1. (30 pts) Consider the 1-dim. wave equation with given boundary and initial values:

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0 \]

**The Boundary Conditions**
\[ u(0, t) = 0 = u(L, t), \quad t > 0 \]

**The Initial Conditions**
\[ u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x), \quad 0 < x < L \]

Provide the details of the derivation of the solution to this boundary-value/initial-value problem. You only need to "quote" the solution of the Sturm-Liouville problem that arises in the derivation.

**SOLUTION:** See your notes and the textbook.

2. (30 pts) Consider the following boundary-value/initial-value problem for heat conduction in a rod of length L:

\[ \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0 \]

**The Boundary Conditions**
\[ u(0, t) = T_1, \quad u(L, t) = T_2, \quad t > 0 \]

**The Initial Conditions**
\[ u(x, 0) = f(x), \quad 0 < x < L \]

where \( T_1 \) and \( T_2 \) are positive constants. Explain how to use the steady-state solution \( u_1(x) = \frac{T_2 - T_1}{L} x + T_1 \) to solve this problem by transforming the problem to a new problem involving a new function \( U(x, t) = u(x, t) - u_1(x) \). You are NOT to explicitly solve the BV-IV problems that arise here (as you did in problem 1), but rather you are to explain the general method of solving this problem. It is appropriate to use the phrase "Let xxx be a solution of ....".

**SOLUTION:** Defining \( U(x, t) = u(x, t) - u_1(x) \) we notice first that \( U \) and \( u \) both satisfy the heat equation. Next we examine the boundary conditions that \( U \) satisfies. We find

\[ U(0, t) = u(0, t) - u_1(0) = T_1 - T_1 = 0 \]

and

\[ U(L, t) = u(L, t) - u_1(L) = T_2 - T_2 = 0 \]

Finally we examine the initial condition for \( U \):

\[ U(x, 0) = u(x, 0) - u_1(x) = f(x) - u_1(x) \]

So the BV-IV problem to solve for \( U(x, t) \) is:

\[ \frac{\partial U}{\partial t} = c^2 \frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0 \]

**The Boundary Conditions**
\[ U(0, t) = 0, \quad U(L, t) = 0, \quad t > 0 \]

**The Initial Conditions**
\[ U(x, 0) = f(x) - u_1(x), \quad 0 < x < L \]

This is the standard BV-IV heat equation problem that we have studied. Writing \( U(x, t) = X(x)T(t) \) and using the methods discussed in class leads to two parameterized ODEs for \( X \) and \( T \). The homogeneous boundary conditions lead to a standard SL-problem for \( X \), which we can solve for the
eigenvalues $\lambda_n$ and eigenfunctions $X_n$. Using the eigenvalues in the $T$ equation we can solve for $T_n$ for each $n \geq 1$. Then forming a formal solution of the form

$$U(x, t) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n^2 t)X_n(x)$$

Applying the initial condition $U(x, 0) = f(x) - u_1(x)$ we can then determine the coefficients $a_n$. So let $U(x, t)$ be the solution of this BV problem with homogeneous boundary conditions. Then solving $U(x, t) = u(x, t) - u_1(x)$ for $u(x, t)$ we find

$$u(x, t) = U(x, t) + u_1(x)$$
as the solution to the original BV problem.

3. (10 pts) Let $\{\phi_1(x), \phi_2(x), \phi_3(x), \ldots\}$ be an orthonormal set of functions on the interval $a \leq x \leq b$ with respect to the weight function $U(x)$, and let $\{\psi_1(y), \psi_2(y), \psi_3(y), \ldots\}$ be another orthonormal set of functions on the interval $c \leq y \leq d$ with respect to the weight function $V(y)$. Let $f(x, y)$ have the correct properties so that it possesses the following double generalized Fourier series expansion:

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_{nm} \phi_n(x) \psi_m(y)$$

Use the method discussed in class and in your textbook to derive the formula for the coefficients $E_{nm}$.

**SOLUTION:** According to the statement of the problem we have the following identities:

$$\int_a^b U(x) \phi_m(x) \phi_n(x) dx = \begin{cases} 0 & \text{for } m \neq n \\ 1 & \text{for } m = n \end{cases} \quad \int_c^d V(y) \psi_m(y) \psi_n(y) dy = \begin{cases} 0 & \text{for } m \neq n \\ 1 & \text{for } m = n \end{cases}$$

So fix an integer $k \geq 1$ and multiply both side of the given equation by $U(x)\phi_k(x)$ and integrate the result from $a$ to $b$. We obtain, assuming that we can interchange summation and integration:

$$\int_a^b U(x)f(x, y)\phi_k(x) dx = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_{nm} \left( \int_a^b U(x)\phi_k(x)\phi_n(x) dx \right) \psi_m(y)$$

Performing the summation on $n$ we see that only one term will survive when $n = k$. We obtain

$$\int_a^b U(x)f(x, y)\phi_k(x) dx = \sum_{m=1}^{\infty} E_{km} \psi_m(y)$$

Next fix an integer $l \geq 1$ and multiply both side of the given equation by $V(y)\psi_l(y)$ and integrate the result from $c$ to $d$. We obtain, assuming that we can interchange summation and integration:

$$\int_c^d V(y)\psi_l(y) \int_a^b U(x)f(x, y)\phi_k(x) dx dy = \sum_{m=1}^{\infty} E_{km} \left( \int_c^d V(y)\psi_l(y)\psi_m(y) dy \right)$$

Again by orthonormality the integrals on the right all vanish except for the one when $m = l$. Then we obtain

$$E_{kl} = \int_c^d \int_a^b V(y)\psi_l(y)U(x)f(x, y)\phi_k(x) dx dy$$
4. (10 pts) The time-dependence of solutions of the wave equation involves cosine and sine functions in the form $a_n \sin(\lambda_n t) + b_n \cos(\lambda_n t)$, whereas the time-dependence of solutions of the heat equation involves decaying exponential functions of the form $e^{-\lambda_n t}$. Explain, in a few brief sentences, how this difference comes about.

**SOLUTION:** In both equations we used the assumption $u(x,t) = X(x)T(t)$ and arrived at the following ODE:

- Wave equation: $X'' + \lambda X = 0$, $T'' + \lambda T = 0$
- Heat equation: $X'' + \lambda X = 0$, $T' + \lambda T = 0$

The only difference is that the $T$ ODE is 2nd order for the wave equation and 1st order for the heat equation. In both cases we had homogeneous boundary conditions for $X$ that led to positive eigenvalues $\lambda_n$. Using positive $\lambda_n$ in the $T$ equation for the wave equation leads to sines and cosines, while using $\lambda_n$ in the $T$ equation for the heat equation led to the decaying exponential time dependence for the heat equation.

5. (10 pts) A rectangular membrane, $0 \leq x \leq 3$ and $0 \leq y \leq 4$, has fundamental modes of vibration given by the functions

$$u_{nm} = \sin\left(\frac{n\pi x}{3}\right) \sin\left(\frac{m\pi y}{4}\right)(a_{nm} \cos(\lambda_{nm} t) + b_{nm} \sin(\lambda_{nm} t))$$

Determine the location of the nodal lines for the mode $u_{32}$.

**SOLUTION:** The nodal lines are determined by solving $\sin\left(\frac{n\pi x}{3}\right) \sin\left(\frac{m\pi y}{4}\right) = 0$ for $0 \leq x \leq 3$ and $0 \leq y \leq 4$. For $u_{32}$ we have:

$$\sin\left(\frac{3\pi x}{3}\right) \sin\left(\frac{2\pi y}{4}\right) = 0$$

For $x$ we find:

$$\pi x = \pi, 2\pi, 3\pi, \ldots \implies \text{the } x \text{ nodal lines are at } x = 1, x = 2$$

For $y$ we find:

$$\pi y/2 = \pi, 2\pi, 3\pi, \ldots \implies \text{the single } y \text{ nodal line is at } y = 2$$

6. (10 pts) Consider the following PDE for the unknown function $u = u(x,y)$.

$$x^2 \frac{\partial^2 u}{\partial x \partial y} + 3y^2 u = 0$$

Write $u(x,y) = X(x)Y(y)$ and then use the method of separation of variables to derive the two parameterized ODEs for $X'$ and $Y'$ that are defined by this PDE.

**SOLUTION:** Inserting $u(x,y) = X(x)Y(y)$ into the PDE we get $x^2X'Y' + 3y^2XY = 0$. Dividing both sides by $XY$ we find $x^2 \frac{X'}{X}Y' + 3y^2 = 0$. Moving the term $3y^2$ to the right-hand side of the equation and multiplying by $\frac{X}{Y}$ we obtain

$$x^2 \frac{X'}{X} = -3y^2 \frac{Y'}{Y}$$

As there is only $x$ dependence on the left and $y$ dependence on the right, both sides must be equal to a constant, say $k$. Then we obtain

$$x^2 \frac{X'}{X} = k \quad \text{and} \quad -3y^2 \frac{Y'}{Y} = k \implies x^2X' - kX = 0, \quad kY'' + 3y^2Y = 0$$