1. **(10 points)** Find the general solution of the following Euler differential equation:

\[ 2x^2y'' + 5xy' - 2y = 0 \]

**SOLUTION:** The indicial equation is \(2r(r - 1) + 5r - 2 = 0 \implies 2r^2 + 3r - 2 = 0\). Using the quadratic formula or factoring we find \(r = -2, 1/2\) so the general solution is \(y = A|x|^{-2} + B|x|^{1/2}\).

2. **(15 points)** The Sturm-Liouville theorem states that eigenfunctions \(y_j\) and \(y_k\) corresponding to **distinct** eigenvalues \(\lambda_j \neq \lambda_k\) of a **regular** SL-problem on the interval \([a, b]\) are orthogonal with weight function \(r(x)\). In proving this one eventually arrives at the equation

\[
(\lambda_j - \lambda_k) \int_a^b r(x)y_j(x)y_k(x)dx = p(b)[y_j(b)y'_k(b) - y'_j(b)y_k(b)] - p(a)[y_j(a)y'_k(a) - y'_j(a)y_k(a)]
\]

Finish the proof by supplying the argument that the right hand side vanishes.

**SOLUTION:** The boundary conditions for a regular SL problem on the interval \([a, b]\) are of the form \(c_1y(a) + c_2y'(a) = 0\) and \(d_1y(b) + d_2y'(b) = 0\), where \(c_1\) and \(c_2\) cannot both be zero, and similarly \(d_1\) and \(d_2\) cannot both be zero. Since both \(y_j\) and \(y_k\) are solutions, they both satisfy both of the boundary conditions. First consider the BC at \(x = a\). We have the following set of equations satisfied by \(y_j\) and \(y_k\):

\[
c_1y_j(a) + c_2y'_j(a) = 0 \quad \text{and} \quad c_1y_k(a) + c_2y'_k(a) = 0
\]

Rewriting these as a matrix equation we find

\[
\begin{pmatrix}
y_j(a) & y'_j(a) \\
y_k(a) & y'_k(a)
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

Since \(c_1\) and \(c_2\) cannot both be zero, the determinant of the coefficient matrix must vanish. Computing the determinant we find that its value is \(y_j(a)y'_k(a) - y'_j(a)y_k(a) = 0\). The left hand side of this equation is precisely the term inside the final set of square brackets in the given equation. Hence that last term vanishes. A similar argument using the boundary condition \(d_1y(b) + d_2y'(b) = 0\) with \(d_1\) and \(d_2\) not both be zero leads to the result \(y_j(b)y'_k(b) - y'_j(b)y_k(b) = 0\). The left hand side of this equation is the term inside the first square brackets in the given equation, and so that term vanishes as well, making the right hand side of the given equation equal to zero.

3. **(15 points)** Find the eigenvalues and eigenvectors of the following Sturm-Liouville problem:

\[ y'' + \lambda y = 0 , \quad y(0) = 0 , \quad y(\pi/2) = 0 \]

**SOLUTION:** Case I: \(\lambda = -\omega^2 < 0\), \(\omega > 0\). The general solution of the ODE with \(\lambda\) in this range is \(y = A\sinh(\omega x) + B\cosh(\omega x)\). Then \(y(0) = 0 \implies 0 = B\), so \(y\) reduces to \(y = A\sinh(\omega x)\). Then \(y(\pi/2) = 0 \implies 0 = A\sinh(\omega \pi/2)\). Since \(\omega > 0\) this equation implies that \(A = 0\) since \(\sinh(x)\) is never zero for \(x > 0\). Hence there are no negative eigenvalues.

Case II: \(\lambda = 0\). The general solution is \(y = Ax + B\). Then \(y(0) = 0 \implies 0 = B\), so \(y\) reduces to \(y = Ax\). Then \(y(\pi/2) = 0 \implies 0 = A\pi/2 \implies A = 0\). Hence \(\lambda = 0\) is not an eigenvalue.
Case III: $\lambda = \omega^2 > 0, \omega > 0$. The general solution of the ODE with $\lambda$ in this range is $y = A\sin(\omega x) + B\cos(\omega x)$. Then $y(0) = 0 \implies 0 = B$, so $y$ reduces to $y = A\sin(\omega x)$. Then $y(\pi/2) = 0 \implies 0 = A\sin(\omega\pi/2)$. Since $\sin(x) = 0$ whenever $x = n\pi$, we find $\omega\pi/2 = n\pi \implies \omega_n = 2n$ where $n = 1, 2, \ldots$. Hence the eigenvalues are $\left[\lambda_n = 4n^2\right]$ and the eigenfunctions are $y_n = c_n\sin(2nx)$ where $n = 1, 2, \ldots$ and $c_n \neq 0$.

4. (15 points) Let $f_1 = x^2 - 1$ and $f_2 = x^3 - x$, let the weight function in this problem be $w(x) = 1$, and let the interval be $[-1, 1]$.

(a) Determine whether or not the functions $f_1$ and $f_2$ are orthogonal.

**SOLUTION:** $\int_{-1}^{1} (x^2 - 1)(x^3 - x)dx = \int_{-1}^{1} (x^5 - 2x^3 + x)dx = 0$ since the integrand is an odd function and the integration is over $-1..1$. Hence YES, the functions are orthogonal.

(b) Compute the norm of $f_1$.

**SOLUTION:** $||f_1||^2 = \int_{-1}^{1} (x^2 - 1)^2dx = \int_{-1}^{1} (x^4 - 2x^2 + 1)dx = 2\left(\frac{1}{3} - \frac{2}{3} + 1\right)$. Hence the norm of $f_1$ is $\sqrt{2\left(\frac{1}{3} - \frac{2}{3} + 1\right)} = \sqrt{\frac{16}{15}}$.

5. (15 points) Consider the differential equation $xy'' + 3y' + (x - \lambda)y = 0$ with $x > 0$.

(a) Put the given equation in Sturm-Liouville form.

**SOLUTION:** $\left[(x^3y)' + (x^3 - \lambda x^2)y\right] = 0$.

(b) Classify, as regular, period or singular, the Sturm-Liouville problem given by the above differential equation and the boundary conditions $y(1) = 0$ and $y'(2) = 0$.

**SOLUTION:** regular

(c) Classify, as regular, period or singular, the Sturm-Liouville problem given by the above differential equation and the boundary conditions $y(0) = 0$ and $y(2) + 5y'(2) = 0$.

**SOLUTION:** singular

6. (15 points) Let $f(x) = x$ be defined on the interval $0 \leq x \leq \pi$. Compute the half-range cosine series of $f(x)$ using the appropriate formula(s) from the following list:

$$\int x \sin(ax) \, dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax), \quad \int x \cos(ax) \, dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

**SOLUTION:** $a_0 = \frac{1}{\pi}\int_{0}^{\pi} x \, dx = \frac{1}{\pi} \left[\frac{x^2}{2}\right]_{0}^{\pi} = \frac{\pi^2}{2}$.

For $n \geq 1$ we have

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} x \cos(nx) \, dx = \frac{2}{\pi} \left[\frac{1}{n^2}\cos(nx) + \frac{x}{n^2}\sin(nx)\right]_{0}^{\pi} = \frac{2}{n^2\pi}(-1)^n - 1 = -\frac{2}{n^2\pi}[1 - (-1)^n]$$

Hence the half-range Fourier Series for the given function is $\frac{\pi^2}{2} - \frac{2}{\pi} \sum_{1}^{\infty} \frac{(1 - (-1)^n)}{n^2} \cos(nx)$.

7. (15 points)

(a) State the Fourier Series Representation theorem for a 2p-periodic piecewise smooth function.

**SOLUTION:** See the textbook.
(b) Use the theorem just stated to sketch the Fourier series of the function shown in the figure on the attached page, which is defined by

\[ f(x) = \begin{cases} 
1/x & -3 \leq x < -1 \\
1 + x^2 & -1 < x \leq 2 \\
3 - x & 2 < x \leq 3
\end{cases} \]

Use the set of axes on the sheet with the plot of the function for your sketch.

**SOLUTION:** The graph of the Fourier series of the function looks exactly like the function where it is continuous, and converges to the average of the left and right limits at \( x = -1 \) and at \( x = 2 \). In the plot given below these average values are shown as small dots that look a little like a small + symbol.