I A theorem on linear homogeneous 2nd order ODE

Theorem Consider the linear second order homogenous ODE

\[ a(x)y'' + b(x)y' + c(x)y = 0 \quad (**) \]

where \( a(x), b(x) \) and \( c(x) \) are continuous on some interval \( I \), and \( a(x) \) never vanishes on \( I \).

(a) (The representation theorem) Let \( y_1 \) and \( y_2 \) be two linearly independent solutions on \( I \) of (**) . Then every solution \( y \) on \( I \) of (**) can be represented in the form

\[ y = c_1y_1 + c_2y_2 \]

for some constants \( c_1 \) and \( c_2 \).

(b) (The linear independence theorem) Two solutions \( y_1 \) and \( y_2 \) on \( I \) of (**) are linearly independent on \( I \) if and only if the Wronskian \( W(y_1, y_2)(x) \) of \( y_1 \) and \( y_2 \) defined below is not zero for some \( x \in I \).

\[ W(y_1, y_2)(x) = \det \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \]

II Constant coefficients

Assume that \( a(x), b(x) \) and \( c(x) \) in (**) are all constants, with \( a \neq 0 \), so the ODE is now in the form

\[ ay'' + by' + cy = 0 \quad (\ddagger) \]

Substituting \( y = e^{rx} \) into (\ddagger) above and canceling the common factor \( e^{rx} \) we obtain the characteristic equation of the ODE (\ddagger):

\[ ar^2 + br + c = 0 \]

The solutions of this equation are given by the quadratic equation:

\[ r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Case I \( b^2 - 4ac > 0 \). Then we have two real distinct roots \( r_1 \) and \( r_2 \), with \( r_1 \neq r_2 \), and two linearly independent solutions on \( \mathbb{R} \) of (\ddagger) are \( y_1 = e^{r_1x} \) and \( y_2 = e^{r_2x} \). The general solution in this case is

\[ y = c_1e^{r_1x} + c_2e^{r_2x} \]

Case II \( b^2 - 4ac = 0 \). Then we have a repeated root \( r_1 = r_2 \). Two linearly independent solutions on \( \mathbb{R} \) of (\ddagger) are \( y_1 = e^{r_1x} \) and \( y_2 = xe^{r_1x} \). The general solution in this case is

\[ y = (c_1 + c_2x)e^{r_1x} \]

Case III \( b^2 - 4ac < 0 \). Then we have a pair of complex conjugate roots \( r_1 = \alpha + i\beta \) and \( r_2 = \alpha - i\beta \). Two linearly independent solutions on \( \mathbb{R} \) of (\ddagger) are \( y_1 = e^{\alpha x}\cos(\beta x) \) and \( y_2 = e^{\alpha x}\sin(\beta x) \). The general solution in this case is

\[ y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)) \]
The non-homogeneous problem

**Theorem** (The representation theorem) Consider the linear second order non-homogenous ODE

\[ a(x)y'' + b(x)y' + c(x)y = f(x) \quad (** \dagger) \]

where \( a(x), b(x), c(x) \) and \( f(x) \) are continuous on some interval \( I \), and \( a(x) \) never vanishes on \( I \). Let \( y_1 \) and \( y_2 \) be two linearly independent solutions on \( I \) of the associated homogeneous equation \((**\)\) above, and let \( y_p \) be a particular solution\(^1\) of the non-homogeneous equation \((** \dagger)\). Then every solution \( y \) on \( I \) of \((** \dagger)\) can be represented in the form

\[ y = c_1 y_1 + c_2 y_2 + y_p \]

for some constants \( c_1 \) and \( c_2 \).

Finding particular solutions

- **Method of undetermined coefficients**
  See the textbook for the general case.

- **Method of variation of parameters for the general case**
  Let \( y_1 \) and \( y_2 \) be two linearly independent solutions on \( I \) of the homogeneous equation \( a(x)y'' + b(x)y' + c(x)y = 0 \) where \( a(x), b(x) \) and \( c(x) \) are continuous on some interval \( I \), and \( a(x) \) never vanishes on \( I \). Then by division by \( a(x) \) we can put the equation into the form

\[ y'' + p(x)y' + q(x)y = g(x) \]

where \( p(x), q(x) \) and \( g(x) \) are continuous on \( I \). The method of variation of parameters yields a particular solution to this non-homogeneous equation.

**NOTE:** The formulas below required that the ODE be put in the above form, where the coefficient of \( y'' \) is 1.

We assume a solution of the form \( y_p = u_1(x) y_1 + u_2(x) y_2 \). The formulas for \( u_1 \) and \( u_2 \) are

\[
 u_1 = \int \frac{-y_2 g(x)}{W(y_1, y_2)} \, dx \quad u_2 = \int \frac{y_1 g(x)}{W(y_1, y_2)} \, dx
\]

where the convention is that we choose 0 for the additive constants. Then the particular solution is given by

\[
 y_p = y_1 \left( \int \frac{-y_2 g(x)}{W(y_1, y_2)} \, dx \right) + y_2 \left( \int \frac{y_1 g(x)}{W(y_1, y_2)} \, dx \right)
\]

\(^1\)i.e. \( y_p \) contains no arbitrary constants