Calculating the Galois group of $L_1(L_2(y)) = 0$, $L_1, L_2$ completely reducible operators

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Abstract

In Calculating Galois groups of completely reducible linear operators, Compoint and Singer describe a decision procedure that computes the Galois group of a completely reducible linear differential operator with rational or algebraic function coefficients (i.e., a linear differential operator that is the least common left multiple of irreducible operators or, equivalently, one whose Galois group is a reductive group). At present, it is unknown how to calculate the Galois group of a general operator. In this paper, we push beyond the completely reducible case by showing how to compute the Galois group of an operator of the form $L_1 \circ L_2$ where $L_1$ and $L_2$ are completely reducible and have rational function coefficients.

We begin by showing how to compute the Galois group of an equation of the form $L(y) = b$ with $L$ completely reducible. This corresponds to the case of $L_1 \circ L_2$ where $L_1 = D - b'/b$. We then show how one can reduce the general case to the above case and give several examples.

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1. Introduction

In [3] a decision procedure is described that computes the Galois group of a completely reducible linear differential operator with rational or algebraic function coefficients (i.e., a linear differential operator that is the least common left multiple of irreducible operators or, equivalently, one whose Galois group is a reductive group). At present, it is unknown how to calculate the Galois group of a general operator. In this paper, we push beyond the completely reducible case by showing how
to compute the Galois group (and Picard–Vessiot extension) of an operator of the form $L_1 \circ L_2$ where $L_1$ and $L_2$ are completely reducible and have rational function coefficients.

The paper is organized as follows. In Section 2 we show how to compute the Galois group and Picard–Vessiot extension of an equation of the form $L(y) = b$ with $L$ completely reducible. This corresponds to the case of $L_1 \circ L_2$ where $L_1 = D - b'/b$. In Section 3, we show how one can reduce the general case to the above case and give several examples.

2. Calculating the Galois group of $L(y) = b$, $L$ completely reducible

Let $k$ be a differential field of characteristic zero with algebraically closed field of constants $\mathcal{C}$ and let $\mathcal{D} = k[D]$ be the ring of differential operators with coefficients in $k$. For any $L \in \mathcal{D}$ and $b \in k$, we shall use the phrase the Picard–Vessiot extension of $L(y) = b$ to denote the Picard–Vessiot extension of $k$ corresponding to $(D - b'/b)L$, that is, the smallest Picard–Vessiot extension of $k$ containing a full set of solutions of $L(y) = 0$ as well as a specific solution of $L(y) = b$. We shall call the Galois group of this Picard–Vessiot extension the Galois group of $L(y) = b$. We will combine the techniques of [1,2] with those of [3] to show how, given $L \in k[D]$, $L$ completely reducible, and $b \in k$, one can calculate the Galois group and Picard–Vessiot extension of $L(y) = b$. Note that an operator is said to be completely reducible if it is the least common left multiple of irreducible operators. An equivalent condition is that the Galois group $G$ be reductive ([7], p. 125), that is, the largest normal subgroup of $G$ consisting of unipotent elements is trivial (for equivalent conditions see Lemma 2.13 of [13] and Proposition 2.2 of [3]). The key is the following proposition, which is a slight modification of Théorème 1 of [2]. We say that an algebraic group is a vector group if it is isomorphic to $(\mathcal{C}^n, +)$ for some $n$.

**Proposition 2.1.** Let $k$, $b$ and $L$ be as above. The Galois group of $L(y) = b$ is isomorphic to the semidirect product $W \rtimes G_L$, of the Galois group $G_L$ of $L(y) = 0$ and a vector group $W$. Furthermore, if $L_1 \in k[D]$ is a monic operator of maximal order satisfying

1. $L_1(y) = b$ has a solution $f_1 \in k$, and
2. $L = L_1 L_0$ for some $L_0 \in k[D]$

then $W$ is $G$-isomorphic to the solution space of $L_0$. In addition such an $L_1$ is unique.

**Proof.** We first note that the operator $L_1 = 1$ satisfies conditions 1. and 2. with $L_0 = L$. Therefore there will exist an operator $L_1$ of maximal order satisfying these conditions.

Let $K$ be the Picard–Vessiot extension of $k$ corresponding to $(D - b'/b)L$ and $K_L \subset K$ be the Picard–Vessiot extension corresponding to $L$. If $L = L_1 L_0$, then $K_L$ will contain
fundamental sets of solutions of both \( L_0(y) = 0 \) and \( L_1(y) = 0 \) and so it makes sense to speak of the action of \( G_L \) on the solution spaces of these two equations.

Let \( f \in K \) be a solution of \( L(y) = b \). For any \( \sigma \in G = \text{Gal}(K/k) \), \( \sigma(f) - f \) is in the solution space \( V_L \) of \( L(y) = 0 \). Let \( \Phi : G \to V_L \) be the map sending \( \sigma \) to \( \sigma(f) - f \). Let \( H \) be the normal subgroup of \( G \) leaving \( K_L \) fixed.

Since \( K = K_L(f) \), we see that \( \Phi \) is injective on \( H \). For any \( \sigma \in G, \tau \in H \), we have that

\[
\Phi(\sigma \tau \sigma^{-1}) = \sigma \tau \sigma^{-1}(f) - f \\
= \sigma[\tau(\sigma^{-1}(f) - f)] - f \\
= \sigma[\tau(\sigma^{-1}(f) - f) + \sigma(f) - f] \quad \text{since } \tau \text{ fixes the elements of } V_L \subset K_L \\
= \sigma[\tau(f) - f] \\
= \sigma \Phi(\tau).
\]

This calculation (from the proof of Théorèm 1 of [2]) shows that \( \Phi \) is a \( G \)-morphism, where the action of \( G \) on \( H \) is given by conjugation. Therefore, \( \Phi \) identifies \( H \) with a \( G \)-invariant subspace \( W \) of \( V_L \). Since \( G/H \) is isomorphic to the reductive group \( G_L \) and \( H \) is unipotent, \( H \) is the unipotent radical of \( G \). Any linear algebraic group may be written as a semidirect product \( G = H \rtimes P \) where \( H \) is the unipotent radical of \( G \) and \( P \) is a reductive group (called a Levi subgroup of \( G \), [11]). Clearly, \( P \) is isomorphic to \( G_L \).

Let \( \tilde{L}_0 \) be the monic operator in \( k[D] \) whose solution space is \( W \) and let \( L = \tilde{L}_1 \tilde{L}_0 \). Since \( \sigma(f) - f \in W \) for all \( \sigma \in H \), we have that \( \tilde{L}_0(\sigma(f)) = \tilde{L}_0(f) \) for all \( \sigma \in H \). Therefore, \( \tilde{L}_0(f) \in K_L \). Let \( W_1 \) be the solution space of \( \tilde{L}_1 \) in \( K_L \) and let \( W_{\tilde{L}_0(f)} \) be the space spanned by \( W_1 \) and \( \tilde{L}_0(f) \). For any \( \sigma \in G_L \), \( \sigma(\tilde{L}_0(f)) \) is again a solution of \( \tilde{L}_0(y) = b \) and so the space \( W_{\tilde{L}_0(f)} \) is left invariant by \( G_L \). Furthermore, \( W_{\tilde{L}_0(f)} \cap W_1 \) is a trivial one-dimensional \( G_L \)-module. Since \( G_L \) is a reductive group, \( W_1 \) has a \( G_L \)-complement in \( W_{\tilde{L}_0(f)} \). This implies that there is an element \( f_0 \in K_L \) such that \( f_0 \equiv \tilde{L}_0(f) \mod W_1 \) and \( f_0 \) is left fixed by \( G_L \). We conclude from this that \( f_0 \in k \) and \( \tilde{L}_1(f_0) = b \).

Now let \( L_1 \) satisfy 1. and 2. above. Since \( L_1(L_0(f)) = b \), we have that \( L_0(f) - f_1 \in W_1 \), where \( W_1 \) is the solution space of \( L_1 \). In particular, \( L_0(f) \in K_L \). Therefore, for any \( \sigma \in H \), \( L_0(\sigma(f) - f) = 0 \). This implies that the image of \( \Phi \) lies in the solution space of \( L_0 \). Therefore, \( \tilde{L}_0 \) divides \( L_0 \) on the right and so the order of \( L_1 \) is at most the order of \( \tilde{L}_1 \). If these two orders are the same and \( L_1 \) is monic, then \( \tilde{L}_0 = L_0 \) and so we must then have that \( L_1 = \tilde{L}_1 \).

The following example illustrates this proposition:

**Example 2.2.** Let \( k = \mathbb{C}(x) \) and \( L = D^2 - 4xD + (4x^2 - 2) = (D - 2x) \circ (D - 2x) \). A basis for the solution space of equation \( L(y) = 0 \) is \( \{e^x, xe^x\} \) so the Galois group of this homogeneous equation over \( k \) is \( \mathbb{C}^* \).
For any \((c, d) \neq (0, 0)\), we have that \((c + dx)e^x\) is a solution of \(L(y) = 0\) and so \(L\) has a right factor of the form \(D - \left(2x + \frac{d}{c + dx}\right)\). Furthermore, all right factors of order one are of this form. Therefore the formula

\[
L = \left(D - \left(2x - \frac{d}{c + dx}\right)\right) \circ \left(D - \left(2x + \frac{d}{c + dx}\right)\right)
\]

with \((c, d) \neq (0, 0)\) yields a parameterization of all irreducible factorizations of \(L\).

We shall now compute the Galois groups of \(L(y) = b\) where \(b = 4x^2 - 2, 1\) and \(\frac{1}{x}\).

1. \(b = 4x^2 - 2\). In this case the equation \(L(y) = b\) has the rational solution \(y = 1\). This implies that the \(W\) of Proposition 2.1 is trivial and so the Galois group of \(L(y) = b\) is \(\mathcal{C}^*\).

2. \(b = 1\). A partial fraction computation shows that \(L(y) = 1\) has no rational solutions. Now let us search for first order left factors \(L_1\) of \(L\) such that \(L_1(y) = 1\) has a rational solution. A calculation shows that the equation

\[
y' - \left(2x - \frac{d}{c + dx}\right)y = 1
\]

has a rational solution \(y = f'\) if and only if \(z = (c + dx)f'\) is a rational solution of

\[
z' - 2xz = c + dx
\]

(cf., Lemma 2.4). The rational solutions of (2) must be polynomials and one sees that this has a polynomial solution if and only if \(c = 0\). Therefore the space \(W\) of Proposition 2.1 is the solution space of \(y' - (2x + \frac{1}{x})y = 0\), that is, the space spanned by \(xe^x\) in the solution space of \(L(y) = 0\). Therefore the Galois group of \(L(y) = 1\) is \(\mathcal{C} \cong \mathcal{C}^*\).

3. \(b = \frac{1}{x}\). We shall show that for any \((c, d) \neq (0, 0)\), the equation

\[
y' - \left(2x - \frac{d}{c + dx}\right)y = \frac{1}{x}
\]

has no rational solution. This implies that \(L(y) = \frac{1}{x}\) also has no rational solution and so the \(W\) of Proposition 2.1 is the solution space of \(L(y) = 0\). Therefore the Galois group of \(L(y) = \frac{1}{x}\) is \(\mathcal{C}^2 \cong \mathcal{C}^*\).

Eq. (3) has a rational solution \(y = f\) if and only if \(z = (c + dx)f\) is a rational solution of

\[
z' - 2xz = \frac{c + dx}{x}.
\]

If \(c \neq 0\) then any rational solution of (4) must have a pole at \(x = 0\). Comparing orders of the left and right-hand side of this equation yields a contradiction. Therefore \(c = 0\). Similar considerations show that \(z' - 2xz = d\) can never have a rational solution if \(d \neq 0\).

Proposition 2.1 allows us to give a detailed description of the Picard–Vessiot extension of \(k\) corresponding to \(L(y) = b\).
Corollary 2.3. Let $k,b,L$ be as above and let $K_L$ be the Picard–Vessiot extension of $k$ corresponding to $L(y) = 0$. Let $L_1 \in k[D]$ be the unique monic operator of maximal order satisfying

1. $L_1(y) = b$ has a solution $f_1 \in k$, and
2. $L = L_1L_0$ for some $L_0 = D' - b_{t-1}D'^{t-1} - \cdots - b_0 \in k[D]$.

Then the Picard–Vessiot extension $K$ of $k$ corresponding to $L(y) = b$ is the field $K_t(z_0,z_1,\ldots,z_{t-1})$ where $z_0,z_1,\ldots,z_{t-1}$ are algebraically independent, $z'_i = z_{i-1}$ for $i = 0,\ldots,t-2$ and $z'_t = f_1 + b_0z_0 + \cdots + b_{t-1}z_{t-1}$.

Proof. Let $V$ be the solution space of $(D - b'/b)L$ in $K$. The linear operator $L_0$ maps $V$ onto the solution space of $(D - b'/b)L_1$. Therefore, there exists a $z \in V$ such that $L_0(z) = f_1$. Since $L(z) = b$, we have that $K = K_L(z)$. Since the Galois group of $K$ over $K_L$ is a vector group of dimension $t$, we have that $K$ is a purely transcendental extension of $K_L$ of transcendence degree $t$. Therefore $K = K_t(z,z',\ldots,z^{(t-1)})$. The elements $z_i = z^{(i)}$ satisfy the conclusion of the Corollary. □

Proposition 2.1 also implies that in order to find the Galois group of $L(y) = b$ we must

1. Calculate the Galois group $G_L$ of $L$, and
2. Find the monic operator $L_1$ of maximal order satisfying 1. and 2. of Proposition 2.1 and identify the action of $G$ on the solution space of $L_0$.

The first task was dealt with in [3]. In this paper it is shown (Theorem 4.1 and its proof) how for all points $z_0 \in C$ outside some finite set (depending on $L$), one can calculate a matrix representation for the Galois group of $L$ in the basis $\{y_0,\ldots,y_{n-1}\}$ of the solution space given by $y_i^{(j)}(z_0) = \delta_{i,j}$.

Dealing with the second task will occupy the remainder of this section. We begin by recalling some basic definitions and facts concerning linear differential equations.

Two operators $L_2$ and $L_1$ are said to be equivalent if the $\mathcal{D}$-modules $\mathcal{D}/\mathcal{D}L_2$ and $\mathcal{D}/\mathcal{D}L_1$ are $\mathcal{D}$-isomorphic (see [13]). This is equivalent to the statement that the two operators have the same order $m$ and that there exist operators $R,S$ of orders at most $m-1$ with $\mathrm{GCRD}(R,L_1) = 1$ such that

$$L_2R = SL_1.$$ \hspace{1cm} (5)

Note that such an operator $R$ can be used to define a map $1 \mapsto R$ which gives the isomorphism from $\mathcal{D}/\mathcal{D}L_2$ to $\mathcal{D}/\mathcal{D}L_1$. We note that the ring $\mathcal{D}$ is a left and right euclidean domain. In particular given operators $U, V \in \mathcal{D}$ an extended euclidean algorithm yields operators $A, B \in \mathcal{D}$, ord $A < \mathrm{ord} V$, ord $B < \mathrm{ord} U$, such that $AU + BV = \mathrm{GCRD}(U,V)$.

Lemma 2.4. Let $L_1, L_2 \in \mathcal{D}$ be equivalent operators and $S \in \mathcal{D}$ as in Eq. (5). The equation $L_1(y) = b$, $b \in k$ has a solution in $k$ if and only if the equation $L_2(y) = S(b)$ has a solution in $k$.

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1 This theorem is stated in terms of matrix systems $Y' = AY$ but the translation to scalar equations $U(y) = 0$ is immediate.
Proof. The extended euclidean algorithm yields $\tilde{R}$ and $\tilde{L}_1$ in $\mathcal{D}$ such that $\tilde{R}R + \tilde{L}_1L_1 = 1$ and $\text{ord } \tilde{R} < \text{ord } L_1$. The map $v \mapsto R(v)$ is an isomorphism of $V_L$ (the solution space of $L_1$) onto $V_{L_2}$ and the map $w \mapsto \tilde{R}(w)$ is the inverse of this isomorphism [13]. Since $L_1\tilde{R}$ and $R\tilde{R} - 1$ vanish on $V_{L_2}$, we have that $L_2$ divides both of these operators. Therefore there exist $\tilde{S}$ and $\tilde{L}_2 \in \mathcal{D}$ such that $L_1\tilde{R} = \tilde{S}L_2$ and $R\tilde{R} + \tilde{L}_2L_2 = 1$.

We now claim that $\tilde{S}S + L_1\tilde{L}_1 = 1$. We have that

\[
(\tilde{S}S + L_1\tilde{L}_1)L_1 = \tilde{S}SL_1 + L_1\tilde{L}_1L_1
\]

\[
= \tilde{S}L_2R + L_1(1 - \tilde{R}R)
\]

\[
= \tilde{S}L_2R + L_1 - L_1\tilde{R}R
\]

\[
= \tilde{S}L_2R + L_1 - \tilde{S}L_2R
\]

\[
= L_1.
\]

and the equation follows after cancelling $L_1$ on the right.

To prove one direction of the lemma, suppose $L_i(f) = b$ for some $f \in k$. If $h = R(f) \in k$, then $L_2(h) = SL_i(f) = S(b)$ as desired. To prove the other direction, suppose $L_2(h) = S(b)$ for some $h \in k$. Let $f = \tilde{R}(h) + \tilde{L}_1(b) \in k$. Then

\[
L_1(f) = L_1\tilde{R}(h) + L_1\tilde{L}_1(b)
\]

\[
= \tilde{S}L_2(h) + (1 - \tilde{S}S)(b)
\]

\[
= \tilde{S}S(b) + b - \tilde{S}S(b)
\]

\[
= b,
\]

completing the proof. □

Any operator can be written as a product of irreducible operators and for any other factorization, one has the same number of irreducible factors. Moreover, after a possible renumbering, the irreducible factors are equivalent. By definition any completely reducible operator $L$ can be written as the least common left multiple of a finite set of irreducible operators. Any left or right factor will therefore be equivalent to the least common left multiple of some subset of these operators. When $k$ is a finite algebraic extension of $\mathbb{C}(x)$, where $\mathbb{C}$ is a computable algebraically closed field of characteristic zero, one can effectively factor any differential operator into a product of irreducible differential operators, determine if an operator is completely reducible and, if so, effectively write it as a least common left multiple of irreducible operators [3].

We shall attack the second task above in the following way. We start by writing $L$ as the least common left multiple of a set of irreducible operators $\mathcal{F} = \{T_1, \ldots, T_n\}$. Any monic operator $L_i$ dividing $L$ on the left is equivalent to a least common left multiple of elements from $\mathcal{F}$. We fix a subset of $\mathcal{F}$ and let $L_2$ be the least common left multiple of elements of this subset. We will show below that one can parameterize all pairs of elements $(L_1, S)$, ord $S < \text{ord } L_1 = \text{ord } L_2$, where $S$ and $L_2$ are as in Eq. (5) and $L_1$ divides $L$ on the left. We furthermore will show that one can decide if there are values of the parameters so that $L_i(y) = b$ has a solution in $k$. Performing these tasks over all subsets of $\mathcal{F}$, we will eventually find an operator $L_1$ of maximal
order satisfying conditions 1. and 2. of Proposition 2.1. We will then show how to describe the action of $G_L$ on the solution space of $L_0$ where $L = L_1L_0$.

The following three lemmas are used to describe the set of pairs $(L_1, S)$ mentioned above. The fourth lemma will be used to decide if there are values of the parameters so that $L_1(\gamma) = b$ has a solution in $k$.

Let $k = C(x)$. For $a = \frac{p}{q} \in k$, $p, q \in C[x]$, $(p, q) = 1$, we define $\deg a = \max(\deg p, \deg q)$. For $L = a_nD^n + a_{n-1}D^{n-1} + \cdots + a_0 \in k[D]$, we define $\deg L = \max_{1 \leq i \leq n}(\deg a_i)$ and ord $L = n$. Given operators $L$ and $L_2$, we will want to parameterize all pairs of operators $(L_1, S)$ satisfying:

1. $L_1$ is a monic operator equivalent to $L_2$ that divides $L$ on the left, and
2. ord $S \leq$ ord $L_2 - 1$ and $L_2R = SL_1$ for some $R \in S$, GCRD$(L_1, R) = 1$.

Lemma 2.5. Let $T_1$ and $T_2$ be operators with coefficients in $C(x)$ of orders $n$ and $m$ and degrees $N$ and $M$ respectively. If $T_3$ is an operator with coefficients in $C(x)$ such that $T_3T_2 = T_1$, then $\deg T_3 \leq (n - m + 1)^2M + N$.

Proof. Let $T_1 = \sum_{i=0}^{n} a_iD^i$, $T_2 = \sum_{i=0}^{m} b_iD^i$ and $T_3 = \sum_{i=0}^{n-m} c_iD^i$. The equation $T_3T_2 = T_1$ yields a system of (algebraic) linear equations

\[
\begin{pmatrix}
a_0 \\
a_{n-1} \\
\vdots \\
a_0
\end{pmatrix} = \begin{pmatrix} c_{n-m} \\
c_{n-m-1} \\
\vdots \\
c_0
\end{pmatrix}
\begin{pmatrix} a_n \\
a_{n-1} \\
\vdots \\
a_{m+1}
\end{pmatrix},
\]

where $B$ is a matrix whose entries are sums of terms of the form $D^j(b_i), 0 \leq j \leq n - m$. Therefore $\deg B \leq (n - m + 1)M$. A solution of this system will be unique so the matrix $B$ has rank $n - m$. Therefore there is an $(n - m) \times (n - m)$ invertible submatrix $\tilde{B}$ of $B$. For convenience of notation we shall assume this is formed by the first $n - m$ rows of $B$. We then have

\[
\begin{pmatrix} c_{n-m} \\
c_{n-m-1} \\
\vdots \\
c_0
\end{pmatrix} = \tilde{B}^{-1}
\begin{pmatrix} a_n \\
a_{n-1} \\
\vdots \\
a_{m+1}
\end{pmatrix},
\]

Since $\deg \tilde{B}^{-1} \leq (n - m + 1)\deg \tilde{B} \leq (n - m + 1)^2M$, we have that

\[
\deg
\begin{pmatrix} c_{n-m} \\
c_{n-m-1} \\
\vdots \\
c_0
\end{pmatrix} = \deg
\begin{pmatrix} a_n \\
a_{n-1} \\
\vdots \\
a_{m+1}
\end{pmatrix}
\leq (n - m + 1)^2M + N. \quad \Box
\]
Lemma 2.6. Let \( k = \mathbb{C}(x) \) and \( L, L_2 \) be monic operators in \( \mathcal{D} \) of orders \( n \) and \( m \), respectively.

1. For any \( i, 0 \leq i \leq \text{ord } L \) one can effectively find an integer \( n_i \) such that if \( L = L_1 L_0 \) with monic \( L_1, L_0 \in \mathcal{D} \) and \( \text{ord } L_1 = i \) then \( \text{deg } L_0 \leq n_i \).

2. One can effectively find an integer \( N \) such that if \( L_2 \mathcal{R} = \mathcal{S} L \) for some \( \mathcal{R}, \mathcal{S} \in \mathcal{D} \) with \( \text{ord } \mathcal{R} < \text{ord } L \) and \( \text{ord } \mathcal{S} < \text{ord } L_2 \), then \( \text{deg } \mathcal{S} \leq N \).

3. One can effectively find an integer \( M \) such that if \( L_1 \) is a monic operator equivalent to \( L_2 \), dividing \( L \) on the left, then there exist \( R \) and \( S \) in \( \mathcal{D} \) such that \( L_2 R = SL_1 \), \( \text{ord } R < \text{ord } L_1 \), \( \text{ord } S < \text{ord } L_2 \) and \( \text{deg } R, \text{deg } S \leq M \).

Proof. 1. This fact is well known (cf., Section 5.1 of [3] and also [5]; the latter paper also contains explicit bounds) and so we only outline the proof. Let \( L = L_1 L_0 \) and \( \text{ord } L_1 = n - i \) and let \( \{ y_1, \ldots, y_{n-i} \} \) be a fundamental set of solutions of \( L_1(y) = 0 \). The coefficients of \( L_1 \) are quotients \( a_i w \) where \( w \) is the wronskian determinant of \( \{ y_1, \ldots, y_{n-i} \} \) and \( a \) is the determinant of some \((n - i) \times (n - i)\) submatrix of \( W = (y_j^{(i)})_{i=1}^{n-i} \) (note that \( w \) is also the determinant of such a matrix). Since the logarithmic derivative of \( w \) is in \( k \), we have that the logarithmic derivative of \( a \) is also in \( k \). Furthermore, the determinants of \((n - i) \times (n - i)\) submatrices of \( W \) satisfy an equation \( \mathcal{L}^{n}(y) = 0 \) where \( \mathcal{L}^{n} \) is an operator that can be effectively constructed from \( L \). Therefore the coefficients of \( L_0 \) are quotients of two solutions of \( \mathcal{L}^{n}(y) = 0 \), each of which has logarithmic derivative in \( k \). One can effectively find sets of rational functions \( \{ g_i \}, \{ f_{rs} \} \) such that the elements \( e^{t g_i} \) are algebraically independent over \( k \) and if \( y \) satisfies \( \mathcal{L}^{n}(y) = 0 \), \( y' / y \in k \), then for some \( r \) there are constants \( \{ c_r \} \) such that \( y = (\sum c_r f_r, e^{t g_r}) \). If a quotient of two such elements lies in \( k \) it will be of the form \( \sum d_r f_r / \sum c_r f_r \) for some constants \( c_r, d_r \), and so \( \text{deg } L_0 \) can be bounded in terms of the degrees of the \( f_r \).

2. We will consider the dual equation \( \mathcal{R}^* L_2^* = L^* \mathcal{S}^* \) formed by taking adjoints (the adjoint of an operator \( L = \sum_{-\infty}^{\infty} a_i D^i \) is the operator \( L^* = \sum_{-\infty}^{\infty} (-1)^i D^i a_i \)). [12], Ch. 10). One showed (see [13] for a modern presentation and references) that there exists an \((m \times m)\) matrix \( \mathcal{A} \) whose entries lie in \( \mathcal{D} \) and can be calculated from the coefficients of \( L \) and \( L_2 \) such that \( \mathcal{S}^* = s_0 + s_1 D + \cdots + s_{m-1} D^{m-1} \) satisfies such an equation if and only if \( \mathcal{A}(s_0, \ldots, s_{m-1}) = 0 \). Using row and column operations one can find a set of linear scalar equations equivalent to the system \( \mathcal{A}(s_0, \ldots, s_{m-1}) = 0 \) (see [4]). Using standard algorithms to find rational solutions of scalar linear differential equations one can therefore find a bound on \( \text{deg } \mathcal{S}^* \) and therefore on \( \text{deg } \mathcal{S} = \text{deg } \mathcal{S}^{**} \).

3. Let \( L = L_1 L_0 \). From 1., one can effectively calculate an integer \( n_i \) such that \( \text{deg } L_0 \leq n_i \). Lemma 2.5 then allows us to calculate an integer \( m_i \) such that \( \text{deg } L_1 \leq m_i \). If \( L_2 R = SL_1 \) with \( \text{ord } R < \text{ord } L_1 \) and \( \text{ord } S < \text{ord } L_2 \) then \( L_2 (RL_0) = SL_1 L_0 = SL \). From 2., we have that there is a computable integer \( N \) such that \( \text{deg } S \leq N \). Let \( i = \text{ord } L_2 = \text{ord } L_1 \). Taking adjoints in the equation \( L_2 R = SL_1 \) we have that \( R^* L_2^* = L_1^* S^* \). One can bound \( \text{deg } L_1^* S^* \) in terms of \( \text{deg } L_1^* \) and \( \text{deg } S^* \). Applying Lemma 2.5 to the equation \( R^* L_2^* = L_1^* S^* \), allows us to bound \( \text{deg } R^* \) and therefore \( \text{deg } R \).
Before stating the next lemma, we introduce the following notion. Let $L = \sum_{i=0}^{n} a_i D^i \in \mathcal{O}(x)[D]$ be an operator of order $n$ and degree at most $m$ where each

$$a_i = \frac{\sum_{j=0}^{m} b_{i,j} x^j}{\sum_{j=0}^{m} c_{i,j} x^j}$$

with $b_{i,j}, c_{i,j} \in \mathcal{O}$. We may identify the operator $L$ with the vector $(b_{0,0}, b_{0,1}, \ldots, c_{n-1,m}) \in \mathcal{O}^{2(n+1) \times (m+1)}$. We say that a set of operators $\mathcal{L}$ is constructible if, under this identification, they form a constructible subset of $\mathcal{O}^{2(n+1) \times (m+1)}$.

**Lemma 2.7.** Let $k$, $L$ and $L_2$ be as in the hypotheses of Lemma 2.6.

1. The set of pairs of monic operators $(L_1, L_0)$, $\text{ord} \ L_1 = m$, $\text{ord} \ L_0 = n - m$ such that $L = L_1 L_0$ forms a constructible set whose defining equations can be explicitly computed.

2. Let $n_m$ be as in Lemma 2.6.1 and $M$ be as in Lemma 2.6.3. The set $\mathcal{P}_{L_2}$ of triples of operators $(L_1, R, S)$ where
   2.1. $\text{ord} \ L_1 = m; \text{deg} \ L_1 \leq n_m; \text{ord} \ R, S \leq m - 1; \text{deg} \ R, S \leq M$,
   2.2. $L_1$ divides $L$ on the the left,
   2.3. $\text{GCRD}(L_1, R) = 1$,
   2.4. $L_2 R = S L_1$ (and so $L_1$ is equivalent to $L_2$)
   is constructible. Furthermore, one can effectively calculate the defining equations of $\mathcal{P}_{L_2}$.

3. The set $\mathcal{H}_{L_2}$ of pairs $(L_1, S)$ such that for some $R \in \mathcal{D}$, $(L_1, R, S) \in \mathcal{P}_{L_2}$ is a constructible set. Furthermore, one can effectively calculate the defining equations of $\mathcal{H}_{L_2}$.

**Proof.** 1. By Lemmas 2.5 and 2.6.1, we can effectively bound the degrees of $L_1$ and $L_0$. Comparing the coefficients of $D^i$ in the equation $L = L_1 L_0$ yields defining conditions for the constructible set.

2. Projecting the set defined in 1. yields the set $\mathcal{L}_1$ of $L_1$ that divide $L$ on the left. Therefore this is a constructible set. Lemma 2.6.3 yields a bound on the degrees of $R$ and $S$. Therefore, the set of $(L_1, R, S)$ such that $L_2 R = S L_1$, $L_1 \in \mathcal{L}_1$ is a constructible set. Imposing the condition that GCRD$(L_1, R) = 1$ yields a constructible subset.

3. This follows from the fact that the projection of a constructible set is constructible.

Let $L = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y \in \mathcal{O}(x)[D]$, let $\alpha \in \mathcal{O}$ and let $m_2 = \min_{0 \leq i \leq n} \{ m_i - i \}$ where $m_i$ is the order of $a_i$ at $\alpha$. Let $S_\alpha$ be the set of pairs $(m, \alpha)$ where $m$ is a negative integer, $\alpha \in \mathcal{O}$ and $L((x - \alpha)^m)$ has order at $\alpha$ strictly larger than $m + m_2$. If $(m, \alpha) \in S_\alpha$, then $\alpha$ must be a finite singular point of $L$. Furthermore, for any finite singular point $\alpha$, there is a nonzero polynomial equation (the indicial equation at $\alpha$) such that the set of $m$ with $(m, \alpha) \in S_\alpha$ is precisely the set of roots of this equation. This equation can be effectively determined from the $(x - \alpha)$-adic expansion of the $a_i$. 

In particular, the set $S_i$ is a finite set that can be effectively determined. Similar calculations can be done at infinity (i.e., at 0 for the equation obtained after replacing $x$ by $1/t$ and $d/dx$ by $-t^2 d/d(t)$) to determine a finite set $S_i^\infty$ of positive integers $m$ having the property that $L(x^m)$ has order at infinity strictly larger than $-m + m_\infty$. Here $m_\infty = \min_{0 \leq i \leq n} (\hat{m}_i + i)$ where $\hat{m}_i$ is the order of $a_i$ at infinity.

**Lemma 2.8.** Let $k = \mathcal{C}(x)$, $N$ an integer and $L = y^{(r)} + \cdots + a_0 y \in \mathcal{D}$. The set $\mathcal{V}$ of $(c_0, \ldots, c_N, d_0, \ldots, d_N) \in \mathcal{C}^{2N+2}$ such that

$$L(y) = \frac{c_N x^N + \cdots + c_0}{d_N x^N + \cdots + d_0}$$

has a solution in $k$, is constructible. Furthermore, one can effectively find the defining equations of $\mathcal{V}$.

**Proof.** Although this result can be deduced from the results of [5], we present a direct proof here. We shall show that there is an a priori bound on the degrees of the denominator and numerator of such a solution. Let $f = \frac{f}{g} \in k$, $(p, q) = 1$ be a solution of (6) for some fixed $(c_0, \ldots, c_N, d_0, \ldots, d_N) \in \mathcal{C}^{2N+2}$. We first claim that $q$ divides

$$Q = \prod_{(m, z) \in S_i} (x - z)^{-m_i} \cdot (d_N x^N + \cdots + d_0).$$

Let

$$q = \prod (x - z)^{r_i}$$

be the factorization of $q$. If $(-n, z) \in S_i$, then $(x - z)^{n_i}$ divides $Q$. If $(-n, z) \notin S_i$ then $L(f)$ has a pole of order $n + n$ at $z$. Therefore $(x - z)^{n_i + n}$ divides $d_N x^N + \cdots + d_0$ and so our first claim is proved, in particular, we have a bound for the degree of the denominator of $f$. To bound the degree of the numerator, we expand at infinity. We have that $f = x^{deg p - deg q} +$ smaller powers of $x$. If $deg p - deg q > deg (c_N x^N + \cdots + c_0) - deg (d_N x^N + \cdots + d_0)$, then $(deg p - deg q) \in S_i^\infty$. Therefore, $deg p - deg q < \max_{m \in S_i} (m, N)$, so we can bound the degree of $p$.

Let $M$ be a bound for the degree of the denominators and numerators of possible $f$. The set of $(h_M, \ldots, h_0, k_M, \ldots, k_0, a_N, \ldots, a_0, d_N, \ldots, d_0) \in \mathcal{C}^{2M+2N+4}$ such that

$$L \left( \frac{h_M x^M + \cdots + h_0}{k_M x^M + \cdots + k_0} \right) = \frac{c_N x^N + \cdots + c_0}{d_N x^N + \cdots + d_0}$$

is clearly a constructible set. Projection yields the set $\mathcal{V}$. \(\square\)

We note that the previous lemma is true when $k$ is replaced by an algebraic extension of $\mathcal{C}(x)$ and the parameterized rational function is replaced by a parameterized set of elements from $k$, but for simplicity we have only stated and proved it in the more restricted context. The fact that the set of factorizations of a linear differential operator constitutes a constructible set is well known (cf., [5] and [14]). Since we need more information, we have presented our method above.
We are now in a position to deal with the second of the above tasks and present the complete algorithm to calculate the Galois group of \( L(y) = b \).

**Algorithm 1**

*Input*: A completely reducible \( n \)th order operator \( L \in \mathfrak{C}(x)[D] \) and an element \( b \in \mathfrak{C}(x) \).

*Output*: A set of equations in \( n^2 \) variables defining the Galois group \( G_L \) of \( L(y) = 0 \), an integer \( t \) and a rational homomorphism \( \Phi : G_L \rightarrow \text{GL}_t(\mathfrak{C}) \) such that the Galois group of \( L(y) = b \) is \( \mathfrak{C}^t \simeq G_L \) where the action of \( G_L \) on \( \mathfrak{C}^t \) by conjugation is given by \( \Phi \).

1. Write \( L \) as a least common left multiple of a set \( \mathcal{F} = \{ T_1, \ldots, T_s \} \) of irreducible operators (using, for example, the algorithms in the Appendix of [3]).
2. If \( L = L_1 L_0 \) then complete reducibility implies that \( L_1 \) is equivalent to the least common left multiple of some subset of \( \mathcal{F} \). Fix some subset of \( \mathcal{F} \) and let \( L_2 \) be the least common left multiple of its elements. For this operator, apply Lemma 2.7.3 to construct the set \( \mathcal{H}_{L_2} \).
3. Let \( (L_1, S) \in \mathcal{H}_{L_2} \). Lemma 2.4 implies that \( L_1(y) = b \) has a solution in \( \mathfrak{C}(x) \) if and only if the equation

\[
L_2(y) = S(b)
\]  

(7)

has a solution \( y \in \mathfrak{C}(x) \). Apply Lemma 2.8 to Eq. (7) to determine the set of \( (L_1, S) \) for which this equation has a rational solution.

4. Repeat steps 2. and 3. until one finds an \( L_2 \) of maximal order so that the set \( \mathcal{H}_{L_2} \) of \( (L_1, S) \in \mathcal{H}_{L_2} \) for which the Eq. (7) has a rational solution is nonempty. In this case, Proposition 2.1 implies that there exists a unique \( L_1 \) such that \( (L_1, S) \in \mathcal{H}_{L_2} \) for some \( S \).

5. We write \( L = L_1 L_0 \) and let \( t \) be the order of \( L_0 \). Find defining equations of the Galois group \( G_L \) of \( L \) with respect to a basis of the solution space that contains a basis of the solution space of \( L_0(y) = 0 \) (the results of [3] allow one to do this). In this basis the Galois group will be in block triangular form. Restriction to the space \( W \simeq \mathfrak{C}^t \) (i.e., selecting an appropriate block) yields the desired rational map \( \Phi : G_L \rightarrow \text{GL}_t(\mathfrak{C}) \) that gives the action of \( G_L \) on \( \mathfrak{C}^t \) in \( \mathfrak{C}^t \simeq G_L \). \( \square \)

**Remarks.** 1. One can easily interpret the calculations in Example 2.2 in terms of the method given above. From the factorization \( L = (D - 2x) \circ (D - 2x) \) one sees that all factors of \( L \) are equivalent to \( D - 2x \). All left first order factors are given parametrically by \( D - (2x - \frac{d}{c+d}) \), \( (c,d) \neq (0,0) \). Therefore to decide if \( L \) has a first order left factor \( L_1 \) such that \( L_1(y) = b \) has a rational solution, we must decide if there exists \( (c,d) \neq (0,0) \) such that \( y' - (2x - \frac{d}{c+d})y = b \) has a rational solution. A calculation using Lemma 2.4 implies that this is equivalent to deciding if \( y' - 2xy = (c + dx) \cdot b \) has a rational solution. Lemma 2.8 gives a method to find all \( (c,d) \) such that this latter equation has a rational solution.

2. The results of [3] allow one to construct a presentation of the Picard–Vessiot extension associated to \( L(y) = 0 \). These can be combined with Corollary 2.3 to give a presentation of the Picard–Vessiot extension associated with \( L(y) = b \).
3. Calculating the Galois group of $L_1(L_2(y)) = 0$, $L_1$, $L_2$ completely reducible

We shall show how to reduce this question to the question dealt with in the previous section. This reduction was made by D. Bertrand in [1] in the context of $G$-modules and this section is devoted to making Bertrand's reduction explicit and effective.

Despite the title of this section, we will find it useful to deal with first order systems $Y' + AY = 0$. We begin by recalling certain facts concerning such systems (cf., [3,6]). Let $k$ be a differential field and let $A_1, A_2 \in M_n(k)$, the ring of $n \times n$ matrices over $k$. We say that the systems $Y' + A_1 Y = 0$ and $Y' + A_2 Y = 0$ are equivalent if there exists a matrix $B \in GL(n,k)$ such that $A_2 = -B'A_2 B^{-1} + BA_1 B^{-1}$. Let $K$ be a Picard–Vessiot extension of $k$ with Galois group $G$ and assume that $Y' + A_1 Y = 0$, $Y' + A_2 Y = 0$ both have fundamental sets of solutions in $K^n$. One can show that these two systems are equivalent if and only if their solution spaces in $K^n$ are isomorphic as $G$-modules (in fact, if $Y_1$ is a fundamental solution matrix of $Y' + A_1 Y = 0$ then $Y_2 = BY_1$ is a fundamental solution matrix of $Y' + A_2 Y = 0$). To each scalar equation $L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y_0 = 0$ one can associate a matrix equation $Y' + A_L Y = 0$ where

$$A_L = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_0 & \cdots & \cdots & \cdots & a_{n-1} \end{pmatrix}.$$ 

The equation $L(y) = b$ can then be written as $Y' + A_L Y = B$ where $B = (0,0,\ldots,b)^T$. Conversely if $k$ contains non-constants then any system is equivalent to one of the form $Y' + A_L Y = 0$ [9]. Finally, it can be shown that if $L = L_1 L_2$ where $L_1$ and $L_2$ are completely reducible operators, then $Y' + A_L Y = 0$ is equivalent to a system of the form

$$Y' + \begin{pmatrix} A_2 & C \\ 0 & A_1 \end{pmatrix} Y = 0,$$  

where the $Y' + A_1 Y = 0$ and $Y' + A_2 Y = 0$ are completely reducible (cf., [3], Section 2.3). In fact, $A_1$ may be taken to be $A_L$, $A_2$ may be taken to be $A_L$, and $C$ may be taken to be the $m \times n$ matrix with entry $-1$ in the $m, 1$ position and 0’s everywhere else, where $n$ is the order of $L_1$ and $m$ is the order of $L_2$. The following result is a concrete realization of Lemma 1 and the discussion in Section 2 of [1].

**Lemma 3.1.** If $Y_1$ and $Y_2$ are fundamental solution matrices of $Y' + A_1 Y = 0$ and $Y' + A_2 Y = 0$ respectively then

$$Y = \begin{pmatrix} Y_2 & U Y_1 \\ 0 & Y_1 \end{pmatrix}$$ 

is a fundamental solution matrix of Eq. (8) if and only if $U$ satisfies $U' + (A_2 U - U A_1) = -C$.

**Proof.** Substitute and calculate. □
We note that the equation \( V' + (A_2V - VA_1) = 0 \) arises naturally when studying systems of linear differential equations. If \( Y' + A_1Y = 0 \) and \( Y' + A_2Y = 0 \) are two such systems then the map \( Y \mapsto VY \) with \( V \) an \( m \times n \) matrix with entries in \( k \) maps solutions of \( Y' + A_1Y = 0 \) to solutions of \( Y' + A_2Y = 0 \) if and only if \( V' + (A_2V - VA_1) = 0 \). This allows one to identify solutions of this latter equation with elements of \( \text{HOM}(W_1, W_2) \) where each \( W_i \) is the solutions space of \( Y' + A_iY = 0 \). One can furthermore identify \( \text{HOM}(W_1, W_2) \) with \( W_1^\ast \otimes W_2 \) and rewrite the differential equation \( V' + (A_2V - VA_1) = 0 \) in terms of this identification. We shall now make this explicit.\(^2\)

Let \( V \) be an \( m \times n \) matrix and let \( v_i \) be the \( i \)th column of \( V \). Let \( \tilde{V} \) be the column vector \( (v_1^T, \ldots, v_n^T)^T \). A calculation shows that \( V \) satisfies \( V' + A_2V - VA_1 = 0 \) if and only if \( \tilde{V} \) satisfies \( \tilde{V}' + (-A_1^T \otimes I_m + I_n \otimes A_2)\tilde{V} = 0 \). Furthermore, if \( Y_1 \) and \( Y_2 \) are fundamental solution matrices of \( Y' + A_1Y = 0 \) and \( Y' + A_2Y = 0 \), respectively, then a calculation shows that \( (Y_1^{-1})^T \otimes Y_2 \) is a fundamental solution matrix of \( \tilde{V}' + (-A_1^T \otimes I_m + I_n \otimes A_2)\tilde{V} = 0 \). Denoting the columns of \( C \) by \( c_i \) we will let \( \tilde{C} \) be the column vector \( (c_1^T, \ldots, c_n^T)^T \).

**Lemma 3.2.** Assume that \( k \) contains a nonconstant. Let \( Y' + A_1Y = 0 \) and \( Y' + A_2Y = 0 \) be completely reducible equations and let \( K \) be the Picard–Vessiot extension of \( k \) corresponding to Eq. (8). Let \( F \subset K \) be the Picard–Vessiot extension corresponding to

\[
Y' + \begin{pmatrix} A_2 & 0 \\ 0 & A_1 \end{pmatrix} Y = 0.
\]

Then

1. \( F \) contains the Picard–Vessiot extension \( E \) corresponding to

\[
\tilde{V}' + (-A_1^T \otimes I_m + I_n \otimes A_2)\tilde{V} = 0
\]

and so the Galois group \( G(E/k) \) is a quotient of \( G(F/k) \).

2. \( K = F(\tilde{V}) \) where \( \tilde{V} \) is a solution of

\[
\tilde{V}' + (-A_1^T \otimes I_m + I_n \otimes A_2)\tilde{V} = -\tilde{C}.
\]

3. The Galois group \( G(E(\tilde{V})/k) \) is the semidirect product \( W \rtimes G(E/k) \) of the Galois group \( G(E/k) \) and a vector group \( W \). Furthermore, the Galois group \( G(K/k) \) is the semidirect product \( W \rtimes G(F/k) \) where the action of \( G(F/k) \) on \( W \) is given by composing the quotient map \( G(F/k) \to G(E/k) \) and the action of \( G(E/k) \) on \( W \).

**Proof.** The first two statements follow from the discussion preceding the lemma.

\(^2\)One can formulate this in terms of \( \mathcal{O} \)-modules (although we shall not need this). Let \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) be the \( \mathcal{O} \)-modules associated to \( Y' + A_1Y = 0 \) and \( Y' + A_2Y = 0 \) (see [3]). One can put a natural \( \mathcal{O} \)-module structure on \( \text{HOM}(\mathcal{O}_1, \mathcal{O}_2) \) such that \( V \in \text{HOM}(\mathcal{O}_1, \mathcal{O}_2) \) defines a \( \mathcal{O} \)-module isomorphism if and only if \( V' + (A_1V - VA_1) = 0 \). The \( \mathcal{O} \)-modules \( \text{HOM}(\mathcal{O}_1, \mathcal{O}_2) \) and \( \mathcal{O}_1^\ast \otimes \mathcal{O}_2 \) are isomorphic \( \mathcal{O} \)-modules and the above identification of the two equations \( V' - (A_1V - VA_1) = 0 \) and \( \tilde{V}' + (-A_1^T \otimes I_m + I_n \otimes A_2)\tilde{V} = 0 \) makes explicit what we need from this identification.
To prove the third statement we note that since \( Y' + A_1 Y = 0 \) and \( Y' + A_2 Y = 0 \) are completely reducible equations, the Galois group \( G(F/k) \) is a reductive group. Since the Galois group \( G(E/k) \) is a quotient of \( G(F/k) \) it is also reductive. If \( \tilde{L} \) is a scalar equation such that \( Y' + A_L Y = 0 \) is equivalent to Eq. (11), then Eq. (12) is equivalent to \( \tilde{L}(y) = \tilde{b} \) for some \( \tilde{b} \in k \). Since \( G(E/k) \) is reductive, \( \tilde{L} \) is a completely reducible operator and we can apply Proposition 2.1 to conclude the first part of the third statement.

To prove the second part, we consider the following diagram:

\[
\begin{array}{c}
F(\tilde{V}) = K \\
\downarrow \downarrow \\
F \downarrow \downarrow E(\tilde{V}) \\
\downarrow \\
k
\end{array}
\]

We will first show that \( G(F(\tilde{V})/F) \simeq G(E(\tilde{V}))/E \). To do this it suffices to show that \( F \cap E(\tilde{V}) = E \) (cf., Lemma 5.10 of [8]). Since \( G(E(\tilde{V}))/E \) is abelian (it is the vector group \( W \)), \( F \cap E(\tilde{V}) \) is a Picard–Vessiot extension of \( k \). The Galois group \( G(F \cap E(\tilde{V}))/E \) is a quotient of \( W \) and so is unipotent. Since \( G(F \cap E(\tilde{V}))/E \) is also a quotient of the reductive group \( G(F/k) \) it is also reductive and therefore must be trivial. Therefore \( F \cap E(\tilde{V}) = E \).

Let us denote by \( \tilde{W} \) the Galois group \( G(K/F) \). As we have just shown this is isomorphic to \( W \), the Galois group \( G(E(\tilde{V}))/E \). Since \( G(K/F)/\tilde{W} \simeq G(F/k) \) is reductive, we have that \( \tilde{W} \) is the unipotent radical of \( G(K/k) \). Therefore \( G(K/k) \) is isomorphic to \( \tilde{W} \rtimes P \) where \( P \) is a Levi factor isomorphic to \( G(F/k) \) (via the map that takes a \( \sigma \in G(K/k) \) and restricts to \( F \)) and \( \tilde{W} \) is isomorphic to \( W \) (via the map that takes \( \sigma \in G(K/k) \) and restricts to \( E(\tilde{V}) \)). We now consider the action of \( P \) on \( \tilde{W} \) by conjugation. Let \( P_E \) be the subgroup of \( P \) that leaves \( E \) elementwise fixed. This is a normal subgroup of \( P \) and so is reductive. If \( \sigma \in P_E \), then \( \sigma \) leaves \( E(\tilde{V}) \) invariant and so restriction gives a homomorphism of \( P_E \) to \( G(E(\tilde{V}))/E \). Since this latter group is unipotent, the homomorphism must be trivial. Therefore any element of \( P_E \) leaves \( \tilde{V} \) fixed and so must commute with any element of \( \tilde{W} \). Therefore, the action of \( P \) on \( \tilde{W} \) factors through the action of \( P/P_E \) on \( \tilde{W} \), and this is the same as the action of \( G(E/k) \) on \( W \).

This last result and its proof tell us how to compute the Galois group of Eq. (8) when \( k = \mathcal{E}(x) \).

**Algorithm II**

*Input*: A system of linear differential equations (8) where \( Y' + A_1 Y = 0 \) and \( Y' + A_2 Y = 0 \) are completely reducible with \( A_1 \in M_n(\mathcal{E}(x)) \), \( A_2 \in M_m(\mathcal{E}(x)) \).

*Output*: A system of equations in \( m + n \) variables defining the Galois group \( G(F/k) \subset GL_{m+n}(\mathcal{E}) \) of the Picard–Vessiot extension corresponding to the system (10),
an integer $t$ and a rational homomorphism $\Phi : G(F/k) \to \text{GL}_r(\mathbb{C})$ such that the Galois group of (8) is $\mathcal{G} \cong G(F/k)$ where the action of $G(F/k)$ on $\mathcal{G}$ by conjugation is given by $\Phi$.

1. One first calculates the Galois group $G(F/k)$ of Eq. (10) using the results of [3]. This Galois group will be represented as matrices acting on $\text{diag}(Y_1, Y_2)$ where $Y_1$ is a fundamental solution matrix of $Y' + A_1 Y = 0$ and $Y_2$ is a fundamental solution matrix of $Y' + A_2 Y' = 0$. One can easily calculate the action of $G(F/k)$ on $(Y_1^{-1})^T \otimes Y_2$ and so calculate the Galois group $G(E/k)$ of equation (11) as well as the map $G(F/k) \to G(E/k)$.

2. Find a scalar equation $\hat{L} \left( \hat{y} \right) = 0$ equivalent to the Eq. (11) as well as an element $\hat{b} \in k$ so that Eq. (12) is equivalent to $\hat{L} \left( \hat{y} \right) = \hat{b}$ (an algorithm to do this is presented in [9]; in the examples below ad hoc methods are used). Using the transformation of $Y' + A_1 Y = 0$ to $\hat{V}' + (-A_1^T \otimes I_n + I_m \otimes A_2)\hat{V} = 0$ allows us to calculate the action of $G(E/k)$ on the solution space of $\hat{L} \left( \hat{y} \right) = 0$.

3. Proposition 2.1 allows us to calculate a vector group $W$ so that the Galois group of $\hat{L} \left( \hat{y} \right) = \hat{b}$ (and so of Eq. (12)) is $W \cong G(E/k)$.

4. Lemma 3.2 now tells us that the Galois group of Eq. (8) is the group $W \cong G(F/k)$ where the action of $G(F/k)$ on $W$ (i.e., the homomorphism $\Phi$) can be calculated from the information we have.

\begin{remark}
As in the case of the equation $L(y) = b$, the algorithms of [3] can be combined with the above to give a presentation of the Picard–Vessiot extension corresponding to $L_1(L_2(y)) = 0$.
\end{remark}

We will now give three examples of this method. In these examples we will start with an equation of the form $L_1(L_2(y)) = 0$ with coefficients in $k = \mathbb{C}(x)$. The Galois group $G(F/k)$ that is the Galois group of equation (10) in Lemma 3.2, is the same as the Galois group of $\text{LCLM}(L_1, L_2)$. In the examples we shall apply ad hoc methods to calculate this Galois group. We will then calculate a scalar equation equivalent to the system (11) as well as the matrix $B$ defining this equivalence. This will allow us to find a scalar equation $L(y) = b$ equivalent to the system (12). We then apply the methods of Section 2 to calculate the vector space $W$.

\begin{example}
Consider the equation $L(y) = 0$, where $L = L_1 \circ L_2$, $L_1 = D^2 - x$, $L_2 = D^2 + \frac{1}{x}D + 1$.

The Galois group of this equation is an extension of the Galois group $G_L$ of $\hat{L} = \text{LCLM}(L_1, L_2)$. Since $L_1$ and $L_2$ are both known to have Galois group isomorphic to $\text{SL}_2(\mathbb{C})$ ($L_1$ is a form of Airy’s equation and $L_2$ is a Bessel equation), $G_L$ is a subgroup of $\text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$.

According to ([10], p. 1158), if $G_L$ is a proper subgroup of $\text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$, then there exist a scalar matrix $R = \text{diag}(x, z)$ with entries in a quadratic extension of $\mathbb{C}(x)$ and a matrix $S \in \text{GL}_2(\mathbb{C}(x))$ such that

\[ \text{Wr}(y_1, y_2) = R \cdot S \cdot \text{Wr}(z_1, z_2), \]

where

\[ \text{Wr}(y_1, y_2) = y_1 y_2' - y_1' y_2, \quad \text{Wr}(z_1, z_2) = z_1 z_2' - z_1' z_2, \]

and $'$ denotes differentiation with respect to $x$.
\end{example}
where \( \{y_1, y_2\} \) (resp., \( \{z_1, z_2\} \)) is a basis for a fundamental solution space of \( L_1 \) (resp., \( L_2 \)). Assuming this is the case, one can then show that
\[
\text{Wr}(y_1^2, y_1 y_2, y_2^2) = \hat{R} \cdot \text{Wr}(z_1^2, z_1 z_2, z_2^2)
\]
for some matrix \( \hat{R} \in \text{GL}_2(\mathcal{O}(x)) \). Since \( \{y_1^2, y_1 y_2, y_2^2\} \) (resp., \( \{z_1^2, z_1 z_2, z_2^2\} \)) is a basis for the fundamental solution space of \( L_1^2 \) (resp., \( L_2^2 \)), we see that such an equation holds if and only if \( L_1^2 \) and \( L_2^2 \) are equivalent over \( \mathcal{O}(x) \).

In our case, we claim that \( L_1^2 \) and \( L_2^2 \) are inequivalent (and therefore that \( G_\ell = \text{SL}_2(\mathcal{O}) \times \text{SL}_2(\mathcal{O}) \)). The expanded version of \texttt{DEtools} developed by Mark van Hoeij for MapleV.5 allows one to calculate symmetric powers, LCLM's and a basis of the ring of \( \mathcal{D} \)-module endomorphisms of \( \mathcal{D} \mathcal{D} L \) for an operator \( L \in \mathcal{D} = \mathcal{O}(x)[D] \). Using this we proceed as follows. A calculation shows that
\[
M = \text{LCLM}(L_1^2, L_2^2)
\]
is of order 4. If \( L_1^2 \) and \( L_2^2 \) were equivalent then \( \mathcal{D} \mathcal{D} M \) would be the direct sum of two isomorphic \( \mathcal{D} \)-modules. The endomorphism ring of \( \mathcal{D} \mathcal{D} M \) would therefore have dimension 4. Using the \texttt{eigenring} command in \texttt{DEtools} one sees that this ring has dimension 2 and the desired result follows.

We now consider the equation
\[
\tilde{V}' + H\tilde{V} = -\tilde{C},
\]
where
\[
H = -A_1^T \otimes I_2 + I_2 \otimes A_2,
\]
\[
A_1 = \begin{bmatrix} 0 & -1 \\ -x & 0 \end{bmatrix},
\]
\[
A_2 = \begin{bmatrix} 0 & -1 \\ 1 & x \end{bmatrix},
\]
\[
\tilde{C} = (0, -1, 0, 0)^T.
\]
A cyclic-vector computation shows that the system \( \tilde{V}' + H\tilde{V} = 0 \) is equivalent to \( Z' + KZ = 0 \), where
\[
K = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ g_1 & g_2 & g_3 & g_4 \end{bmatrix},
\]
\[
g_1 = \frac{x^6 + 3x^5 + 3x^4 + 5x^3 + 6x^2 + 3x - 3}{x(x^3 + x^2 + 1)},
\]
\[
g_2 = -\frac{x^6 + 5x^5 + 3x^3 - 7x^2 - 1}{x^3(x^3 + x^2 + 1)},
\]
\[ g_3 = -\frac{2x^3 - 2x^2 + 1}{x^2}, \]
\[ g_4 = -\frac{x^3 - 2}{x(x^3 + x^2 + 1)}. \]

The equivalence is given by the equation \( Z = B\bar{V} \), where

\[
B = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -x & 0 \\
-1 + x & -\frac{1}{x} & -1 & -2x \\
2 + \frac{1}{x} & \frac{3x^2 - x^2 + 2}{x^3} & 3x - x^2 & 0
\end{bmatrix}.
\]

Therefore, the equation \( \bar{V}' + H\bar{V} = -\hat{C} \) is equivalent to

\[ Z' + KZ = -B\hat{C}. \quad (13) \]

(The reader can verify that \( K = -B'B^{-1} + BHB^{-1} \).) Since \( K \) is in companion-matrix form, it is easy to convert (13) into the inhomogeneous scalar equation \( \hat{L}(y) = \hat{b} \), where

\[ \hat{L} = D^4 + g_4 D^3 + g_3 D^2 + g_2 D + g_1 \quad (g_i \text{ as above}), \]
\[ \hat{b} = \frac{x^4 + 2x^3 + x^2 + 4x + 3}{x^3 + x^2 + 1}. \]

Computations using the \texttt{DFactor} and \texttt{ratpols} commands in \texttt{DEtools} show that \( \hat{L} \) is irreducible over \( \mathbb{C}(x) \) and that this equation admits no rational solutions. Thus, the vector space \( W \) referred to in Lemma 3.2(3) is all of \( \mathbb{C}^4 \). We conclude that the Galois group \( G_L \) is \( \mathbb{C}^4 \cong (\text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})). \)

**Example 3.4.** Consider the equation \( L(y) = 0 \), where \( L = L_1 \circ L_2, \)
\( L_1 = D^2 + \frac{1}{2} D + 1, \)
\( L_2 = D^2 - D. \)

The Galois group \( G_L \) of \( L \) is once again an extension of \( G_{L_2} \), the group of \( \hat{L} = \text{LCLM}(L_1, L_2) \). To calculate \( G_{\hat{L}} \) note that the Galois group \( G_{L_1} \) of \( L_1 \) is \( \text{SL}_2 \) and the Galois group \( G_{L_2} \) of \( L_2 \) is the multiplicative group \( \mathbb{C}^\times \). The group \( G_{\hat{L}} \) is a subgroup of \( G_{L_1} \times G_{L_2} \) that projects surjectively onto each factor. The Theorem of [10] implies that, in this case, \( G_{\hat{L}} = G_{L_1} \times G_{L_2} \).

We now consider the equation

\[ \bar{V}' + H\bar{V} = -\hat{C}, \]

where

\[ H = -A_1^T \otimes I_2 + I_2 \otimes A_2, \]
\[ A_1 = \begin{bmatrix} 0 & -1 \\ 1 & \frac{1}{x} \end{bmatrix}, \]
\[
A_2 = \begin{bmatrix}
0 & -1 \\
0 & -1 \\
\end{bmatrix},
\]

\[
\tilde{C} = (0, -1, 0, 0)\top.
\]

A cyclic-vector computation shows that the system \( \tilde{V}' + H \tilde{V} = 0 \) is equivalent to \( Z' + KZ = 0 \), where

\[
K = \begin{bmatrix}
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
g_1 & g_2 & g_3 & g_4 \\
\end{bmatrix},
\]

\[
g_1 = \frac{10x^4 + 5x^3 - 6x^2 + 6x + 3}{x(5x^2 - 3)},
\]

\[
g_2 = -\frac{10x^3 + 15x^2 + 9x + 12}{x(5x^2 - 3)},
\]

\[
g_3 = 3\frac{5x^2 + 5x + 2}{5x^2 - 3},
\]

\[
g_4 = -2\frac{5x^2 + 5x - 3}{5x^2 - 3}.
\]

The equivalence is given by the equation \( Z = B\tilde{V} \), where

\[
B = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
-1 & 1 & \frac{1}{x} & 2 \\
-\frac{1}{x} & -2 & -1 & 3 + \frac{3}{x} \\
\end{bmatrix}.
\]

Therefore, the system \( \tilde{V}' + H \tilde{V} = -\tilde{C} \) is equivalent to

\( Z' + KZ = -B\tilde{C}. \)

(The reader can again verify that \( K = -B' B^{-1} + BHB^{-1}. \)) Conversion to an inhomogeneous scalar equation yields \( \tilde{L}(y) = \tilde{b} \), where

\[
\tilde{L} = D^4 + g_4 D^3 + g_3 D^2 + g_2 D + g_1 \quad (g_i \text{ as above}),
\]

\[
\tilde{b} = -2 + \frac{5x^2 + 5x + 12}{5x^2 - 3}.
\]

Using the \texttt{eigenring} command of \texttt{DDEtools}, one sees that the dimension of the endomorphism ring of \( D/D\tilde{L} \) is two. Since \( \tilde{L} \) is completely reducible, this implies that \( D/D\tilde{L} \) is the direct sum of two nonisomorphic irreducible \( D \)-modules. This furthermore implies that \( \tilde{L} \) has exactly two nontrivial irreducible right (resp., left) factors. A computation using the command \texttt{endomorphism_charpoly} yields two different right factors. From these one calculates the unique left factors and then one can show that for neither of
these left factors \( \tilde{L} \) does the equation \( \tilde{L}(y) = \tilde{b} \) have a rational solution. Since \( \tilde{L}(y) = \tilde{b} \) also has no rational solutions, we conclude that \( G_L \) is \( \mathbb{G}^4 \simeq (SL_2(\mathbb{G}) \times \mathbb{G}^*). 

**Example 3.5.** Consider the equation \( L(y) = 0 \), where \( L = L_1 \circ L_2 \), \( L_1 = \text{LCLM} \), \( (D - 2x, D) \), \( L_2 = D^2 \).

Here it is clear that \( G_L \), the group of \( \tilde{L} = \text{LCLM}(L_1, L_2) \), is \( \mathbb{G}^* \).

We now consider the equation

\[
\vec{V}' + H\vec{V} = -\vec{C},
\]

where

\[
H = -A_1^T \otimes I_2 + I_2 \otimes A_2,
\]

\[
A_1 = \begin{bmatrix}
0 & -1 \\
0 & 2x - \frac{1}{x}
\end{bmatrix},
\]

\[
A_2 = \begin{bmatrix}
0 & -1 \\
0 & 0
\end{bmatrix},
\]

\[
\vec{C} = (0, -1, 0, 0)^T.
\]

A cyclic-vector computation shows that the system \( \vec{V}' + H\vec{V} = 0 \) is equivalent to \( Z' + KZ = 0 \), where

\[
K = \begin{bmatrix}
0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & h_1 & h_2
\end{bmatrix},
\]

\[
h_1 = 2\frac{8x^6 - 12x^4 + 18x^2 + 9}{4x^4 + 3},
\]

\[
h_2 = 4\frac{x(4x^4 - 4x^2 + 3)}{4x^4 + 3}.
\]

The equivalence is given by the equation \( Z = B\vec{V} \), where

\[
B = \begin{bmatrix}
0 & -x & -x & 0 \\
x & -1 & 2x^2 & -x \\
-2x^2 + 1 & 2x & -2x(2x^2 - 1) & 4x^2 \\
2x(2x^2 - 3) & -6x^2 + 3 & 4x^2(2x^2 - 3) & -6x(2x^2 - 1)
\end{bmatrix}.
\]

Therefore, the equation \( \vec{V}' + H\vec{V} = -\vec{C} \) is equivalent to

\[
Z' + KZ = -B\vec{C}.
\]
(The reader can once again verify that $K = -B'B^{-1} + BHB^{-1}$.) In this example, the equivalent inhomogeneous scalar equation is $\ddot{\tilde{L}}(y) = \tilde{b}$, where
\[
\ddot{\tilde{L}} = D^4 + h_2D^3 + h_1D^2,
\]
\[
\tilde{b} = 3 - 6\alpha^2 + 6\frac{4\alpha^4 - 8\alpha^2 - 5}{4\alpha^4 + 3}.
\]

The eigenring command shows that the corresponding endomorphism ring has dimension 10 and yields a basis of this ring. Applying the endomorphism characterpoly to each of these will yield a list of right factors and a simple calculation yields their corresponding left factors. Despite the fact that in this case there is an infinite set of left factors, there is a third order operator
\[
L_0 = D^3 + \frac{8\alpha^6 - 12\alpha^4 + 6\alpha^2 + 3}{x(4\alpha^4 + 3)}D^2 - 2\frac{8\alpha^6 + 3}{x^2(4\alpha^4 + 3)}D + 2\frac{8\alpha^6 + 12\alpha^4 - 6\alpha^2 + 3}{x^3(4\alpha^4 + 3)}
\]
on this list of left factors such that $L_0(y) = \tilde{b}$ admits the rational solution $y = -\frac{1}{2}x(6\alpha^2 + 5)$. Meanwhile, another computation shows $\ddot{\tilde{L}}(y) = \tilde{b}$ admits no rational solutions. We are therefore able to avoid a calculation involving parameterized operators. Thus, we have
\[
G_L = \mathcal{G} \gg \mathcal{G}^*.
\]

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References