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Diff. Eq. test 2 Key

10pts 1a $y'' - 3y' - x' - 10y + 10x = \sin t \Rightarrow y'' = 3y' + x' + 10y - 10x + \sin t$

$x'' + 2x - y = t^2 \Rightarrow x'' = -2x + y + t^2$

let $x_1 = x$, $x_2 = x' = x_1'$, $x_3 = x'' = x_2'$

$y_1 = y$, $y_2 = y' = y_1'$, $y_3 = y'' = y_2'$

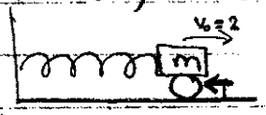
so $y_2' = 3y_2 + x_2 + 10y_1 - 10x_1 + \sin t$, $y_1' = y_2$

$x_2' = -2x_1 + y_1 + t^2$, $x_1' = x_2$

1 so $\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 1 & 10 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ t^2 \\ 0 \\ \sin t \end{bmatrix}$

10pts b $x_p = At^2 + Bt + C + D\cos t + E\sin t$

2 $m = 1/2$, $K = 5$, $B = 3$, $v_0 = 2$, $pos_0 = -1$

5pts @ 

5pts (b) $\frac{1}{2}y'' + 3y' + 5y = f(t)$

$\frac{1}{2}r^2 + 3r + 5 = 0 \Rightarrow r = \frac{-3 \pm \sqrt{9 - 4(1/2)(5)}}{1} = -3 \pm \sqrt{-1} = -3 \pm i$

$y(t) = c_1 e^{-3t}\cos t + c_2 e^{-3t}\sin t$

$y(0) = c_1 = -1$

$y(t) = -e^{-3t}\cos t + c_2 e^{-3t}\sin t$

$y'(t) = 3e^{-3t}\cos t + e^{-3t}\sin t - 3c_2 e^{-3t}\sin t + c_2 e^{-3t}\cos t$

$y'(0) = 3 + c_2 = 2 \Rightarrow c_2 = -1$

$y(t) = -e^{-3t}\cos t - e^{-3t}\sin t$

3pts © $b^2 = 9$ $4mK = 4(\frac{1}{2})(5) = 10$ so $b^2 < 4mK$

underdamped motion

5pts @ $\omega = \sqrt{K/m} = \sqrt{5/1/2} = \sqrt{10} \approx 3.1623$ $\theta = \tan^{-1}(\frac{-1}{-1}) = .7854$

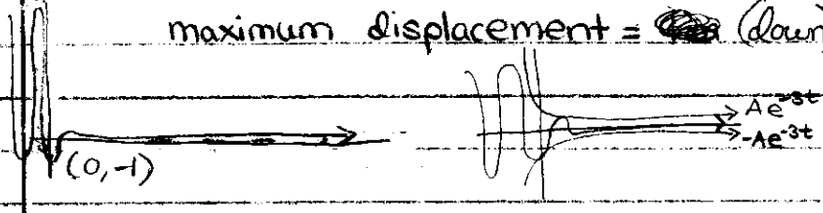
period $= \frac{2\pi}{\sqrt{10}} = .63\pi \approx 1.9869$

frequency $= \frac{\sqrt{10}}{2\pi} = \frac{1.5811}{\pi} \approx .5033$

amplitude $= -\theta/\sqrt{10} \approx -.2484$

maximum displacement = (down) 26.565 sec or .334m

or $y' = 0$



5 pts © $F_0 = 4\cos(3t)$ $y_p = A\cos(\gamma t) + A_2\sin(\gamma t)$

$y_p = \frac{F_0}{(K - m\gamma^2) + B^2\gamma^2}[\sin(\gamma t + \theta)] = \frac{4}{(5 - \frac{1}{2}9)^2 + (9.9)}[\sin(3t + \theta)] = .4438\sin(3t + \theta)$

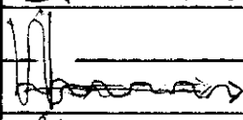
$B_1 = 8/325$, $B_2 = 144/325$

$\tan\theta = \frac{k-m\gamma^2}{b\gamma} = \frac{5-\frac{1}{2}(2)}{3(3)} = \frac{1}{18} \Rightarrow \theta \approx .055499$

so $y_p = .4438(\sin(3t + .055499))$

$y_t = -e^{-3t}\cos t - e^{-3t}\sin t + .4438\sin(3t + .055499)$

It causes it to start oscillating a little at the end



1 ③ $\frac{dx}{dt} = x' = -3x - 2y = -3x_1 - 2y_1$ $\quad f = \begin{bmatrix} t \\ 2e^t \end{bmatrix}$

$\frac{dy}{dt} = y' = 4x + y = 4x_1 + y_1$

5pts ⓐ let $x_1 = x$, $x_2 = x' = x_1'$, $y_1 = y$, $y_2 = y' = y_1'$

$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}' = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} t \\ 2e^t \end{bmatrix}$

5pts ⓑ $A = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix}$

eigenvalues: $-1 \pm 2i$

$-1+2i$: $\begin{bmatrix} -3+1-2i & -2 \\ 4 & 1+1-2i \end{bmatrix} = \begin{bmatrix} -2-2i & -2 \\ 4 & 2-2i \end{bmatrix} = \begin{bmatrix} 1+i & 1 \\ 2 & 1-i \end{bmatrix} = \begin{bmatrix} i & \frac{1}{2}+\frac{1}{2}i \\ 2 & 1-i \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2}-\frac{1}{2}i \\ 2 & 1-i \end{bmatrix}$

$= \begin{bmatrix} 1 & \frac{1}{2}-\frac{1}{2}i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_a \\ u_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u_a + (\frac{1}{2}-\frac{1}{2}i)u_b = 0 \Rightarrow u_a = (-\frac{1}{2}+\frac{1}{2}i)u_b \Rightarrow$ if $u_b = s$

$u_a = -\frac{1}{2}+\frac{1}{2}i$ so $u_1 = s\begin{bmatrix} -\frac{1}{2}+\frac{1}{2}i \\ 1 \end{bmatrix} = s\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + is\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$

let $s = 1$ then $z = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + i\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$

5pts ⓒ gen. sol'n: $x = c_1 e^{\alpha t}(\cos\beta t\, a - \sin\beta t\, b) + c_2 e^{\alpha t}(\sin\beta t\, a + \cos\beta t\, b)$

$\Rightarrow x = c_1 e^{-t}(\cos 2t\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} - \sin 2t\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}) + c_2 e^{-t}(\sin 2t\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \cos 2t\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix})$

6pts ⓓ $x_p = \begin{bmatrix} -1/5 \\ 4/5 \end{bmatrix}t + \begin{bmatrix} 7/25 \\ -8/25 \end{bmatrix} + \begin{bmatrix} -2/3 \\ -4/9 \end{bmatrix}e^t$

5pts ⓔ critical pts: $\begin{cases} -3x - 2y = 0 \Rightarrow -3x = 2y \Rightarrow x = -\frac{2}{3}y \Rightarrow x = 0 \\ 4x + y = 0 \Rightarrow -\frac{2}{3}y + y = 0 \Rightarrow \frac{1}{3}y = 0 \Rightarrow y = 0 \end{cases}$

$(0,0)$

Stable

5pts ④ $\frac{dx}{dt} = -2x$ $\frac{dy}{dt} = y$ critical pt $= (0,0)$

(A) b/c $\frac{dx}{dt}$ is neg when x is pos & $\frac{dy}{dt}$ is pos when y is pos

5pts ② $\frac{dx}{dt} = x(y-1)$ $\frac{dy}{dt} = y(x+1)$ critical pts: $(0,0)$, $(-1,1)$

(C) b/c only plot w/ 2 critical pts

5pts ③ $\frac{dx}{dt} = -x+1$ $\frac{dy}{dt} = x+y$ critical pt: $(1,-1)$

(B) b/c only plot w/ critical pt $(1,-1)$

5pts ④ $\frac{dx}{dt} = -3x-2y$ $\frac{dy}{dt} = 4x+y$ critical pt: $(0,0)$

(D) b/c both are diagonally traveling