the blow-up technique (see, [7, 21, 32]). This technique reveals an intermediate region near the sonic surface which serves as the link between the fast layer solution to the reduced flow on the slow manifold for \( \varepsilon \neq 0 \) (see also [2, 34]). In §6, we
outline the proof that is based on the blow-up technique (see [32] for details) and describe the flow near the sonic surface. We provide some final remarks in §7.

2. A GEOMETRIC SINGULAR PERTURBATION APPROACH

Since \( \varepsilon \) is a small parameter, we will use singular perturbation techniques [14, 16, 19, 34] to find standing waves for system (1.6). To convert the system into a fast-slow form, we introduce the new variables

\[
\begin{align*}
m &:= \rho u, \quad \theta := \varepsilon b \rho \lambda y, \\
n &:= -mu - p + \epsilon(u_y + 2u/r), \quad r = y.
\end{align*}
\]

The dummy independent variable \( y \geq 0 \) allows us to rewrite (1.6) as an autonomous system. The change of variables \( (\rho, u, \dot{u}, \lambda, \dot{\lambda}) \rightarrow (m, n, u, \lambda, \theta) \) is non-singular when \( u > 0 \), which is the case considered in this paper. This leads to the following slow system on the slow scale \( y \):

\[
\begin{align*}
r_y &= 1, \\
m_y &= -\frac{2m}{r}, \\
n_y &= \frac{2mu}{r}, \\
\epsilon u_y &= n + mu + p(\lambda, \frac{m}{u}) - \epsilon \frac{2u}{r}, \\
\epsilon \lambda_y &= \frac{\theta u}{b m}, \\
\epsilon \theta_y &= \frac{\theta u}{b} - aw(\lambda, \frac{m}{u}) - \epsilon \frac{2\theta}{r}.
\end{align*}
\]

An equivalent system on the fast scale \( z = y/\epsilon \) is given by the fast system

\[
\begin{align*}
r_z &= \epsilon, \\
m_z &= -\epsilon \frac{2m}{r}, \\
n_z &= \epsilon \frac{2mu}{r}, \\
u_z &= n + mu + p(\lambda, \frac{m}{u}) - \epsilon \frac{2u}{r}, \\
\lambda_z &= \frac{\theta u}{b m}, \\
\theta_z &= \frac{\theta u}{b} - aw(\lambda, \frac{m}{u}) - \epsilon \frac{2\theta}{r}.
\end{align*}
\]
The singular nature of these systems is revealed by taking the limit \( \epsilon \to 0 \). For the fast system (2.3) this yields the layer problem

\[
\begin{align*}
   r_z &= m_z = n_z = 0, \\
   u_z &= n + mu + p(\lambda, \frac{m}{u}), \\
   \lambda_z &= \frac{\theta u}{bm}, \\
   \theta_z &= \frac{\theta u}{b} - aw(\lambda, \frac{m}{u}),
\end{align*}
\]

(2.4)

for the evolution of the fast variables \((u, \lambda, \theta)\). Note that the slow variables \((r, m, n)\) are parameters in the layer problem. On the other hand, for the slow system (2.2) the result of taking the limit \( \epsilon \to 0 \) is the reduced problem

\[
\begin{align*}
   r_y &= 1, \\
   m_y &= \frac{2m}{r}, \\
   n_y &= \frac{2mu}{r}, \\
   0 &= n + mu + p(\lambda, \frac{m}{u}), \\
   0 &= \frac{\theta u}{bm}, \\
   0 &= \frac{\theta u}{b} - aw(\lambda, \frac{m}{u}),
\end{align*}
\]

(2.5)

which is a differential-algebraic system for the evolution of the slow variables \((r, m, n)\). The phase space of the reduced problem is defined by the algebraic constraints

\[
\begin{align*}
   0 &= n + mu + p(\lambda, \frac{m}{u}), \\
   0 &= \frac{\theta u}{bm}, \\
   0 &= \frac{\theta u}{b} - aw(\lambda, \frac{m}{u}),
\end{align*}
\]

(2.6)

or equivalently

\[
\begin{align*}
   n &= -mu - p(\lambda, \frac{m}{u}), \\
   \theta &= 0, \\
   w(\lambda, \frac{m}{u}) &= 0.
\end{align*}
\]

(2.7)

Assuming \( \rho > 0 \), it follows that \( w = 0 \) has three possible solutions: \( \lambda = 0 \), \( \lambda = 1 \) and \( p = p_{eq} \) which correspond to the three branches of the three-dimensional critical manifold (or slow manifold in some literature) \( S = S_0 \cup S_1 \cup S_e \). These branches can be expressed as functions of the \((r, m, u)\) variables:

\[
S_{0,1,e} := \{(r, m, n, u, \lambda, \theta) \in \mathbb{R}^6 : \lambda = \Lambda(m, u), \theta = 0, n = N(m, u)\}.
\]

(2.8)
Under Assumption 2, the surface $S_e$ will not be of interest to us. The functions $\Lambda(m, u)$ and $N(m, u)$ for $S_0$ and $S_1$ are given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>$\Lambda(m, u)$</th>
<th>$N(m, u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>0</td>
<td>$-mu - p(0, m/u)$</td>
</tr>
<tr>
<td>$S_1$</td>
<td>1</td>
<td>$-mu - p(1, m/u)$</td>
</tr>
</tbody>
</table>

An example of the graphs of $S_0$ and $S_1$ is shown in Figure 4 where we chose $p = (1 + \lambda)p^2$ as a representative pressure function in (1.3). For fixed $(\lambda, m)$, since the function $p(\lambda, m/u)$ is concave upward with $p(\lambda, 0) = 0, p(\lambda, \infty) = \infty$, the graph of $N(m, u)$ is concave downward, has a vertical asymptote as $u \to 0$, and approaches $n = -mu$ as $u \to \infty$. The maximum value of $N(m, u)$ is reached on the codimension-one sonic surface where $u^2 = p_\rho$.

**Definition 1.** For a fixed $m$, let $\bar{n}$ be the maximum value of $N(m, u)$ on $S_1$.

For a given pair of $m$ and $r$, the number of equilibrium points of the layer problem on $S_0$ or $S_1$ is a function of $n$ only. Since equilibria do not depend on $r$, we will often not mention $r$ in the future. There is no equilibrium point on $S_1$ for $n > \bar{n}$. Therefore we shall only consider $n \leq \bar{n}$. Depending on $n = \bar{n}$ or $n < \bar{n}$, the number of equilibrium points on $S_0 \cup S_1$ can be three or four as illustrated in Figure 4.

From the layer problem (2.4) we find that

$$u_z = n + mu + p(\lambda, \frac{m}{u}) = n - N(m, u).$$

By flipping the graph of $N(m, u)$ to $-N(m, u)$ then shifting by the constant $n$, we can easily determine the sign of $u_z$ for any $(\lambda, u)$; see Figure 5. This will be very useful when constructing the layer solutions.

### 2.1 Stability analysis of the layer problem.

We study the eigenvalues and eigenfunctions for the equilibrium points of the the layer problem (2.4) on $S_0 \cup S_1$. The Jacobian of (2.4) is given by

$$J = \begin{pmatrix} m + p & p & 0 \\ \frac{p}{u} & 0 & \frac{p}{u} \\ 0 & \frac{p}{u} & \frac{p}{u} \end{pmatrix}.$$

The derivatives of $p$ and $w$ with respect to $\lambda$ and $u$ are given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>$p_\lambda$</th>
<th>$p_w$</th>
<th>$w_\lambda$</th>
<th>$w_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>$-\frac{m}{u^2}p_\rho &lt; 0$</td>
<td>$&gt; 0$</td>
<td>$0$</td>
<td>$-\frac{m}{u}(p - p_{eq})$</td>
</tr>
<tr>
<td>$S_1$</td>
<td>$-\frac{m}{u^2}p_\rho &lt; 0$</td>
<td>$&gt; 0$</td>
<td>$0$</td>
<td>$+\frac{m}{u}(p - p_{eq})$</td>
</tr>
</tbody>
</table>

On the branches $S_0$ and $S_1$, the Jacobian reduces to

$$J = \begin{pmatrix} \frac{m}{u^2}(u^2 - p_\rho) & p & 0 \\ 0 & 0 & \frac{p}{u} \\ 0 & \pm a\frac{m}{u}(p_{eq} - p) & \frac{p}{u} \end{pmatrix},$$

where the minus sign is associated to $S_0$ and the plus sign to $S_1$. The eigenvalues $\{l_1, l_2, l_3\}$ of the matrix $J$ satisfy the characteristic equation

$$\left(\frac{m}{u^2}(u^2 - p_\rho) - l\right) \cdot \left(l(l - \frac{u}{b}) \pm \frac{a}{b}(p - p_{eq})\right) = 0,$$
where the minus sign belongs to $S_1$ and the plus sign to $S_0$. On both branches $S_0, S_1$, one eigenvalue is given by $l_1 = \frac{m}{n^2}(u^2 - p_\rho)$. On the branch $S_1$ and under Assumption 2, $p < p_{eq}$, the other two eigenvalues are real and satisfy $l_2 < 0 < l_3$. On the other branch $S_0$, if $p < p_{eq}$ and $u^2 + 4ab(p - p_{eq}) > 0$ then the other two eigenvalues are real and satisfy $0 < l_2 < l_3$. On the other hand, if $p < p_{eq}$ and $u^2 + 4ab(p - p_{eq}) < 0$, the eigenvalues $l_2, l_3$ are complex conjugates. A direct calculation of the corresponding eigenspace of this complex conjugate pair of eigenvalues $\{l_2, l_3\}$ implies that layer solutions will enter the physically unrealistic region $\lambda < 0$ as $z \to -\infty$, due to the oscillations of these layer solutions near $S_0$ caused by the complex eigenvalues.
Figure 5. The two concave-up curves correspond to \( n = -N(m, u) \) with \( \lambda = 0, 1 \) respectively. By choosing \( \lambda \in [0, 1] \) and \( u \), we can find \( u_z \) from the figure. For example, given \( n < \bar{n} \), we have \( u_z < 0 \) if \( u_2 < u < u_3 \) and \( u_z > 0 \) if \( u < u_1 \) or \( u > u_4 \).

The evaporation waves discussed in this paper occur in a bounded subset \( \Omega \) of the phase space, defined as

\[
\Omega := \{(r, m, n, u, \lambda, \theta) : \bar{u} \leq u \leq \bar{u}, \bar{m} \leq m \leq \bar{m}, r_0 \leq r \leq r_1\}.
\]

The set \( \Omega \) and the positive constants \( \bar{u}, \bar{u}, \bar{m} \) and \( \bar{m} \) depend on the types of evaporation waves we try to construct and are not difficult to determine in each case. Throughout this paper we assume that the variables \( (u, \lambda, m) \) satisfy the following condition:

**Assumption 3.** \( u^2 + 4ab(p(\lambda, m/u) - p_{eq}) > 0 \) for \( \bar{u} \leq u \leq \bar{u}, \bar{m} \leq m \leq \bar{m} \) and \( \lambda = 0 \).

Although this assumption is only given for \( \lambda = 0 \), it is easy to show the following result:

**Lemma 2.1.** If \( (u, \lambda = 0, m) \) satisfy Assumption 3, then \( u^2 + 4ab(p(\lambda, m/u) - p_{eq}) > 0 \) for \( 0 \leq \lambda \leq 1 \).

**Proof.** Since \( p_\lambda > 0 \), then we have \( p(\lambda, m/u) > p(0, m/u) \) for \( \lambda > 0 \). \( \square \)

Notice that if \( \bar{u} > 0 \) is given, Assumption 3 holds if \( a \) and \( b \) are small enough such that \( ab < \bar{u}^2/(4p_{eq}) \).

Under Assumption 3, the eigenvalues of the Jacobian (2.11) are real. The corresponding eigenvectors to the eigenvalues \( \{l_1, l_2, l_3\} \) are given by

\[
v_1 = (1, 0, 0) \quad v_j = \left( \frac{p_\lambda}{l_j - l_1}, 1, \frac{bn}{u} l_j \right), j = 2, 3,
\]

where we assume that \( l_j \neq l_1, j = 2, 3 \) for convenience. We remark that if \( l_1 = l_j \) for \( j = 2 \) or \( 3 \), then the expression for \( v_j \) can be complicated. However, most of the results of this paper do not depend on this expression and are still valid.

An important observation is that the eigenvalue \( l_1 \) vanishes if \( u^2 = p_\rho \), i.e. along the sonic surface of \( S_0 \) respectively \( S_1 \). Geometrically, the branches \( S_0 \) and \( S_1 \) are three-dimensional folded manifolds within the six-dimensional phase space where the sonic surface corresponds to the fold where the layer flow is tangent to the branches \( S_0 \) and \( S_1 \) along the eigendirection
spanned by the nullvector $v_1$. Consequently, the layer problem projected onto the one-dimensional nullspace undergoes a saddle-node bifurcation along the sonic surface as can be seen in Figure 4.

Under the Assumptions 2 and 3, the signs of the eigenvalues for the layer problem on $S_0$ and $S_1$ can be summarized in the following table:

<table>
<thead>
<tr>
<th></th>
<th>$S_0$</th>
<th>$S_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>subsonic, $u^2 &lt; p\rho$</td>
<td>$-$, $+$, $+$</td>
<td>$-$, $-$, $+$</td>
</tr>
<tr>
<td>sonic, $u^2 = p\rho$</td>
<td>$0$, $+$, $+$</td>
<td>$0$, $-$, $+$</td>
</tr>
<tr>
<td>supersonic, $u^2 &gt; p\rho$</td>
<td>$+$, $+$, $+$</td>
<td>$+$, $-$, $+$</td>
</tr>
</tbody>
</table>

Since the critical manifolds consist of equilibrium points of the layer problem, the fiber dimensions of the associated unstable and stable manifolds of $S_0$ and $S_1$ on the subsonic and supersonic regions, and the center(-stable) manifolds of $S_0$ and $S_1$ on the sonic surface can be determined by the signs of eigenvalues of the layer problem and are listed in the following table:

<table>
<thead>
<tr>
<th></th>
<th>$\dim W^u(S_0)$</th>
<th>$\dim W^s(S_0)$</th>
<th>$\dim W^a(S_1)$</th>
<th>$\dim W^s(S_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>subsonic</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>supersonic</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>sonic</td>
<td>$\dim W^c(S_0) = 1$</td>
<td>$\dim W^{cs}(S_1) = 2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Existence of evaporation layer solutions

Evaporation layer solutions are heteroclinic connections from $S_0$ to $S_1$. Assuming that Assumptions 2 and 3 are satisfied, then the types of connections are listed in the following table, where we refer to the fiber dimensions of $S_0$ and $S_1$:

<table>
<thead>
<tr>
<th></th>
<th>$\dim W^u(S_0) \rightarrow \dim W^{(c)s}(S_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>subsonic</td>
<td>$2 \rightarrow 2$</td>
</tr>
<tr>
<td>supersonic</td>
<td>$3 \rightarrow 1$</td>
</tr>
<tr>
<td>transonic (sub to super)</td>
<td>$2 \rightarrow 1$</td>
</tr>
<tr>
<td>transonic (super to sub)</td>
<td>$3 \rightarrow 2$</td>
</tr>
<tr>
<td>sub (or supersonic) to sonic</td>
<td>$2 \rightarrow 2$ (or $3 \rightarrow 2$)</td>
</tr>
</tbody>
</table>

The existence of non-transonic evaporation waves, i.e. subsonic to subsonic and supersonic to supersonic waves, has been proved in [14], where the transverse intersection of $W^u(S_0)$ and $W^s(S_1)$ for subsonic waves was checked numerically. Using the method presented in the proof of Lemma 3.3, it can also be proved rigorously.

**Definition 2.** For any $0 < \alpha < \beta$, a corresponding rectangle $R(\alpha, \beta)$ can be defined in the $(u, \lambda)$ plane:

$$R(\alpha, \beta) := \{(u, \lambda) | 0 \leq \lambda \leq 1, \ \alpha \leq u \leq \beta\}.$$
A pentahedron shaped solid $W$ (Figure 6) in $(u, \lambda, \theta)$ space is said to be based on the rectangle $R(\alpha, \beta)$ if the five surfaces of $W$ are as follows:

\[
\begin{align*}
F_b &= \{(u, \lambda, \theta) : \alpha < u < \beta, 0 < \lambda < 1, \theta = 0\} \\
F_r &= \{(u, \lambda, \theta) : \alpha < u < \beta, \lambda = 1, 0 < \theta < \theta_m/2\} \\
F_s &= \{(u, \lambda, \theta) : \alpha < u < \beta, 0 < \lambda < 1, \theta = \lambda_m/2\} \\
F_k &= \{(u, \lambda, \theta) : u = \beta, 0 < \lambda < 1, 0 \leq \theta \leq \lambda_m/2\} \\
F_f &= \{(u, \lambda, \theta) : u = \alpha, 0 < \lambda < 1, 0 < \theta < \theta_m/2\}
\end{align*}
\]  
(3.1)

**Lemma 3.1.** For a given $m$, assume that $(u, \lambda, m)$ satisfy Assumption 2 for $\alpha \leq u \leq \beta$. If $W$ is a pentahedron shaped solid based on the rectangle $R(\alpha, \beta)$, then at points on the planes $F_b, F_r$ and $F_s$, the flow of the layer problem (2.4) leaves $W$.

Moreover, if $(\alpha, 0, 0)$ and $(\beta, 0, 0)$ are two equilibrium points for the layer problem on $S_0$, then at points on the plane $F_f$, the flow of the layer problem (2.4) enters $W$ while at points on the plane $F_k$, the flow of the layer problem (2.4) leaves $W$.

**Proof.** We show that the layer flow leaves $W$ through the interior of $F_x$, $x = b, r, s$, i.e. $g \cdot n_x > 0$ evaluated in the interior of $F_x$ where $g$ denotes the vector field of the layer problem (2.4) and $n_x$ denotes the corresponding outer normal vector of $F_x$ given by

\[
\begin{align*}
   n_b &= (0, 0, -1), \quad n_r = (0, 1, 0), \quad n_s = (0, -m/2, 1).
\end{align*}
\]  
(3.2)

We have the following results for $p < p_{eq}$:

\[
\begin{align*}
   (n_b \cdot g)|_{F_b} &= aw(\lambda, \frac{m}{u}) > 0, \quad \alpha < u < \beta, 0 < \lambda < 1 \\
   (n_r \cdot g)|_{F_r} &= \frac{\theta u}{bm} > 0, \quad \alpha < u < \beta, \theta > 0 \\
   (n_s \cdot g)|_{F_s} &= \frac{\lambda m u}{4b} - a(p - p_{eq})\lambda(\lambda - 1)\frac{m}{u} = \frac{\lambda m}{4bu} (u^2 - 4ab(p - p_{eq})(\lambda - 1)) \\
   &\geq \frac{\lambda m}{4bu} (u^2 + 4ab(p - p_{eq})) > 0, \quad \alpha < u < \beta, 0 < \lambda < 1
\end{align*}
\]

which follows from Assumption 3 and Lemma 2.1.
The outer normal vector of $F_k$ and $F_f$ are $n_k = (1, 0, 0)$, $n_f = (-1, 0, 0)$ respectively. We have $n + mu + p = 0$ at the equilibria $(\alpha, 0, 0)$ and $(\beta, 0, 0)$. Due to the fact that $p_\lambda > 0$ and $\lambda > 0$, we have

$$(n_f \cdot g)|_{F_f} = (-n - mu - p)|_{F_f} < 0, \quad \theta > 0, 0 < \lambda < 1$$

$$(n_k \cdot g)|_{F_k} = (n + mu + p)|_{F_k} > 0, \quad \theta > 0, 0 < \lambda < 1.$$ 

This proves the last assertion of the lemma. \qed

We should always assume that $n \leq \bar{n}$ (otherwise, evaporation layers are not possible). If $n < \bar{n}$, then there are two equilibrium points $E_1^+, E_2^- \in S_0$ and two equilibrium points $E_1^+, E_2^+ \in S_1$. Also $E_1^+, E_2^+$ are on the subsonic and $E_2^-, E_2^+$ are on the supersonic branches respectively. If $n = \bar{n}$, there are two equilibrium points $E_1^+, E_2^+ \in S_0$ and a unique equilibrium point $E_+ \in S_1$. Also $E_1^-, E_+^+, E_2^-$ are on the subsonic, sonic and supersonic branches respectively; see Figures 7 and 8. The signs of $u_z$ can easily be determined from these figures as well, as described in Section 2. Let us define the $u$-nullsurface in the $(u, \lambda, \theta)$-space along which $u_z = 0$ in (2.9). Since $p_\lambda > 0$ it follows that $u_z = 0$ can be solved for $\lambda = \Lambda^c(u)$. Furthermore, we know that

$$\frac{d\Lambda^c}{du} = \begin{cases} m (p_\rho - u^2) / p_\lambda & > 0 \text{ if } u^2 < p_\rho, \\ 0 & = 0 \text{ if } u^2 = p_\rho, \\ < 0 & < 0 \text{ if } u^2 > p_\rho. \end{cases}$$

Note that the right hand side of (2.9) is independent of $\theta$. Hence, we can define the projection of a $u$-nullsurface along the $\theta$-direction, onto $(u, \lambda)$-space and call it a $u$-nullcline. For simplicity we sometimes call such $u$-nullsurface a $u$-nullcline too. In Figures 7(b) and 8(b), curves in $(u, \lambda)$-space connecting equilibrium points diagonally inside the rectangle boxes are segments of the $u$-nullcline.

If $n < \bar{n}$, then there are two trivial transonic heteroclinic orbits from supersonic to subsonic regions: One satisfies $\lambda = 0, \theta = 0$ and connects $E_2^-$ to $E_1^+$ on $S_0$; the other satisfies $\lambda = 1, \theta = 0$ and connects $E_2^+$ to $E_1^+$ on $S_1$. Following the terminology of Haitao Fan [11], we will call them ‘wet shock’ and ‘dry shock’. With $(\lambda, \theta)$ given, a wet or dry shock is determined by a scalar ODE for $u(z)$ and is easy to find. From Figures 7 and 8, it is clear that $u_z < 0$ for all $z \in \mathbb{R}$ for both wet shocks or dry shocks. If $n = \bar{n}$, there is only one trivial transonic heteroclinic orbit connecting $E_2^+ to E_1^+$ on $S_0$, which is a wet shock. All those trivial transonic layers are monotone decreasing in $u$ and go from supersonic to subsonic regions.

3.1. Existence of supersonic to subsonic layer solutions. For $n < \bar{n}$, there are four equilibrium points $\{E_1^-, E_1^+, E_2^-, E_2^+\}$ on $S_0 \cup S_1$ marked by their $u$ coordinates $u_1^- < u_1^+ < u_2^+ < u_2^-$ in Figure 7.

**Theorem 3.1.** Assume $n < \bar{n}$ and the Assumptions 2 and 3 are satisfied with $\bar{u} < u_1^- < u_2^- < \bar{u}$. Then there exists a one-parameter family of supersonic to subsonic heteroclinic solutions to the layer problem (2.4) connecting $E_2^-$ to $E_1^+$ (see Figure 7). This one-parameter family of $E_2^- \to E_1^+$ heteroclinic solutions are bounded by two pairs of heteroclinic solutions. At one end the boundary is a pair of heteroclinic orbits connecting $E_2^-$ to $E_1^+$ along the line $\lambda = 0$ (wet shock) and then connecting $E_1^+$ to $E_1^+$ (subsonic layer). At the other end the boundary is a pair of heteroclinic orbits connecting $E_2^-$ to $E_2^+$ (supersonic layer) and then connecting $E_2^+ to E_1^+$ along the line $\lambda = 1$ (dry shock).
Proof. Among the \( u \)-coordinates of the four equilibrium points, \( u_1^- \) is the minimum and \( u_2^- \) is the maximum. Based on the two-dimensional rectangle \( R(u_1^-, u_2^-) \), one can construct a pentahedron shaped solid \( W \) in \((u, \lambda, \theta)\)-space as in Definition 2. Then by Lemma 3.1, for any point \( P \in W \), the forward flow through \( P \) leaves \( W \) through the surfaces \( F_b, F_r, F_s, F_k \). On the other hand the backward flow through \( P \) can leave \( W \) only through \( F_f \). It is also straightforward to show that the backward layer flow cannot leave \( W \) through the six edges of \( W \) that do not form the boundary of \( F_f \).

The projection of the \( u \)-nullsurface onto the \((u, \lambda)\)-plane, the \( u \)-nullcline, divides the base rectangle \( R(u_1^-, u_2^-) \) into three components: the most left one \( R_1 \) above the segment of the \( u \)-nullcline that connects \( E_1^- \) and \( E_1^+ \), the most right one \( R_3 \) above the segment of the \( u \)-nullcline that connects \( E_2^+ \) and \( E_2^- \) and the middle one \( R_2 \) between the two segments of the \( u \)-nullcline. Define

\[
V(R_j) := \{(u, \lambda, \theta) : (u, \lambda) \in R_j\}, \quad j = 1, 2, 3.
\]

From Figures 7 (a) and 7 (b), \( V(R_2) \) is backward invariant with respect to the layer problem.

We now prove the existence of a heteroclinic solution. Note that if a heteroclinic connection exists, it has to be solely in \( W \). The point \( E_2^- = (u_2^-, 0, 0) \) is on the supersonic branch of \( S_0 \) and is fully repelling. Thus in backward time \( E_2^- \) is attracting all the points in a neighborhood of \( E_2^- \). If the order of the corresponding eigenvalues is \( 0 < l_1 < l_2 < l_3 \) then the eigenvector \( v_1 \) points along one edge of \( W \) while the others point not into \( W \) (see the definitions of the
Proof. Eigenvectors in (2.13). On the other hand, if \( 0 < l_2 < l_1 \) then the eigenvector \( v_2 \) points into \( W \) as well. Hence the weakest eigendirection of \( E^2_+ \) always points into (or at least along) \( W \).

The point \( E^1_+ = (u_+, 1, 0) \) is on the subsonic branch of \( S_1 \). Thus \( W^s(E^1_+) \) is two dimensional and is spanned by the eigenvectors \( (v_1, v_2) \). The local stable manifold can be expressed as

\[
W^s(E^1_+) = \{(u, \lambda, \theta) | \theta = \theta^s(u, \lambda), \| (u, \lambda) \| < \delta_0 \}.
\]

Consider a semicircle of radius \( \delta \) in the half plane \( \lambda \leq 1 \) where \( \phi \) is the angle of \( v \) measured from the negative \( u \)-axis:

\[
(3.3) \quad v(\phi) = (u(\phi), \lambda(\phi)) = \delta(\cos \phi)(-1, 0) + \delta(\sin \phi)(0, -1), \quad 0 \leq \phi \leq \pi.
\]

Then \( \{P(\phi) = (u(\phi), \lambda(\phi), \theta(\phi)) | \theta(\phi) = \theta^s(u(\phi), \lambda(\phi))\}, \) \( 0 < \phi < \pi \} \) describes a smooth arc on the local stable manifold \( W^s(E^1_+) \). Its projection on the \((u, \lambda)\) plane intersects the \( u \)-nullcline at \( 0 < \phi = \phi_1 < \pi/2 \) and the line \( u = u^1_1 \) at \( \phi = \pi/2 \), where \( \phi_1 = \phi_2 \). Now let \( \phi_2 \) be the inifimum of the angles such that the backward orbit through \( P(\phi) \) will enter \( V(R_2) \). It is clear that \( 0 < \phi_2 < \phi_1 \). For \( \phi_2 < \phi < \pi \), the backward orbit through \( P(\phi) \) will enter the region \( V(R_2) \) which is backward invariant. Since \( d\lambda/dz > 0 \), it follows that \( \lambda(z) \rightarrow 0 \) as \( z \rightarrow -\infty \).

The invariant set on the surface \( \lambda = 0 \) consists only of \( E^1_+ \) and \( E^2_+ \). Hence, the backward orbit must approach \( E^2_+ \). The forward orbit of course must approach \( E^1_+ \) so we have a layer connection \( E^2_+ \rightarrow E^1_+ \).

If \( \phi = \phi_2 \), then the backward orbit should stay above the \( u \)-nullcline and connect to \( E^1_+ \). This is a subsonic to subsonic layer solution and has been discussed in [14] by using the principle of Wazewski. In other words, as \( \phi \rightarrow \phi_2 \), the limit of the one-parameter family of \( E^2_+ \rightarrow E^1_+ \) layers is the union of two heteroclinic orbits: \( E^2_+ \rightarrow E^1_+ \) (wet shock) and \( E^1_+ \rightarrow E^1_+ \).

Observe that not all of the super to subsonic layers are monotone in \( u \). The \( u \) component of some solutions can go below \( u_+ \) then return back to \( u_+ \).

On the other hand as \( \phi \rightarrow \pi \), the limit of the one-parameter family of \( E^2_+ \rightarrow E^1_+ \) layers is the unique pair of two heteroclinic orbits: \( E^2_+ \rightarrow E^2_+ \), which is a supersonic to supersonic connection as studied in [14], followed by the transonic layer solution \( E^2_+ \) to \( E^1_+ \) (dry shock).

\[\square\]

**Corollary 3.1.** Assumption 2 of Theorem 3.1 can be weakened to

\[\text{(H1) } p < p_{eq} \text{ for } (u, \lambda) = (u^1_-, 1)\]

**Proof.** Note that Assumption 2 still holds within the pentahedron \( W \) since \( p_\lambda > 0 \) and \( \partial_u p(\lambda, m/u) < 0 \) which follows from \( p_{\rho} > 0 \) by Assumption 1.

\[\square\]

Complementing to the results of Theorem 3.1, we can show that

**Theorem 3.2.** There does not exist any sub- to supersonic evaporation layer solutions connecting \( E^1_- \in S_0 \) to \( E^2_+ \in S_1 \).

**Proof.** This can be seen from the fact that \( u_z < 0 \) for \( u^1_+ < u < u^2_+ \).

An indirect proof of Theorem 3.2 is to use a result from [14] which shows that the branch of the one-dimensional stable manifold \( W^s(E^2_+) \) that stays in the pentahedron \( W \) will intersect with \( W^s(E^2_+) \) and form a supersonic to supersonic layer solution. Therefore \( W^s(E^2_+) \) will not intersect with \( W^u(E^1_-) \).
3.2. Existence of subsonic(supersonic) to sonic layer solutions. For \( n = \bar{n} \), there are three equilibrium points \( \{E_1^-, E_+, E_2^-\} \) on \( S_0 \cup S_1 \) marked by their \( u \) coordinates \( u_1^- < u_+ < u_2^- \). Among them \( E_1^- = (u_1^-, 0, 0) \in S_0 \) is subsonic, \( E_+ = (u_+, 1, 0) \in S_1 \) is sonic, and \( E_2^- = (u_2^-, 0, 0) \in S_0 \) is supersonic.

**Theorem 3.3.** Assume \( n = \bar{n} \) and the Assumptions 2 and 3 are satisfied with \( \bar{u} < u_1^- < u_2^- < \bar{\bar{u}} \). Then there exists a one-parameter family of non-monotone supersonic to sonic heteroclinic solutions to the layer problem (2.4) connecting \( E_2^- \) to \( E_+ \) (see Figure 8). Every solution in that family approaches \( E_+ \) by being tangent to the part of the \( u \)-axis where \( u < u_+ \).

There is also a unique subsonic to sonic layer \( E_1^- \to E_+ \). Together with the layer solution \( E_2^- \to E_1^- \) along the line \( \lambda = 0 \) (wet shock), this pair of heteroclinic solutions is a boundary of the one-parameter family of \( E_2^- \to E_+ \) layer solutions.

The other boundary of that one-parameter family of \( E_2^- \to E_+ \) layer solutions is a unique monotone supersonic to sonic layer solution \( E_2^- \to E_+ \) that approaches \( E_+ \) by being tangent to the part of \( W^*(E_+) \) where \( \lambda < 1 \).

**Proof.** Consider a pentahedron shaped solid \( W \) in \((u, \lambda, \theta)\)-space based on the rectangle \( R(u_1^-, u_2^-) \) as in Definition 2. Then by Lemma 3.1, for any point \( P \in W \), the forward flow through \( P \) leaves \( W \) through the surfaces \( F_b, F_r, F_s, F_h \). On the other hand the backward flow through \( P \) can leave \( W \) only through \( F_f \). It is also straightforward to show that the backward layer flow cannot leave \( W \) through the six edges of \( W \) that do not form the boundary of \( F_f \).
The \( u \)-nullcline divides the base rectangle \( R(u_1^1, u_2^1) \) into three components: the most left one \( \mathcal{R}_1 \) above the segment of the \( u \)-nullcline that connects \( E_+^1 \) and \( E_+ \), the middle one \( \mathcal{R}_2 \) below the \( u \)-nullcline, and the most right one \( \mathcal{R}_3 \) above the segment of the \( u \)-nullcline that connects \( E_+^2 \) and \( E_+ \). Note that \( V(\mathcal{R}_2) \) is backward invariant under the flow of the layer problem.

The equilibrium point \( E_+^2 = (u_2^2, 0, 0) \) on the supersonic branch of \( S_0 \) is an unstable node. The equilibrium point \( E_+^1 = (u_1^1, 0, 0) \) on the subsonic branch of \( S_0 \) has a two-dimensional unstable manifold \( W^u(E_+^1) \). The corresponding unstable eigenspace is spanned by the eigenvectors \( \{v_2, v_3\} \). Only the eigenvector \( v_2 \) corresponding to the weaker unstable eigenvalue points into \( W \) (we have \( l_1 < 0 < l_2 < l_3 \)); see the definitions of eigenvectors in (2.13). This already guarantees that part of the (local) manifold \( W^u(E_-) \) lies within the pentahedron \( W \).

The point \( E_+ = (u_+, 1, 0) \) on the sonic surface of \( S_1 \) has a two-dimensional center-stable manifold \( W^{cs}(E_+) \). The corresponding center-stable eigenspace is spanned by the eigenvectors \( \{v_1, v_2\} \) (we have \( l_2 < l_1 = 0 < l_3 \)); see the definitions of eigenvectors in (2.13). The eigenvector \( v_1 \) points along an edge of \( W \) while the eigenvector \( -v_2 \) points into \( W \). The signs of the components of \( -v_2 \) are \((+, -, +)\), therefore there is a branch of the stable manifold \( W^s(E_+) \) that enters the region \( V(\mathcal{R}_2) \) which is invariant with respect to the backward flow. The backward flow of that branch of \( W^s(E_+) \) will remain in \( V(\mathcal{R}_2) \) and its \( \alpha \)-limit set is the point \( E_+^2 \). This proves the existence of a unique layer connection \( E_+^2 \to E_+ \) that approaches \( E_+ \) exponentially fast and is monotone in \( u \).

The local center-stable manifold of \( W^{cs}(E_+) \) can be expressed as

\[
W^{cs}(E_+) = \{(u, \lambda, \theta) \mid \theta = \theta^s(u, \lambda), \| (u, \lambda) \| < \delta_0 \}.
\]

Note that \( W^c(E_+) \) is semistable: it is stable from the side \( u < u_+ \) and is unstable from the side \( u > u_+ \). The center-stable manifold \( W^{cs}(E_+) \) is divided by \( W^s(E_+) \) into two components, call it the stable and unstable parts of \( W^{cs}(E_+) \). An orbit on \( W^{cs}(E_+) \) must follow the stable fibers of the foliation of \( W^{cs}(E_+) \) and, hence, approaches \( W^c(E_+) \) exponentially. On the other hand, the motion of the projection of such an orbit along the stable fibers on \( W^c(E_+) \) is only algebraic (nonexponential). If the projection of such an orbit along the stable fibers is on the stable part of \( W^{cs}(E_+) \), then it will approach \( E_+ \) nonexponentially and tangent to the branch of \( W^c(E_+) \) that is below \( u_+ \). If it is on the unstable part of \( W^{cs}(E_+) \), then it will leave \( E_+ \) following the flow of the unstable branch of \( W^c(E_+) \) where \( u > u_+ \).

To study these two possibilities, define the semi-circle on \( W^{cs}(E_+) \) as in (3.3). Then

\[
P(\phi) := \{(u(\phi), \lambda(\phi), \theta(\phi)) \mid 0 \leq \phi \leq \pi\}
\]

is a smooth arc on \( W^{cs}(E_+) \). Its projection to the \((u, \lambda)\) plane intersects the subsonic part of the \( u \)-nullcline at \( \phi = \phi_2 \) and the projection of \( W^s(E_+) \) at \( \phi = \phi_3 \). Let \( \phi_1 \) be the infimum of all the angles \( \phi \) such that the backward orbit through \( P(\phi) \) will enter the region \( V(\mathcal{R}_2) \). Note that \( 0 < \phi_1 < \phi_2 - \pi/2 < \phi_3 < \pi \).

If \( \phi_1 < \phi < \pi \), then the orbit passing through \( P(\phi) \) approaches \( E_+^2 \) in backward time due to the facts the region \( V(\mathcal{R}_2) \) is backward invariant and the \((\lambda, u)\) components in \( V(\mathcal{R}_2) \) are monotone. Consider the \( \omega \)-limit set of an orbit through \( P(\phi) \). If \( \phi < \phi_3 \) then the \( \omega \)-limit set of the orbit through \( P(\phi) \) is \( E_+ \), while if \( \phi > \phi_3 \) the \( \omega \)-limit set does not exist since the orbit leaves \( W \) through the side \( F_k \) and is unbounded. This shows if \( \phi_1 < \phi < \phi_3 \), the one-dimensional family of supersonic to sonic connection \( E_+^2 \to E_+ \) exists. Each orbit in that family is not monotone in \( u \) and does not approach \( E_+ \) exponentially. As \( \phi \to \phi_3 \), the
Corollary 3.2. The Assumption 2 of Theorem 3.3 can be weakened to (H1) \( p < p_{eq} \) for \( (u, \lambda) = (u_1^-, 1) \).

Similar to Corollary 3.1, we have the following
4. Analysis of the reduced (outer) problem on slow manifolds

The phase space of the reduced system (2.5) is the critical manifold $S$ (2.8). We only focus on the reduced flow on $S_0$ and $S_1$ since $S_c$ does not play a role in our analysis of evaporation waves (Assumption 2).

**Proposition 4.1.** The reduced vector field on $S_0$ and $S_1$ points away from the sonic fold surface $u^2 = p_\rho$. 

**Proof.** Since $S_0$ and $S_1$ are given as graphs over $(r, m, u)$-space, we study the reduced flow in this single $(r, m, u)$-chart. Hence, we differentiate $n = N(m, u)$ with respect to the slow scale $y$ to obtain the reduced system (2.5) in the $(r, m, u)$-chart:

\begin{align}
  ry &= 1 \\
  my &= \frac{2m}{r} \\
  Nu_y &= \frac{2mu}{r} + \frac{N_m 2m}{r}.
\end{align}

(4.1)

Since $n = N(m, u) = -mu - p(\Lambda(m, u), \frac{mu}{r})$, we have $N_u = -m - p\Lambda u - p_\rho a$ and $N_m = -u - p\Lambda_m - p_\rho p_m$. The derivatives of the function $\Lambda$ evaluated on $S_0$ respectively $S_1$ are $\Lambda_m = \Lambda_u = 0$. Using $\rho = m/u$ to evaluate $\rho_u$ and $\rho_m$, we obtain the reduced system on $S_0$ respectively $S_1$:

\begin{align}
  ry &= 1 \\
  my &= \frac{2m}{r} \\
  (u^2 - p_\rho)u_y &= \frac{2u}{r}p_\rho.
\end{align}

(4.2)

Note that this system is singular along the sonic surface defined by $u^2 = p_\rho$. We desingularize the system by the rescaling $dy = (u^2 - p_\rho)d\bar{y}$ which gives the desingularised system:

\begin{align}
  \bar{r}_\bar{y} &= (u^2 - p_\rho) \\
  \bar{m}_\bar{y} &= \frac{-2m}{r}(u^2 - p_\rho) \\
  \bar{u}_\bar{y} &= \frac{2u}{r}p_\rho > 0.
\end{align}

(4.4)

The phase portraits of the reduced and desingularised system are equivalent up to a change of orientation in the subsonic domain, i.e. $u^2 < p_\rho$. Hence $u_y < 0$ in the subsonic domain and $u_y > 0$ in the supersonic domain which implies that the reduced flow moves away from the sonic fold surface $u^2 = p_\rho$ on $S_0$ respectively $S_1$. Note, this follows because the sonic surface is given as a graph $u = U(m)$ for $m > 0$ independent of $r$ (see also Proposition 5.1). Since we assume $u > 0$, this also implies that there exists no ordinary or folded singularities on $S_0$ or $S_1$. 

**Remark 4.1.** Observe that $S_0$ and $S_1$ represent manifolds of equilibria for the layer problem (2.4). The part of $S_0$ and $S_1$ that satisfies $u^2 \neq p_\rho$ consists of hyperbolic equilibrium points of the first three equations of the layer problem. Thus, on $S_0$ or $S_1$, if $u^2 \neq p_\rho$, $u$ can be solved
as a function of \((m, n, r)\) and it has two smooth branches. One is subsonic and the other is supersonic according to the signs of \(u^2 - p\). If we use \(d = u^2 - p\) to measure the distance to the sonic surface, it is shown in [14] that when restricted to the super or subsonic branch of \(S_0 \cup S_1\), \((d^2)_y > 0\) with respect to the reduced problem (2.5). Therefore, a solution staying on the critical manifold \(S_0\) (or \(S_1\)) will not correspond to a transonic wave.

5. Existence of Transonic Evaporation Waves for \(0 < \varepsilon \ll 1\)

From Proposition 4.1 it follows that a transonic evaporation wave must have an internal layer at some \(r = r_0 \in (r_0, r_1)\) as shown in Figure 2. To form a singular standing wave profile \(\Gamma\) for \(r \in [r_0, r_1]\) such an internal layer solution \(\gamma_i\) has to be concatenated with two (outer) solutions from the reduced problem, one on the subsonic (supersonic) branch of \(S_0\) for \(r \in [r_0, r_1]\) denoted by \(\gamma_0\) and the other on the supersonic (subsonic) branch of \(S_1\) for \(r \in [r_1, r_0]\) denoted by \(\gamma_1\). In the following, we show the existence of (non-degenerate) transonic evaporation waves for \(0 < \varepsilon \ll 1\) that exist near a singular \((\varepsilon = 0)\) wave profile \(\Gamma = \gamma_0 \cup \gamma_i \cup \gamma_1\) (see Figure 3) in the supersonic to subsonic case (section 5.1) and the subsonic to supersonic case (section 5.2).

The sub- and supersonic branches of \(S_0\) (respectively \(S_1\)) away from the sonic surface are normally hyperbolic, with three zero eigenvalues whose eigenspace is tangent to \(S_0\) (respectively \(S_1\)). Fenichel’s theory [16] implies that these normally hyperbolic manifolds perturb smoothly to \(O(\varepsilon)\)-close slow manifolds \(S_0(\varepsilon)\) (respectively \(S_1(\varepsilon)\)), and the slow flow on these manifolds is an \(O(\varepsilon)\) smooth perturbation of the corresponding reduced flow. In the same manner, the end points \(P_0\) and \(P_1\) of \(\Gamma\) perturb smoothly to \(P_0(\varepsilon)\) and \(P_1(\varepsilon)\), respectively.

Remark 5.1. It is well known that the slow manifolds \(S_0(\varepsilon)\) respectively \(S_1(\varepsilon)\) are not unique but they represent a family of slow manifolds that lie within \(O(\exp(-K/\varepsilon))\) distance from each other for some \(K > 0\). We make an arbitrary choice of \(S_0(\varepsilon)\) respectively \(S_1(\varepsilon)\) and show that the results obtained in this section are independent of such choice.

5.1. Supersonic to Subsonic Evaporation Waves. Assume that an internal layer solution \(\gamma_i\) from Theorem 3.1 is given which connects the three-dimensional manifold \(W^u(E^2)\) on the supersonic region of \(S_0\), denoted by \(S_0^{\text{sup}}\), to the two-dimensional manifold \(W^s(E^1)\) on the subsonic region of \(S_1\), denoted by \(S_1^{\text{sub}}\) (see Figure 7). To construct the concatenated orbit \(\Gamma\), we must find \(\gamma_1\) as a forward orbit from \(E^1_+\) and \(\gamma_0\) as a backward orbit from \(E^2_\). The existence of such a solution \(\gamma_1\) that stays on the subsonic region \(S_1^{\text{sub}}\) is guaranteed since the reduced flow of (4.3) moves away from the sonic fold. We assume that \(\gamma_1\) forward connects \(E^1_+\) to \(P_1\) at \(r = r_1\). On the other hand, the backward solution \(\gamma_0\) on \(S_0^{\text{super}}\) with initial condition given by \(E^2_\) might reach the sonic fold for some \(r \in (r_0, r_1)\). To avoid this, the following assumption will be used:

Assumption 4. The distance \(|r_1 - r_0|\) is sufficiently small such that the backward solution \(\gamma_0\) of the reduced problem (4.3) starting at \(r = r_1\) with initial condition \(E^2_\in S_0^{\text{sup}}\) given by the internal layer \(\gamma_i\) stays in the supersonic region \(S_0^{\text{sup}}\) for all \(r \in [r_0, r_1]\).

By Assumption 4, we assume \(\gamma_0\) backward connects \(E^2_-\) to \(P_0\) at \(r = r_0\) following the reduced flow on \(S_0^{\text{sup}}\).

Theorem 5.1. Under Assumptions 1 - 4, let \(\Gamma = \gamma_0 \cup \gamma_i \cup \gamma_1\) denote a singular evaporation wave in the phase space of system (2.3) which is a concatenation of an internal layer solution

\[\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho u \end{pmatrix} = \nabla \cdot \begin{pmatrix} \rho u \\ \rho u^2 \end{pmatrix} + \begin{pmatrix} 0 \\ -p \\ 0 \end{pmatrix}, \quad \frac{\partial p}{\partial t} = \frac{\partial^2}{\partial x^2} \left( \frac{\partial}{\partial x} \left( \frac{\gamma - 1}{\gamma} p \right) \right) \]
\( \gamma_i \) of (2.4) at \( r = r_i \) connecting the supersonic branch \( S_0^{\text{super}} \) with the subsonic branch \( S_1^{\text{sub}} \), an outer solution \( \gamma_0 \) of the reduced problem (2.5) on the supersonic branch \( S_0^{\text{super}} \) for \( r \in [r_0, r_1] \) and another outer solution \( \gamma_1 \) of the reduced problem (2.5) on the subsonic branch \( S_1^{\text{sub}} \) for \( r \in [r_i, r_1] \).

If \( \epsilon_0 > 0 \) is sufficiently small, then for all \( \epsilon \in (0, \epsilon_0) \) there exists a two-dimensional family of standing wave solutions \( \Gamma(\epsilon) \) of system (2.3) lying within \( O(\epsilon) \) of \( \Gamma \).

**Proof.** The normally hyperbolic supersonic branch \( S_0^{\text{super}} \) has an associated local unstable manifold \( W^u(S_0^{\text{super}}) = \bigcup_{p_0 \in S_0^{\text{super}}} W^u(p_0) \) (unstable layer fibration) and the normally hyperbolic subsonic branch \( S_1^{\text{sub}} \) has an associated local stable manifold \( W^s(S_1^{\text{sub}}) = \bigcup_{p_1 \in S_1^{\text{sub}}} W^s(p_1) \) (stable layer fibration). Fenichel’s theory \([16]\) implies that these local unstable and stable manifolds (fibrations) \( W^u(S_0^{\text{super}}) \) and \( W^s(S_1^{\text{sub}}) \) perturb smoothly to \( O(\epsilon) \)-close local unstable and stable manifolds (fibrations) \( W^u(S_0^{\text{super}}(\epsilon)) \) and \( W^s(S_1^{\text{sub}}(\epsilon)) \).

Observe that the two projections of the outer solutions \( \gamma_j \), \( j = 0, 1 \), of \( \Gamma(\epsilon) \) onto the slow variable space \((m, n, r)\) intersect at the common \( n(r_i) \), which is the \( n \)-coordinate for \( E^2 \) and \( E^1_+ \). From the third equation of (2.5) it is easily proved that the two (projected) solutions intersect transversely. Thus for \( 0 < \epsilon \ll 1 \), the two (projected) solutions intersect transversely \( O(\epsilon) \) nearby, i.e. at the common \( n \)-coordinate \( n^\epsilon = n(r_i^\epsilon = r_i + O(\epsilon)) \) and we denote the corresponding common points by \( E^2(\epsilon) \) and \( E^1_+(\epsilon) \). The outer trajectory \( \gamma_0 \in S_0^{\text{super}} \), connecting \( E^2 \) to \( P_0 \), perturb \( O(\epsilon) \) smoothly to the trajectory \( \gamma_0(\epsilon) \in S_0^{\text{super}}(\epsilon) \), connecting \( E^2_-(\epsilon) \) to \( P_0(\epsilon) \), while \( \gamma_1 \in S_1^{\text{sub}} \), connecting \( E^1_+ \) to \( P_1(\epsilon) \), perturb \( O(\epsilon) \) smoothly to the trajectory \( \gamma_1(\epsilon) \in S_1^{\text{sub}}(\epsilon) \), connecting \( E^1_+(\epsilon) \) to \( P_1(\epsilon) \).

In Theorem 3.1 we have shown that there exists a supersonic to subsonic evaporation layer solution \( \gamma_i \) connecting base points \( E^2_- \in S_0^{\text{super}} \) and \( E^1_+ \in S_1^{\text{sub}} \). Since this layer fiber intersection of \( W^u(E^2_-) \subset W^u(S_0^{\text{super}}) \) and \( W^s(E^1_+) \subset W^s(S_1^{\text{sub}}) \) is transverse and the projection of \( \gamma_0(\epsilon) \) and \( \gamma_1(\epsilon) \) onto the slow variable space is transverse at the common base points \( E^2(\epsilon) \) and \( E^1_+(\epsilon) \), this fiber intersection will persist for \( 0 < \epsilon \ll 1 \), i.e. the unstable fibers \( W^u(E^2_-(\epsilon)) \) transversely intersect with the stable fibers \( W^s(E^1_+(\epsilon)) \). Thus \( W^u(E^2_-(\epsilon)) \cap W^s(E^1_+(\epsilon)) \) is a two-dimensional submanifold.

For each point \( \zeta \) on the two-dimensional intersection surface \( W^u(E^2_-(\epsilon)) \cap W^s(E^1_+(\epsilon)) \), we construct an orbit \( \Gamma(\epsilon) \) which exponentially approaches \( S_0^{\text{super}}(\epsilon) \) in backward time along its unstable fibers and exponentially approaches \( S_1^{\text{sub}}(\epsilon) \) in forward time along its stable fibers. Since the unstable fibers of \( S_0^{\text{sub}}(\epsilon) \) are backward invariant with respect to the flow, \( \Gamma(\epsilon) \) is on the unstable fibers of \( P_0(\epsilon) \) at \( r = r_0 \). Also the stable fibers are forward invariant to the flow. Thus \( \Gamma(\epsilon) \) is on the stable fibers of \( P_1(\epsilon) \) at \( r = r_1 \). Let the solution that passes through \( \zeta \) at \( r = r_1 \) be denoted by \( q(r, \zeta, \epsilon) \). Then \( q(r_0, \zeta, \epsilon) = (m, n, u, \lambda, \theta)(r_0) \) is exponentially close to \( P_0(\epsilon) \), and is in the supersonic region with \( \lambda(r_0) \approx 0 \). Also \( q(r_1, \zeta, \epsilon) = (m, n, u, \lambda, \theta)(r_1) \) is exponentially close to \( P_1(\epsilon) \), and is in the subsonic region with \( \lambda(r_1) \approx 1 \). Thus \( q(r, \zeta, \epsilon) \), whose orbit is \( \Gamma(\epsilon) \), is a super to subsonic evaporation wave. Such waves form a two-dimensional family determined by the choice of \( \zeta \).

As mentioned in Remark 5.1, the solution \( \Gamma(\epsilon) \) is unique once the choices \( S_0^{\text{super}}(\epsilon) \) and \( S_1^{\text{sub}}(\epsilon) \) have been made, and the differences due to choices are only exponentially small. \( \square \)

**5.1.1. Boundary conditions that determine a unique super to subsonic wave.** Theorem 5.1 proved the existence of a two-dimensional family of super to subsonic evaporation waves.

\(^1\)Note, since \( y = r \), and \( z = r/\epsilon \), we write solutions as functions of \( r \).
Figure 10. Only the fast variables are plotted. The forward image of the manifold $B_0$ is $B'_0$ while the backward image of the manifold $B_1$ is $B'_1$. At $r = r^e$, these manifolds intersect transversely.

It does not address the uniqueness of these super to subsonic evaporation waves. In the following we prescribe boundary conditions at $r = r_0$ and $r = r_1$ to obtain a unique transonic evaporation wave. The boundary condition shall be given by two boundary manifolds $B_j, j = 0, 1$:

$$(r, m, n, u, \lambda, \theta)(r_j) \in B_j, \quad j = 0, 1.$$ 

For simplicity, we assume that $B_0$ and $B_1$ are linear affine planes that are defined by splitting the conditions on slow and fast variables. First, we define the following ‘slow’ submanifolds $B^s_j, j = 0, 1$:

**Assumption 5.** For the slow variables $(r, m, n) \in B^s_j$, either $n(r_0) = n_0$ or $n(r_1) = n_1$ is given. Also, either $m(r_0) = m_0$ or $m(r_1) = m_1$ is given.

The conditions given in Assumption 5 provide four choices of putting constraints on the slow part $B^s_j$ of the boundary manifolds $B_j, j = 0, 1$. There is an obvious constraint on the variable $r$, that is, at $y = r_0, r(r_0) = r_0$. Using, $dr/dy = 1$, at the other end of the boundary we will have $r(r_1) = r_1$. All together we have introduced three conditions on the slow variables ($m, n, r$).

For $\epsilon = 0$, we assume an internal layer at $r = r_i$. Thus for a given layer position $r = r_i$ and for boundary conditions on $(m, n)$ given by Assumption 5, the end points $\{P_0, P_1, E_2, E_1^\epsilon\}$ of the segments of the singular orbit $\Gamma = \gamma_0 \cup \gamma_i \cup \gamma_1$ defined in Theorem 5.1 are uniquely determined.

For the fast variables $(u, \lambda, \theta)$, we define the following ‘fast’ submanifolds $B^f_j, j = 0, 1$:

**Assumption 6.** The manifold $B^f_0$ is one-dimensional and is transverse to the the weakest two unstable eigenspaces of $P_0$, spanned by $\{v_1, v_2\}$. The manifold $B^f_1$ is two-dimensional and is transverse to the local unstable manifold of $P_1$.

**Definition 3** (Boundary Manifolds, super to sub). Under Assumptions 5-6, let $B^*_0 = B^s_0 \times B^f_0$ and $B^*_1 = B^s_1 \times B^f_1$, be the product of slow and fast planes. The boundary manifolds are given by $B_0 = B^*_0 + q(r_0, \zeta, \epsilon), \zeta \in W^u(E^2_\epsilon) \cap W^s(E^1_\epsilon), \text{ and } B_1 = B^*_1 + Q_1$ where $Q_1$ is any point near $P_1$. 

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Remark 5.2. In a typical linear PDE, the boundary conditions are given first then the existence of solutions are proved. If we assign boundary conditions without any pre-knowledge of the solutions of the nonlinear problem, the solution of the BVP may not exist. So our boundary conditions are based on the knowledge of the singular limit solution of the transonic waves.

Note, $B_0$ depends on the prescribed $\epsilon$ while $B_1$ is independent of $\epsilon$. If $B_1$ happens to pass through $q(r_1, \zeta, \epsilon)$ then $q(r, \zeta, \epsilon)$ obviously satisfies the boundary conditions defined by $B_0$ and $B_1$ (and is the unique such solution). However, we do not want to define the boundary manifolds to pass the end points of the standing wave, then claim that there exists a solution. We want to maintain physical conditions at both ends of the nozzle with some error and still have a solution. This explains why we assume that $Q_1$ is close to but not equal to $P_1$.

Recall that for the supersonic branch $S_0^{\text{sup}}$, the eigenvalues for the fast system with $(m, n, r) \in S_0$ satisfy the condition $0 < \ell_1$, $0 < \ell_2 < \ell_3$. The following assumption shall be used in the next theorem.

Assumption 7. For the singular wave $\gamma_0$ on $S_0^{\text{sup}}$ defined from $r = r_0$ to $r = r_1$, the eigenvalues for the fast system where $(m, n, r) \in \gamma_0$ satisfy $\ell_1 < \ell_3$, i.e., $\ell_3$ is the strongest unstable eigenvalue.

Theorem 5.2. Let $q(r, \zeta, \epsilon)$ be a family of evaporation waves that passes the initial value $\zeta \in W^u(E^2_1(\epsilon)) \cap W^s(E^1_3(\epsilon))$ at $r = r^*_1 = r_1 + O(\epsilon)$. Assume that the family of solutions are tangent to the two weaker unstable fibers based at $P_0(\epsilon)$ and that Assumption 7 is satisfied. Then the boundary manifolds $B_0$ and $B_1$ given by Definition 3 determine a unique super to subsonic evaporation wave.

Proof. The standing wave that satisfies the boundary conditions (Definition 3) can be expressed as

$$U(r, \epsilon) := \Phi(r - r_0, \epsilon)B_0 \cap \Phi(r - r_1, \epsilon)B_1, \quad r_0 \leq r \leq r_1.$$  \hfill (5.1)

It suffices to show that for one particular $r$, the intersection defined by the right hand side of (5.1) is a unique nonempty point. To this end, we shall use Fenichel theory on the persistence of hyperbolic part of the slow manifolds and their fibrations (or foliations).

Fenichel theory implies for sufficiently small $0 < \epsilon \ll 1$ that $B_0$ transversely intersects the ‘fast’ subspace spanned by the two weakest unstable fiber directions of $P_0(\epsilon)$ as well as $B_1$ transversely intersects the unstable fiber of $P_1(\epsilon)$.

First, we focus on the intersection of the slow components of $B_0$ and $B_1$ projected onto the slow variable space. Under Assumption 5, the singular limit segments $\gamma_j$, $j = 0, 1$, of $\Gamma$ are uniquely determined.

As shown in the proof of Theorem 5.1, the two projections of $\gamma_j$, $j = 0, 1$, onto the slow variable space $(m, n, r)$ intersect transversely at the common $n(r_i)$. Thus for $0 < \epsilon \ll 1$, the two (projected) solutions intersect transversely $O(\epsilon)$ nearby, i.e. at the common $n$-coordinate $n^c = n(r^*_1)$ and, as before, we denote the corresponding common points by $E^2_1(\epsilon)$ and $E^1_3(\epsilon)$.

Next, we shall use the theorem of graph transformation to complete the proof. When the flow is near a saddle equilibrium point, such theorem is called the Lambda Lemma, or
Inclination Lemma. In singular perturbed systems, such theorem is also called the Exchange Lemma [19]. Let $\Phi(y, \epsilon)$ denote the flow of the nonlinear system (2.2) for $\epsilon > 0$. Since $B_1$ transversely intersects the unstable fiber $W^u(P_1(\epsilon))$, the backward image $B'_1 = \Phi(r_1^r - r_1, \epsilon)B_1$ is $C^1$ close to the stable fibers at $E^u_1(\epsilon)$.

On the other hand, since the unstable fibers of $P_0(\epsilon)$ are three-dimensional, the regular Lambda Lemma or Exchange Lemma does not apply. A ‘Generalized Lambda Lemma’ was proved in [25] which can be extended to apply to this case. Recall that $E^u_2(\epsilon) = \Phi(r_1^r - r_0, \epsilon)P_0(\epsilon)$. From Assumption 7, the eigenvalues for the fast system do not change order following the slow flow from $P_0(\epsilon)$ to $E^u_2(\epsilon)$. Therefore the Generalized Lambda Lemma should also apply to our fast system. Since $B_0$ is transverse to the two weakest eigenvectors of $P_0(\epsilon)$, its forward image under the flow, $B'_0 = \Phi(r_1^r - r_0, \epsilon)B_0$ is $C^1$ close to the strongest unstable fiber of $E^u_2(\epsilon)$. In particular, $B'_0$ is transverse to the two weakest unstable fibers based at $E^u_2(\epsilon)$, which is the same as the two-dimensional stable fiber space based at $E^u_1(\epsilon)$, as assumed in this theorem. The fact $B'_1$ is $C^1$ close to the stable fibers at $E^s_1(\epsilon)$ ensures the transverse intersection of $B'_0$ and $B'_1$ at $r = r_1^r$.

Finally, at $r = r_1^r$, the transverse intersection of $\Phi(r_1^r - r_0, \epsilon)B_0 \cap \Phi(r_1^r - r_1, \epsilon)B_1$, at a unique point near $\zeta$ determines the evaporation wave solution that satisfies the boundary values defined by $B_0$ and $B_1$ at $r_0$ and $r_1$, respectively. This is illustrated in Figure 10.

Note that due to the transverse intersection $\Phi(r_1^r - r_0, \epsilon)B_0 \cap \Phi(r_1^r - r_1, \epsilon)B_1$, the existence and uniqueness of the transonic evaporation wave $U(r, \epsilon)$ is independent of the choice of $S^\text{super}_0(\epsilon)$ or $S^\text{sub}_1(\epsilon)$, as seen from definition (5.1) and as mentioned in Remark 5.1. □

Remark 5.3. The heteroclinic orbits in the fast system are connections from node to saddle equilibrium points, similar to the waves in KPP/Fisher equations. Based on the observation made in [25], it is necessary to assume that $B_0$ passes through $q(r_0, \zeta, \epsilon)$ for some $\zeta \in W^u(E^u_2(\epsilon)) \cap W^s(E^s_1(\epsilon))$. In fact, if the initial manifold $B_0 \cap W^u(P_0(\epsilon))$ does not intersect with the family of waves $q(r, \zeta, \epsilon)$ under consideration, then the boundary value problem does not have a solution near such waves. The reason is simply because the weakest unstable fibers are unstable in the three-dimensional unstable fiber space of $P_0(\epsilon)$. Any point on $B_0 \cap W^u(P_0(\epsilon))$ that is not on the weakest unstable fiber gets pushed further away if $r > r_0$, so it will never be a boundary manifold for any evaporation wave that stays near $q(r, \zeta, \epsilon)$. See [25] for a proof of this on the node-to-saddle waves in KPP/Fisher type equations, which is similar to our system.

We also remark that Assumption 7 used in Theorem 5.2 is not the most general one. As long as the eigenvalues for the fast system do not change order along $\gamma_0$, the conclusion of the theorem still holds.

Remark 5.4. Note, if $Q_1$ happens to be $P_1$, then as $\epsilon \to 0$, the solution satisfying the boundary conditions will approach $\Gamma$ which has $P_1$ as its right-end-point at $r = r_1$. On the other hand, if $Q_1$ is near $P_1$, and the tangent plane of $B_1$ is specified, the boundary surface will intersect the unstable fiber of $P_1$ at a unique point $Q'_1$. As $\epsilon \to 0$, the wave solution will approach the union of $\Gamma$ and part of the unstable manifold connecting $P_1$ to $Q'_1$. Such boundary layer like behavior have been discussed in [25] if the solution has a node to saddle type internal layer, and in [24] if the solution has a saddle to saddle type internal layer. General boundary conditions for singularly perturbed slow-fast system are that $B_1$ must intersect transversely with the unstable fibers that passes through $P_1$, see [24]. We did
not assume that $Q_1$ in Definition 3 is on the unstable fiber of $P_1$, so the construction of $B_1$ does not require the precise knowledge of the unstable fiber of $P_1$.

5.2. **Subsonic to supersonic evaporation waves.** This section is dedicated to subsonic to supersonic evaporation waves for $0 < \varepsilon \ll 1$. Although there exists no subsonic to supersonic evaporation layer solution (Theorem 3.2), we are still able to construct a singular ($\varepsilon = 0$) subsonic to supersonic evaporation wave profile $\Gamma^s = \gamma_0 \cup \gamma_i \cup \gamma_1$ by concatenation of the (unique) critical internal layer solution $\gamma_i$ from Theorem 3.3 (which connects the base point $E_i^1$ on the subsonic region $S_{0}^{\text{sub}}$ to the base point $E_+^1$ on the sonic region $S_{1}^{\text{super}}$ (see Figure 8) and outer solutions $\gamma_0$ on the subsonic branch $S_{0}^{\text{sub}}$ and $\gamma_1$ on the supersonic branch $S_{1}^{\text{super}}$ starting on the sonic fold. The existence of such a supersonic $\gamma_1$ that stays on $S_{1}^{\text{super}}$ is always guaranteed. However, to ensure the existence of a subsonic $\gamma_0$ that stays on $S_{0}^{\text{sub}}$ until it reaches $r = r_0$, we need Assumption 4 as stated before Theorem 5.1, adapted to the subsonic case:

**Assumption 8.** The distance $|r_i - r_0|$ is sufficiently small such that the backward solution $\gamma_0$ of the reduced problem (4.3) starting at $r = r_i$ with initial condition $E^1_{s} \in S_{0}^{\text{sub}}$ given by the internal layer $\gamma_i$ stays in the subsonic region $S_{0}^{\text{sub}}$ for all $r \in [r_0, r_i]$.

As in Section 5.1.1, we prescribe boundary conditions at $r = r_0$ and $r = r_1$ via boundary manifolds $B_j$, $j = 0, 1$, to obtain a unique transonic evaporation wave. Again for simplicity, we assume that $B_0$ and $B_1$ are linear affine planes that are defined by splitting the conditions on slow and fast variables. The ‘slow’ submanifolds $B_j^s$, $j = 0, 1$, are defined as in Assumption 5. For the fast variables $(u, \lambda, \theta)$, we define the following ‘fast’ submanifolds $B_j^f$, $j = 0, 1$:

**Assumption 9.** The manifold $B_0^f$ is two-dimensional and is transverse to the local stable manifold $W^s(P_0)$. The manifold $B_1^f$ is one-dimensional and is transverse to the local unstable manifold $W^u(P_1)$.

**Definition 4 (Boundary Manifolds, sub to super).** Under Assumptions 5, 9, let $B_0^s = B_0^s \times B_0^f$ and $B_1^s = B_1^s \times B_1^f$, be the product of slow and fast planes. The boundary manifolds are given by $B_0 = B_0^s + Q_0$ and $B_1 = B_1^s + Q_1$, where $Q_0$ is any point near $P_0$ and $Q_1$ is any point near $P_1$.

**Theorem 5.3.** Under Assumptions 1-3, 8, let $\Gamma^s = \gamma_0 \cup \gamma_i \cup \gamma_1$ denote a singular evaporation wave in the phase space of system (2.3) which is a concatenation of an internal layer solution $\gamma_i$ of (2.4) at $r = r_i$ connecting the subsonic branch $S_{0}^{\text{sub}}$ with the sonic fold of $S_1$, an outer solution $\gamma_0$ of the reduced problem (2.5) on the subsonic branch $S_{0}^{\text{sub}}$ for $r \in [r_0, r_i]$ and another outer solution $\gamma_1$ of the reduced problem (2.5) on the supersonic branch $S_{1}^{\text{super}}$ for $r \in [r_i, r_1]$ starting at the sonic fold of $S_1$ for $r = r_1$.

If $\varepsilon_0 > 0$ is sufficiently small, then for all $\varepsilon \in (0, \varepsilon_0)$ there exists a standing wave solution $\Gamma^s(\varepsilon)$ of system (2.3) lying within $O(\varepsilon^{2/3})$ of $\Gamma^s$. The boundary manifolds $B_j$, $j = 0, 1$, given by Definition 4, determine a unique sub to supersonic evaporation wave.

**Remark 5.5.** The $O(\varepsilon^{2/3})$ neighborhood statement in Theorem 5.3 is due to the boundary layer position near the sonic surface of $S_1$. This leads to a turning point problem in terms of classic matched asymptotic expansions.

**Proof.** Fenichel theory [16] implies that the trajectory $\gamma_0 \subset S_{0}^{\text{sub}}$ perturbs to $\gamma_0(\varepsilon) \subset S_{0}^{\text{sub}}(\varepsilon)$ and $O(\varepsilon)$ close to $\gamma_0$. We denote the endpoints of this segment $\gamma_0(\varepsilon)$ by $P_0(\varepsilon)$ at $r = r_0$
and \( E^1(\varepsilon) \) at \( r = r_s^1 \). Similarly, the segment of the trajectory \( \gamma_1 \subset S_1^{super} \) which is bounded away from the sonic surface, i.e., for \( r \in [r_s^1 + \eta, r_1] \) and \( \eta > 0 \), perturbs by Fenichel theory to \( \gamma_1(\varepsilon) \subset S_1^{super}(\varepsilon) \) and \( O(\varepsilon) \) close to \( \gamma_1 \). We denote the endpoints of this segment \( \gamma_1(\varepsilon) \) by \( P_1(\varepsilon) \) at \( r = r_1 \) and \( P_0(\varepsilon) \) at \( r = r_s^1 + \eta \). Furthermore, Fenichel theory implies that the manifolds \( W^i(S_j^k) \), \( i = s, u, j = 0, 1 \), and \( k = sub, super \), perturb smoothly to \( O(\varepsilon) \) close local manifolds \( W^i(S_j^k(\varepsilon)) \).

Recall that \( \Phi(y, \varepsilon) \) denotes the flow of the nonlinear system (2.2) for \( \varepsilon > 0 \). Since the boundary manifold \( B_1 \) transversely intersects the unstable fibers \( W^u(P_1(\varepsilon)) \), the backward image \( B_1' = \Phi(r^s_1 + \eta - r_1, \varepsilon)B_1 \) is \( C^1 \) close to the stable fiber at \( P_0(\varepsilon) \). This follows from the Exchange Lemma. Similarly, since the boundary manifold \( B_0 \) transversely intersects the stable fiber \( W^s(P_0(\varepsilon)) \), the forward image \( B_0' = \Phi(r^s_1 + r_0, \varepsilon)B_0 \) is \( C^1 \) close to the unstable fibers \( W^u(E^1(\varepsilon)) \).

The aim of the rest of the proof is to track \( \gamma_1(\varepsilon) \) and hence the manifold \( B_1' \) backward around the sonic surface of \( S_1 \) and towards the subsonic branch \( S_0^{sub} \) to show that the image transversely intersect with \( B_0' \) at \( r = r_s^1 \). A similar analysis is presented in [2] in the context of electrical waves in cardiac tissue that are modeled by a reaction-diffusion system. The proof is divided into three parts. In parts A and B, we study the local dynamics near the sonic surface of \( S_1 \). This enables us to track, in part C, the boundary manifold of the supersonic branch \( S_1^{super} \) (‘backwards’) towards the subsonic branch \( S_0^{sub} \), to show the transverse intersection of the boundary manifolds and hence the existence of a unique sub-to-supersonic wave.

5.2.1. Part A: Isolating the nonhyperbolic dynamics near the sonic surface of \( S_1 \). At the sonic surface of \( S_1 \) one of the non-zero eigenvalues of the layer problem, \( l_1 \), becomes zero which corresponds to an eigenvector tangent to the \( u \) axis. Normal hyperbolicity is lost along the sonic surface, i.e., each singularity on the sonic surface has a zero eigenvalue of algebraic multiplicity four, and Fenichel’s theory does not apply anymore. To be able to extend the slow manifold \( S_1^{super}(\varepsilon) \) and its foliation (backwards) into the neighborhood of the sonic surface of \( S_1 \), we apply the blow-up technique [7, 21, 32] in the neighborhood of the sonic surface. This geometric technique is commonly used to prove complex oscillatory behaviour in singularly perturbed systems.

To be able to apply the blow-up technique, we have to isolate first the nonhyperbolic dynamics locally near the sonic surface of \( S_1 \) which are constrained to a four-dimensional center manifold \( W^c \). The corresponding generalised center eigenspace \( E^c \) is spanned by the basis of the coordinates \( (r, m, n, u) \). Hence, the four-dimensional invariant center manifold \( W^c \) is a graph over the basis of the coordinates \( (r, m, n, u) \). In fact, the center manifold \( W^c \) is defined by \( \lambda = 1; \theta = 0 \) since this defines a four-dimensional invariant subspace of (2.3) for \( \varepsilon \geq 0 \) and the flow on \( W^c \) is given by

\[
\begin{align*}
  r_z &= \varepsilon \\
  m_z &= -\frac{2m}{r} \\
  n_z &= \frac{2mu}{r} \\
  u_z &= n + mu + p(1, \frac{m}{u}) - \frac{2u}{r} := f(r, m, n, u, \varepsilon).
\end{align*}
\]
Proposition 5.1. For \( m, r, u > 0 \), there exists a smooth change of coordinates that transforms system (5.2) locally near the sonic surface of \( S_1 \) to

\[
\begin{align*}
\dot{r}_z &= \epsilon \mathcal{O}(\tilde{u}, \tilde{u}) =: \varepsilon \tilde{g}_3(r, m, \tilde{n}, \tilde{u}) \\
\dot{m}_z &= \epsilon \mathcal{O}(\tilde{u}, \tilde{u}) =: \varepsilon \tilde{g}_2(r, m, \tilde{n}, \tilde{u}) \\
\dot{\tilde{n}}_z &= -\epsilon(M_1(\tilde{m}) + O(\tilde{u})) =: \varepsilon \tilde{g}_1(r, m, \tilde{n}, \tilde{u}) \\
\dot{\tilde{u}}_z &= \tilde{n} + M_2(\tilde{m}) \tilde{u}^2 + O(\epsilon, \tilde{n} \tilde{u}^2, \tilde{u}^3) =: \tilde{f}(r, m, \tilde{n}, \tilde{u})
\end{align*}
\]

where \( M_1(\tilde{m}) > 0 \) and \( M_2(\tilde{m}) > 0 \) for \( \tilde{m} > 0 \).

Proof. System (5.2) is a singularly perturbed system with three slow variables \((r, m, n)\) and one fast variable \(u\) and it has a three-dimensional folded critical manifold \( S_1 \) (2.8) given by \( n = N(m, u) = -mu - p(1, \frac{m}{u}) \). The sonic surface is defined by \( f_u = \frac{m}{u^2}(u^2 - p_\rho) = 0 \). Since

\[
f_{uu} = \frac{m^2}{u^4} p_{pp} + \frac{2m}{u^3} p_\rho > 0, \quad \forall m, u > 0
\]

it follows by the implicit function theorem that the sonic surface equation \( f_u = 0 \) can be solved for \( u = U(m) \), i.e. the sonic surface is a graph over \((r, m)\)-space (although independent of \(r\)). The coordinate transformation

\[
\tilde{n} = n - N(m, U(m)), \quad \tilde{u} = u - U(m),
\]

rectifies the sonic surface to its base, the \((r, m)\)-space. After applying the Taylor expansion of \( p(1, \frac{m}{u}) \) at \( \tilde{u} = 0 \), we arrive at system

\[
\begin{align*}
\dot{r}_z &= \epsilon \\
\dot{m}_z &= -2m \frac{r}{r} =: \varepsilon g_2(r, m) \\
\dot{\tilde{n}}_z &= -\epsilon(M_1(m) + O(\tilde{u})) =: \varepsilon g_1(r, m, \tilde{u}) \\
\dot{\tilde{u}}_z &= \tilde{n} + M_2(m) \tilde{u}^2 + O(\epsilon, \tilde{n} \tilde{u}^2, \tilde{u}^3)
\end{align*}
\]

where

\[
M_1(m) := \frac{2m}{r} U(m) > 0, \quad M_2(m) := \frac{1}{2}(\rho^2 p_{pp} + \rho_{uu} p_\rho) > 0.
\]

To prove \( M_1 \) and \( M_2 \) are positive, we observe that \( \rho = m/u \) so that \( \rho_{uu} = \frac{2m}{u^2} > 0 \) for \( m, u > 0 \) and \( p_\rho, p_{pp} > 0 \) by (1.2). In a last step we apply the transformation

\[
\tilde{r} = r - \frac{1}{g_1(m, r, 0)} \tilde{n}, \quad \tilde{m} = m - \frac{g_2(m, r)}{g_1(m, r, 0)} \tilde{n}
\]

and the Taylor expansion along the sonic surface to obtain the result. \(\square\)

System (5.3) is a normal form of a singularly perturbed system with a three-dimensional folded critical manifold. The point \( E^+ \) on the folded surface \( F \) satisfies the transversality condition

\[
\begin{align*}
\left( \begin{array}{c}
\tilde{f}_\tilde{r} \\
\tilde{f}_\tilde{m} \\
\tilde{f}_\tilde{n}
\end{array} \right) \cdot \left( \begin{array}{c}
\tilde{g}_3 \\
\tilde{g}_2 \\
\tilde{g}_1
\end{array} \right) &|_{E^+ \in F} \neq 0
\end{align*}
\]

which is also known as the normal switching condition. Recall that the reduced flow of (5.3) is away from the folded surface. Hence \( E^+ \in F \) is a regular jump point for the backward
flow of (5.3). In fact, any point on $F$ has this property. Note that $\hat{f}(\tilde{r}, \tilde{m}, 0, 0) = 0$ defines the sonic surface $F$.

5.2.2. Part B: Transition map and the blow-up. Note that the center manifold $W^c$ itself is normally hyperbolic. Hence its two-dimensional foliation perturbs smoothly for $0 < \varepsilon \ll 1$. This result includes, in particular, the one-dimensional stable foliation which is an important observation, since we are interested in extending the supersonic manifold $S_1^{\text{super}}(\varepsilon)$ and its stable foliation into the neighborhood of the sonic surface $F$ (and beyond). If we are able to track the position of the base points of these fibers, then we know how to track the corresponding stable foliation. This defines the aim of this section, i.e. to track the base points in the center manifold $W^c$.

For small $d > 0$, we define a transition map $\Pi : \Delta_{\text{out}} \to \Delta_{\text{in}}$ for the backward flow of (5.3) where

$$(5.9) \Delta_{\text{out}} = \{ (\tilde{r}, \tilde{m}, -d^2, \tilde{u}) : (\tilde{r}, \tilde{m}, \tilde{u}) \in J_{\text{out}} \}$$

is a section transverse to the supersonic branch $S_1^{\text{super}}$ where $J_{\text{out}} \in \mathbb{R}^3$ is a suitable domain and

$$(5.10) \Delta_{\text{in}} = \{ (\tilde{r}, \tilde{m}, \tilde{n}, -d) : (\tilde{r}, \tilde{m}, \tilde{n}) \in J_{\text{in}} \}$$

is a section transverse to the fast fibers where $J_{\text{in}} \in \mathbb{R}^3$ is a suitable domain.

**Proposition 5.2.** For system (5.3) there exists an $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ the following holds:

1. There exists a suitable rectangle $I_{\text{in}}$ such that for $(\tilde{r}, \tilde{m}) \in I_{\text{in}}$ the manifold $S_1^{\text{super}}(\varepsilon)$ intersects $\Delta_{\text{in}}$ in a smooth surface which is a graph $\tilde{n} = h_{\text{in}}(\tilde{r}, \tilde{m}, \varepsilon)$.

2. The section $\Delta_{\text{out}}$ is mapped to an exponentially thin strip around $S_1^{\text{super}}(\varepsilon) \cap \Delta_{\text{in}}$, i.e. its width in $\tilde{n}$ direction is $O(e^{-k/\varepsilon})$ where $k$ is a positive constant.

3. The map $\Pi : \Delta_{\text{out}} \to \Delta_{\text{in}}$ has the form

$$(5.11) \Pi \begin{pmatrix} \tilde{r} \\ \tilde{m} \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} G_r(\tilde{r}, \tilde{m}, \tilde{u}, \varepsilon) \\ G_m(\tilde{r}, \tilde{m}, \tilde{u}, \varepsilon) \end{pmatrix} + O(e^{-k/\varepsilon})$$

with $h_{\text{in}}(G_r(\tilde{r}, \tilde{m}, \tilde{u}, \varepsilon), G_m(\tilde{r}, \tilde{m}, \tilde{u}, \varepsilon), \varepsilon) = O(\varepsilon^{2/3})$, $G_r(\tilde{r}, \tilde{m}, \tilde{u}, \varepsilon) = G_{r,0}(\tilde{x}) + O(\varepsilon \ln \varepsilon)$, $x = r, m$, where $G_{r,0}(\tilde{x}) = \tilde{x} + O(d^2)$ is induced by the (backward) reduced flow on $S_1^{\text{super}}$ from $\Delta_{\text{out}}$ to the fold surface $F$.

**Proof.** This is an extension of the proof of Theorem 1 presented in [32] that studied singularly perturbed systems with two-dimensional folded critical manifold consisting of regular jump points. The increase of dimension of the folded critical manifold does not change the nature of the blow-up analysis presented in [32]. In fact, the above result can be extended to any $l$-dimensional folded critical manifold, $l \geq 2$, that consists of regular jump points. In section 6 we present the basic concept of the (rather lengthy) proof and refer to [32] for details. □

The local result of the transition map $\Pi$ in Proposition 5.2 has been obtained after preliminary transformations of system (5.2) to system (5.3) by a local diffeomorphism. Hence there exists a section $\tilde{\Delta}_{\text{in}}$ respectively $\tilde{\Delta}_{\text{out}}$ in system (5.2) which is the pre-image of $\Delta_{\text{in}}$ respectively $\Delta_{\text{out}}$ in system (5.3). Without loss of generality, let us assume that the parameter $d > 0$ that defines $\Delta_{\text{in}}$ respectively $\Delta_{\text{out}}$ and the parameter $\eta$ that defines $r = r_1 + \eta$
are chosen in a way such that $\gamma_1(\varepsilon)$, viewed as a trajectory of system (5.2) crosses $\tilde{\Delta}_{\text{out}}$ for $r = r^*_1 + \eta$ and $\tilde{\Delta}_{\text{in}}$ for $r = r^*_1$.

Thus the transition map $\Pi$ tells us how to track $\gamma_1(\varepsilon)$ from the section $\tilde{\Delta}_{\text{out}}$ backwards around the fold surface $F$ and gives us its position in the section $\tilde{\Delta}_{\text{in}}$. Recall that the manifold $B'_1$ is $C^1$ close to the stable fiber at $P_0^s(\varepsilon)$. Hence, the manifold $B'_1$ will track (backward) $C^1$ close the unstable fiber along $\gamma_1(\varepsilon)$ to section $\tilde{\Delta}_{\text{in}}$\(^2\), and we denote this manifold by $B''_1$. So, based on the above analysis we see that on leaving the neighborhood of the fold $F$ at $\tilde{\Delta}_{\text{in}}$, the tangent plane to $W^s(\gamma_1(\varepsilon))$ is spanned by a vector tangent to the (fast) stable fiber of the center manifold and a vector in the direction of the flow. By Proposition 5.2, $\gamma_1(\varepsilon)$ is $O(\varepsilon^{2/3})$ close to the unperturbed $\gamma_1$ and, hence, the manifold $W^s(\gamma_1(\varepsilon))$ will be $C^1$ $O(\varepsilon^{2/3})$ to the unperturbed one, and so will $B''_1$.

5.2.3. Part C: Transverse intersection of the tracked boundary manifolds. In Theorem 3.3 we have shown that there exists a unique subsonic to sonic evaporation layer solution $\gamma_i$ connecting the base point $E^1_\varepsilon \in S_0^{\text{sub}}$ to $E_+ \in F \subset S_1$ on the sonic surface, i.e. the corresponding layer fiber intersection $W^u(E^1_\varepsilon) \cap W^{cs}(E_+) = \gamma_i$ is transverse. This unique solution $\gamma_i$ is $C^1$ close to the center direction of $E_+$. In Part B we found that $W^s(\gamma_1(\varepsilon))$ is $C^1$ $O(\varepsilon^{2/3})$ close to $W^{cs}(E^+)$.

Corollary 5.1. The reduced flow on the subsonic branch $S_0^{\text{sub}}$ is transverse to the projection of the sonic surface $F$ of $S_1$ onto $S_0^{\text{sub}}$ along the unstable layer fibers of $S_0^{\text{sub}}$.

Corollary 5.1 guarantees that the projection of $\gamma_1(\varepsilon) \subset S_1^{\text{super}}(\varepsilon)$ can be traced uniquely in an $O(\varepsilon^{2/3})$ neighborhood of $\gamma_1(\varepsilon)$ towards $\gamma_0(\varepsilon) \subset S_0^{\text{sub}}(\varepsilon)$.

We are able to identify a fast fiber of $S_0^{\text{sub}}(\varepsilon)$ that has $\gamma_i(\varepsilon)$ on it at $\tilde{\Delta}_i$ and the corresponding base point on $S_0^{\text{sub}}(\varepsilon)$ is $E^1_\varepsilon(\varepsilon)$. The corresponding unstable manifold $W^u(E^1_\varepsilon(\varepsilon))$ is $C^1$ $O(\varepsilon)$ close to $W^u(E^1_\varepsilon)$. Hence, $B'_0$ is $C^1$ $O(\varepsilon)$ close to $W^u(E^1_\varepsilon)$.

The transverse intersection $W^u(E^1_\varepsilon) \cap W^{cs}(E_+)$ and the sufficiently small perturbation $0 < \varepsilon \ll 1$, imply the transverse intersection of $B'_0$ and $B''_1$. Thus a unique sub to supersonic evaporation wave $\Gamma^s(\varepsilon)$ has been found that is within an $O(\varepsilon^{2/3})$ neighborhood of $\Gamma^s$. The proof of Theorem 5.3 has been completed.

6. Outline of proof of Proposition 5.2

Since normal hyperbolicity of the critical manifold $S_1$ is lost near the fold surface $F$ one has to use the blow-up technique [7, 21, 32] to analyse the flow near the fold. To be more precise, the blow-up analysis is applied to the extended system \{(5.3), $\varepsilon_2 = 0$\}. We define the blow-up transformation $\Phi : \mathbb{S}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$

\begin{equation}
\tilde{u} = \delta^2 \hat{u}, \quad \hat{u} = \delta \tilde{u}, \quad \varepsilon = \delta^3 \hat{\varepsilon},
\end{equation}

where $\delta \geq 0$ represents the ‘radial’ component of the blow-up. This leads to a blow-up manifold $\mathbb{S}^2 \times \mathbb{R}^3$ with $(\tilde{u}, \hat{u}, \hat{\varepsilon}) \in \mathbb{S}^2$, i.e. the fold surface $F$ is blown-up to a cylinder $\mathbb{S}^2 \times J$ with $(\tilde{r}, \tilde{m}) \in J$. Figure 11 shows the action of the blow-up $\Phi$, namely it inflates the singularity (black dot, left) to a two-sphere (compact object, right). Note that this

\(^2\)to be more precise: to section $\tilde{\Delta}_{\text{in}} \times (\lambda, \theta)$
Figure 11. A sketch of the action of the blow-up $\Phi$. The sonic surface of $S_1$ (here a black dot, left) is blown-up to a half-cylinder (here a half-sphere, right). Three charts are needed ($\kappa_1$-$\kappa_3$, right) to cover the essential flow in the blown-up locus.

Figure 12. A sketch of the supersonic slow manifold $M_1$ of the extended system \{\eqref{5.3}, $\varepsilon_z = 0$\} in $(\tilde{u}, \tilde{n}, \varepsilon)$-space (left). The blow-up of the region near the fold surface (black-dot, left) and a sketch of the extension of the slow (blown-up) manifold $\hat{M}_1$ over the blown-up sphere (right).

cylindrical blow-up $\Phi$ acts only transverse to the sonic surface (the fold) which indicates that this procedure is independent of the dimension $l$ of the fold.\footnote{This points to the generality of the blow-up analysis presented in [32].}

In the extended phase space of \{\eqref{5.3}, $\varepsilon_z = 0$\} the slow manifold $S_1^{\text{super}}(\varepsilon)$ corresponds to a slice $\varepsilon = \text{const.}$ of a four-dimensional (supersonic) slow manifold which we denote by $M_1$ (see Figure 12, left). With the blow-up technique, we are able to extend this four-dimensional manifold (denoted $\hat{M}_1$ in the blow-up) near the sonic surface backwards onto the blown-up sphere and we are able to show that this manifold $\hat{M}_1$ exits the blown-up sphere almost parallel to the ‘center fibers’ as shown in Figure 12. This is possible, because we have gained normal hyperbolicity at both branches of $S_1$ at the equator as indicated by double arrows near $\hat{E}_+^1$ and $\hat{E}_2^+$ in Figure 12.
Figure 13. The local flow near the sonic surface of $S_1$ is given by a Riccati equation (6.3). It represents the flow in chart $\kappa_2$ shown in Figure 11. The pink curve $\ell_0$ connects to the supersonic branch of $S_1$ at the equator of the blown-up sphere (to $E^2_+\ell_0$ in Figure 12).

To rigorously calculate the extension of $\hat{M}_1$ by the (backward) flow on the upper blown-up cylinder one would need three charts, denoted $\kappa_1$ to $\kappa_3$, to cover the essential flow on the cylinder. These charts are defined by $\kappa_1 : \hat{n} = -1$, $\kappa_2 : \hat{\epsilon} = 1$ and $\kappa_3 : \hat{u} = -1$ (see Figure 11). We do not present a full blow-up analysis here which can be found in [32], but show only the core of the analysis which is covered by chart $\kappa_2$. In this chart, the blow-up is simply an $\epsilon$-dependent rescaling of the dependent variables $\tilde{n} = \epsilon^{2/3}\hat{n}$, $\tilde{u} = \epsilon^{1/3}\hat{u}$ and the independent variable $z = \epsilon^{-1/3}\zeta$ which transforms (5.3) to

$$
\begin{align*}
\hat{r}' &= O(\epsilon) \\
\hat{m}' &= O(\epsilon) \\
\hat{n}' &= -M_1(\hat{m}) + O(\epsilon^{1/3}) \\
\hat{u}' &= \hat{n} + M_2(\hat{m})\hat{u}^2 + O(\epsilon^{1/3}).
\end{align*}
$$

Note, system (6.2) is a regularly perturbed system (on a compact domain) since $M_1 > 0$ by (5.6). As $\epsilon^{1/3} \to 0$, this yields system

$$
\begin{align*}
\hat{n}' &= -M_1(\hat{m}) \\
\hat{u}' &= \hat{n} + M_2(\hat{m})\hat{u}^2
\end{align*}
$$

which is known as a Riccati equation (see, e.g. [29]). Since we are dealing with a regular perturbation problem (on a compact domain), this also implies that the local dynamics of (5.3) are governed (to leading order) by the Riccati equation (6.3) in a region where $(\hat{n}, \hat{u})$ are $O(\epsilon^{2/3}) \times O(\epsilon^{1/3})$.

The dynamics of system (6.3) are well studied (see e.g. [29]) and Figure 13 shows a phase portrait of the Riccati flow. Note that ‘infinity’ in chart $\kappa_2$ corresponds to the ‘equator’ of the blown-up sphere (see Figure 11). Thus, the flow in Figure 13 can be divided into three types depending on how the Riccati flow connects to the equator of the blown-up sphere:
(1) There exists a unique solution $\ell_0(\zeta)$ (shown in pink) that approaches the ‘positive’ branch ($\hat{u} > 0$) of the parabola $\hat{u} + M_2(\hat{m})\hat{u}^2 = 0$ as $\zeta \to \infty$. This solution connects asymptotically to the supersonic branch $S^\text{super}_1$ at the equator of the blown-up sphere (to $\hat{E}^2_-\hat{E}^2_1$ in Figure 12).

(2) There is a one-dimensional family of solutions $\ell_1(\zeta)$ whose $\hat{u}$ coordinate becomes unbounded ($+\infty$) as $\zeta \to \infty$. These solutions connect asymptotically to the (fast) outgoing center fiber at the equator of the blown-up sphere.

(3) There is a one-dimensional family of solutions $\ell_2(\zeta)$ that approaches the ‘negative’ branch ($\hat{u} < 0$) of the parabola $\hat{u} + M_2(\hat{m})\hat{u}^2 = 0$ as $\zeta \to \infty$. All these solutions connect asymptotically to the subsonic branch $S^\text{sub}_1$ at the equator of the blown-up sphere (to $\hat{E}^1_1$ in Figure 12).

All three types of solutions connect in backward ‘time’ to the (fast) incoming center fiber at the equator of the blown-up sphere.

Note further that system (6.2) has an intermediate $O(\varepsilon^{1/3})$ scale. Thus chart $\kappa_2$ corresponds to a matching region of an inner solution (chart $\kappa_3$, fast $O(1)$ scale) and an outer solution (chart $\kappa_1$, slow $O(\varepsilon)$ scale) of this ‘turning point’ problem. The flow in the rescaling chart $\kappa_2$ indicates how the manifold $\hat{M}_1$ and hence the manifold $S^\text{super}_1(\varepsilon)$ can be extended from the supersonic regime around the sonic surface $F$ and how it then exits almost parallel to the (‘fast’) center fibers. It also hints to the $O(\varepsilon^{2/3})$ displacement stated in this Proposition 5.2 due to the different weights (powers of $\varepsilon$) of the blow-up transformation which is well-known for this type of turning point problem. The details of the entry and the exit of the manifold $S^\text{super}_1(\varepsilon)$ near the blown-up cylinder are covered by the dynamics in the entry and exit charts $\kappa_1$ and $\kappa_3$. As mentioned before, we omit the corresponding analysis in these charts. It can be found in [32].

**Remark 6.1.** The result of Proposition 5.2 is independent of the choice of $S^\text{super}_1(\varepsilon)$; see also Remark 5.1.

7. Concluding remarks

To the best of our knowledge, this paper proves for the first time the existence of transonic evaporation waves in a spherical symmetric nozzle. The key in proving the existence of these evaporation waves is the application of recent results and techniques within the field of dynamical systems. In the case of super to subsonic evaporation waves, a generalization of the Exchange Lemma [19] is needed that deals with a node to saddle transition that describes the evaporation layer. This Generalized Exchange Lemma was recently shown in the case of the Fisher-KPP equation [25].

In the case of sub to supersonic evaporation waves, the application of a geometric desingularization technique, known as the blow-up technique [32], is needed to deal with loss of normal hyperbolicity near the sonic surface. This blow-up technique was also successfully used in a different physical context to prove the existence of electrical waves in cardiac tissue that are modelled by a reaction-diffusion system [2].

It is worthwhile to mention the deep insights one can gain from this geometric desingularization procedure. The blow-up technique allows us to identify the pre-images of solutions in a small neighborhood of the sonic surface for $0 < \varepsilon \ll 1$, which already exists on the blown-up cylinder for $\varepsilon = 0$. These are the solutions of the Riccati equation (6.3) in chart.
κ_2 compactified onto the blown-up cylinder as schematically shown in Figure 12. They are connected to the sub- or supersonic branch of S_1.

For example, we are able to identify the pre-images of the one-parameter family of supersonic to sonic evaporation layer solutions E^2 → E_+ shown in Figure 8 locally near E_+. These are all solutions on the blown-up cylinder that connect from the fiber point \( \hat{E}_0 \) on the equator to the subsonic point \( \hat{E}_1^+ \) on the equator. These solutions correspond to the family of solutions \( \ell_2 \) of the Riccati equation shown in Figure 13.

Note that the unique layer solution \( E^2 \rightarrow E_+ \) shown in Figure 8 does not show up directly in the blow-up (Figure 12) because it is not part of the center manifold, i.e. it approaches the sonic point \( E_+ \) in a transverse direction to the center manifold. Obviously, we cannot identify the heteroclinic connection from \( E^2 \rightarrow E_1^- \) (wet shock, shown in Figure 8) in Figure 12 because none of the equilibria is part of the blow-up.

More importantly, we can also identify the pre-images of the layer solutions shown in Figure 7 in the blow-up shown in Figure 12 as \( n \rightarrow \bar{n} \): The limit of the subsonic evaporation wave \( E_1^- \rightarrow E_1^+ \) corresponds to the connection \( \hat{F}_0 \rightarrow \hat{E}_1^+ \) along the equator. The limit of the dry wave \( E_+ \rightarrow E_+^2 \) corresponds to the connection \( \hat{E}_1^+ \rightarrow \hat{E}_2^+ \) along the equator. Finally, the limit of the supersonic evaporation wave \( E_2^- \rightarrow E_2^+ \) can be thought of as a connection to \( \hat{E}_2^+ \) on the equator transverse to the center manifold. By continuity, the (local) solutions in the interior of Figure 7 near \( E_1^+ \) have pre-images that correspond to solution \( \ell_2 \) on the blown-up sphere.

**References**


Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA
E-mail address: xblin@ncsu.edu

School of Mathematics and Statistics, University of Sydney, Sydney, NSW 2006, Australia
E-mail address: wm@maths.usyd.edu.au

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