Exercises

Nil Hecke

Consider \( V = \mathbb{C}[x_1, x_2, \ldots, x_m] \).

The operator \( x_k \) acts by left mult. Operators \( \sigma \in S_m \) act by permutation:

\[ x_k \sigma(x_1, x_2, \ldots, x_m) = x_{\sigma(k)} \text{ but if } f \in V, \quad \sigma f(x_1, x_2, \ldots, x_m) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)}) \]

Exercise [Verify ?]

Define operators \( \partial_k \) on \( V \) by

\[ \partial_k f = \frac{f - x_k f}{x_k - x_k + 1} \]

Exercise [Verify \( \partial_k f \) is a polynomial].

Exercise [Verify \( \partial_k \) satisfy the braid relations: What is \( \partial_k^2 \)? How do \( \partial_k \) and \( x_k \) interact?]

We say \( \text{NilH}_m = \langle \partial_k | 1 \leq k \leq m, x_k | 1 \leq k \leq m \rangle / \text{relations} \)

Let \( M \) be a finite dim rep of \( \text{NilH}_m \):

\[ \deg(x_k) = 2, \quad \deg(\partial_k) = -2 \]
Exercises

Let $M$ be an irrep of $\text{NilH}_2$.
Can $\dim M = 1$?
If so $M = \text{span} \{ v \}$ and $x_1 v = av$, $x_2 v = bv$, $\delta v = cv$ and what can you say about $a,b,c$?

Let $L(a)$ be the 1-dim rep of $\text{NilH}_1 \otimes \mathbb{C} x_1$ in which $x-a$ vanishes.
Note $L(a) \otimes L(b)$ denotes the 1-dim rep of $\text{NilH}_1 \otimes \text{NilH}_1 \cong \mathbb{C} x_1 \otimes \mathbb{C} x_2$ on which $x_1-a$, $x_2-b$ vanish.

Exercise. Compute a basis of $\text{Ind}_{\text{NilH}_2 \otimes \text{NilH}_2}^{\text{NilH}_1 \otimes \text{NilH}_1} L(a) \otimes L(b)$ and the explicit action of generators of $\text{NilH}_2$ on this basis.

In the case $a=b=0$, you should observe this agrees with $
\frac{V}{V_{S_2}}$

Exercise. What is its graded character?
Compare to quantum shuffle.
Nil Hecke

Exercises

Note symmetric polynomials \( S^m \subseteq V \) generate an \( \text{Nil} H_m \) - invariant subspace \( V_+^m \) so \( V/V_+^m \) is a \( m! \) - dim repr of \( \text{Nil} H_m \).

On which \( (\ell x_r - x_m)^m = Z(\text{Nil} H_m) \) acts trivially (in fact, the unique such repr).

**Exercise** For \( m = 2 \) write out the action of the generators explicitly on some basis.

\( \text{Nil} H_m \subseteq \text{Nil} H_m \) making \( \text{Nil} H_m \) into a free right (or left) \( \text{Nil} H_m \) - module

**Exercise** Write down a basis.

Let's call \( V/V_+^m = L(0^m) \).
Then we can restrict \( \text{Res}_m^{\infty} L(0^m) \) to a \( \text{Nil} H_m \) - module.

**Exercise** What is its structure as a \( \text{Nil} H_m \) - module?

*Hint: Think about the uniqueness of \( L(0^{m+1}) \).

*Does your guess work for \( m = 2 \)?

*Hint: Expand.
Exercises

Claim \( \text{soc} \ Res_{m}^{m+1} L(0^{m}) = L(0^{m+1}) \)

and in particular is irreducible.
(soc stands for "socle" which is the largest semisimple submodule and coincides with the sum of all simple submodules;
dually, the cosocle is the largest semisimple quotient.)

This phenomenon is one big ingredient in defining our crystal operators.

Claim \( \text{cosoc} \ Ind_{m+1}^{m} L(0^{m}) \otimes (0) = L(0^{m+1}) \)

If we declare \( M \) for an irreps of \( \text{Nil} \)

Recall if \( A \otimes B \) are nice algebras
\[ \text{Ind}_{A}^{B} M = B \otimes_{A} M \]
Exercises

KLR-algebra for \( \mathfrak{sl}_2 \)

Dynkin *
\[ I \ell = 1 \] , let's call \( I = 3 \ell \).

The handout/slide has the presentation of \( R(\nu) \).
Write on the generators for
- \( \nu = 0 \)
- \( \nu = i \)
- \( \nu = mi \)
(aka \( \nu = 0 \); \( \nu = \omega \); \( \nu = m \omega^{\pm} \))

Let \( \text{Pol}_m = \text{subalgebra} \text{ generated by } x_1, \ldots, x_m \)

Exercise: What are all the irreps of \( \text{Pol}_m \)?
- (a) for \( m = 1 \);
- (b) for general \( m \)?

Consider \( M = \text{Ind}_{\text{Pol}_m}^{R(\nu)} L(1) \oplus \cdots \oplus L(1) \)

Exercise: What is \( \text{dim} M \)?
1. Write down a basis of \( M \).
2. Do the \( \mathfrak{g} \) act triangulantly on this basis?
3. For \( m = 2 \) with an explicitly how generators act.
   In that case, what is \( \text{Res}_{\text{Pol}_m} M \)?
$\chi^{LR}$ for $SL_2$

Let $M$ be an irreducible $R(m_i)$-module. Set $M = \text{Ind}_{\mathfrak{m}} M \otimes L(i)$.

Declare $i: M = \text{Ind}_{\mathfrak{m}} M \otimes L(i)$.

Build a crystal

$1(0) \rightarrow 1(1) \rightarrow 1(2) \rightarrow 1(3) \rightarrow \cdots$

Exercise: What crystal is this?

Grothendieck groups

If $\mathcal{R} = \text{Rep} R$ = category of fin dim $R$-modules

$G_0(\text{Rep} R)$ = $\mathbb{Z}$-module generated by $[N]$, $N$ = fin dim $R$-mod

and relations $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ exact $\implies [B] = \left[ N \right] + \left[ C \right]$.

If $R$ is graded, we get a $\mathbb{Z}(q, q^*)$-module

In the above, our crystal came from an underlying space $\bigoplus_{m} G_0(\text{Rep} R(m_i))$. 