Fractional Evolution Equations and Applications

Kazufumi Ito*

October 15, 2013

Abstract

In recent years increasing interests and considerable researches have been given to the fractional differential equations both in time and space variables. These are due to the applications of the fractional differential operators to problems in a wide areas of physics and engineering science and a rapid development of the corresponding theory. Motivating examples include the so-called continuous time random walk process and the Levy process model for the mathematical finance. Basset integral is appearing in the equation of motion of a particle moving through a fluid. A fractional diffusion equation is derived as a homogenization of heterogeneous groundwater flow. In this lecture we develop solution methods based on the linear and nonlinear semigroup theory and apply it to solve the corresponding inverse and optimal control problems. The theory is applied to concrete examples including fractional diffusion equation, Navier-Stokes equations and conservation laws. For the linear case we develop the operator theoretic representation of solutions and the sectorial property of the fractional operator in time is used to establish the regularity and asymptotic of the solutions. The property and stability of the solutions as well as numerical integration methods are discussed. The lecture also covers the basic theory and application of the so-called Crandall-Ligget theory and the DS-approximation theory developed by Kobayashi-Kobayashi-Oharu for evolution operator and the semi-linear theory based on the sectorial estimates of the fractional equation.

1 Introduction

In this monograph we consider the fractional power equation of the form

\[ D_t^\alpha u = \int_0^t (t-s)^{-\alpha} u'(s) \, ds = Au(t) + f(t), \quad u(0) = x \quad \text{for} \quad 0 < \alpha < 1 \]

in a Banach space \( X \), where \( A \) is an \( m \)-dissipative linear or nonlinear operator in \( X \). Here, \( D_t^\alpha \) is the Caputo fractional derivative of order \( \alpha \) with \( 0 < \alpha < 1 \). Our analysis

*Department of Mathematics, North Carolina State University, Raleigh, North Carolina, USA; research was partially conducted while the author was visiting the University of Tokyo
will focus on fractional power equations in time but $A$ may represent the fractional power operator in space such as fractional power Laplacean $(-\Delta)^{\beta}$ (also, see Section ). In general we consider equations of the form

$$
\int_0^t g(t-s)u'(s)\,ds = Au(t) + f(t), \quad u(0) = x, \quad (1.2)
$$

where we assume $g(s) \geq 0$ is monotonically deceasing and integrable on any finite interval $(0, R)$. Equation (1.1) is the special case of with

$$
g(t) = g_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad (1.3)
$$

As shown in Appendix, such an equation arises in the continuous time random walk model for groundwater movement in naturally fractured and heterogeneous porous aquifers [3] and tick-by-tick dynamics of financial markets [5]. Boussinesq (1885) and Basset (1888) found that the force $F$ on an accelerating spherical particle in a viscous fluid is given by

$$
F = \frac{3}{2}D^2 \sqrt{\pi \rho \mu c} \int_0^t \frac{Du(s) - v'(s)}{\sqrt{t-s}} \, ds,
$$

where $D$ is the particle diameter, $\frac{Du}{Ds}$ is the material derivative, and $u$ and $v$ are the fluid and particle velocity vectors, respectively. Thus, one can the Basset equation as

$$
u'(t) + kD^\frac{3}{2} u = Au(t) + f(t), \quad u(0) = x.
$$

Or, in general

$$
g(t) = \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} e^{-\beta t} \, d\mu(\alpha)
$$

where $\mu$ is a positive measure on $(0, 1]$. The exponential decay model is

$$
g(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} e^{-\beta t}
$$

for some $\beta > 0$. Or, in general

$$
g(t) = \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} e^{-\beta t} \, d\tilde{\mu}(\beta)
$$

where $\tilde{\mu}$ is a positive measure on $[0, \infty)$.

Since by the change of variable $t - s = -\theta$

$$
\int_0^t g(t-s)u'(s)\,ds = \int_{-t}^0 g(-\theta)u'(t+\theta)\,d\theta,
$$

we have

$$
\int_0^t g(t-s)u'(s)\,dx = \frac{d}{dt} \int_0^t g(t-s)u(s)\,ds - g(t)u(0). = \frac{d}{dt} \int_{-\infty}^t g(t-s)u(s)\,ds, \quad (1.4)
$$
where we set \( u(s) = x, \ s \leq 0 \). Equivalently,

\[
\int_0^t g(t - s)u'(s) \, dx = \frac{d}{dt} \int_0^t g(t - s)(u(s) - u(0)) \, ds.
\]

Thus, (\text{frac} 1) is written as the fractional differential equation of the Riemann-Liouville form:

\[
\frac{d}{dt} \int_0^t (t - s)^{-\alpha} u(s) \, ds = Au(t) + f(t) + \frac{(t - s)^{-\alpha}}{\Gamma(1 - \alpha)} u(0),
\]

or

\[
\frac{d}{dt} \int_0^t g(t - s)(u(s) - u(0)) \, ds = Au(t) + f(t).
\]

There are several approaches to define the solution to (\text{frac} 1.1). One uses the Mittag-Leffler function \( E_{\alpha,1}(t) \) defined by

\[
E_{\alpha,1}(\lambda t) = \sum_{n=0}^{\infty} \frac{\lambda^n t^{n\alpha}}{\Gamma(n\alpha+1)}.
\]

That is, \( E_{\alpha,1} \) satisfies

\[
D_{t}^{\alpha} E_{\alpha,1}(\lambda t) = \lambda E_{\alpha,1}(\lambda t).
\]

Suppose \( A \) has a spectral resolution

\[
A\phi = \int_C \lambda d\mathcal{E}(\lambda)\phi,
\]

then it can be shown that for \( f = 0 \) the solution to (\text{frac} 1.1) is given by

\[
u(t) = \int_C E_{\alpha,1}(\lambda t) d\mathcal{E}(\lambda) x.
\]

The other approach is the integral equation approach. Let \( J_{t}^{\alpha} \) be the operator defined by

\[
J_{t}^{\alpha} \phi = \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s) \, ds.
\]

Then, \( J_{t}^{\alpha} D_{t}^{\alpha} u = u(t) - x, \ t \geq 0 \) and we obtain the integral equation for \( u; \)

\[
u(t) = x + \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} (Au(s) + f(s)) \, ds,
\]

Thus, one can apply the maximum monotone operator theory to establish the existence and regularity of solutions \( [1, \ 4] \).

Our approach is based on the following observation. One can write (\text{frac} 1.1) as the functional differential equation

\[
\int_{-\infty}^0 g(\theta)u'(t + \theta) \, d\theta = Au(t) + f(t), \quad \text{with initial value } u(\theta) = \phi(\theta), \ \theta \leq 0.
\]

where \( t - s = -\theta \leq 0 \) and \( \theta \in \mathbb{R}^{-} \rightarrow g(\theta) = g(-\theta) \) is the even extension of \( g \) and is monotonically increasing on \( \mathbb{R}^{-} \). Note that if we let \( \phi(\theta) = x, \ \theta \leq 0 \) and
\[ g(\theta) = \frac{1}{\pi (1-\alpha)} |\theta|^{\alpha}; \text{then } (\text{1.11}) \text{ reduces to } (\text{1.11}^\text{frags}). \text{ That is, if } \theta \to u(t+\theta) \text{ is absolute continuous, it follows from (1.7) (see, Section 2.2 for the precise discussions) that} \]

\[ \int_{-\infty}^{0} g(\theta)u'(t+\theta) \, d\theta = \int_{0}^{t} g(t-s)u'(s) \, ds. \]

We then embed the solution \( u(t) = z(t,0) \) in the state (history) space \( Z(t,\theta) = u(t+\theta) \in Z = C([-\infty,0],X). \)

Then, (1.1) has the Markovian form as the evolution equation in \( Z \):

\[ \frac{d}{dt} z(t) = A(t)z(t), \quad (1.7) \]

where the operator \( A(t) \) is defined by

\[ A(t)\phi = \phi'(\theta), \quad \theta \in (-\infty,0] \quad (1.8) \]

in \( Z \) with domain

\[ \text{dom}(A(t)) = \{ \phi \in Z : \phi' \in Z \text{ and } \int_{-\infty}^{0} g(\theta)\phi'(\theta) \, d\theta = A\phi(0)+f(t), \quad \phi(0) \in \text{dom}(A) \}. \]

\[ (1.9) \]

Dynamics (1.1) is embedded in (1.7) as the non-local boundary value condition as \( \theta = 0^+ \) for the first order differential operator \( A(t) \). We analyze the well-posedness and the property of solutions to (1.7) based on the semigroup generated by (1.1), i.e., show that the solution map \( (x,f) \in X \times C(0,T;X) \to u(t) \in C(0,T;X) \) exists and continuous. It will be shown that if \( A \) is dissipative and maximal monotone in \( X \), then \( A(t) \) is dissipative and maximal monotone in \( Z \). We then use the semigroup generation theory to define the solution \( z(t) \in C(0,T;Z) \) to (1.7) and the solution to (1.1) by \( u(t) = z(t,0) \in C(0,T;X) \). In this way we can define the solution to a more general class of equations of the form (1.1). In the case of (1.1) with a closed linear operator \( A \), we have

\[ u(t) = x + A \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) \, ds + \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) \, ds. \]

for all \( x \in X \) and \( f \in C(0,T;X) \), i.e.,

\[ \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) \, ds \in C(0,T;\text{dom}(A)). \]

For the case of a closed linear operator \( A \), we also develop the operator theoretic approach to (1.1) and (1.7). If we take the Laplace transform of (1.1)

\[ \lambda^\alpha \hat{u} = \lambda^\alpha x + \hat{f} \]

and thus

\[ \hat{u} = (\lambda^\alpha I - A)^{-1}(\lambda^\alpha x + \hat{f}) \]

assuming \( A \) is maximal monotone. Here, for \( 0 < \alpha < 1 \) there exists \( \theta > 0 \) such that

\[ |(\lambda^\alpha I - A)^{-1}| \leq \frac{M_\alpha}{|\lambda|^\alpha} \quad \text{on the sector } \Sigma_\theta = \{ z \in C : \arg(z) \leq \frac{\pi}{2} + \theta \} \cap \{ z \neq 0 \}. \]
Thus, we have the representation of the solution operator $u(t) = S(t)x$:

$$S(t)x = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (\lambda^\alpha I - A)^{-1} \lambda^{\alpha - 1} x \, d\lambda.$$ 

where $\Gamma_{\gamma,\delta}$ be the integration path defined by

$$\Gamma^\pm = \{ z \in C : |z| \geq \delta, \arg(z) = \pm (\frac{\pi}{2} + \gamma) \}, \quad \Gamma_0 = \{ z \in C : |z| = \delta, |\arg(z)| \leq \frac{\pi}{2} + \gamma \}.$$ 

For $t > 0$ define $P(t) \in \mathcal{L}(X)$ by

$$P(t)x = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (\lambda^\alpha I - A)^{-1} x \, d\lambda.$$ 

Then, we have the solution representation

$$u(t) = S(t)x + \int_0^t P(t-s)f(s) \, ds.$$  \hspace{1cm} (1.10)  \hspace{1cm} \text{solr}$$

We will analyze the solution properties based on this operator representation of $S(t)$ and $P(t)$ and establish the regularity and asymptotic property as $t \to \infty$ of $S(t)$ and $P(t)$. For the sectorial operator $A$ we analyze the properties of $S(t)$ and $P(t)$ based on the fractional operator calculus.

For the semilinear equation with $Au = A_0u + F(u)$, we define the mild solution by the solution representation;

$$u(t) = S(t)x + \int_0^t P(t-s)(F(u(s) + f(s)) \, ds.$$  \hspace{1cm} (1.11)  \hspace{1cm} \text{semil}$$

where $S(t)$, $P(t)$ are generated by $A_0$. We establish the existence of local and global solutions to (1.11) based on the properties $S(t)$ and $P(t)$ for the general case and the case when $A_0$ is a sectorial operator.

In summary the following is an outline of our presentation.

Plan of the Manuscript

1. Well-posedness of (1.6) using $C_0$-semigroup theory for linear case, Section 2.
2. Well-posedness of (1.6) using Crandall-Ligget theory for nonlinear monotone graph, Section 3.
4. Operator theoretic method and Sectorial Calculus based on the resolvent, Section 5.
5. Solution Representations for Caputo equation and Riemann-Liouville equation, Section 5.
7. Dual system, weak solutions and Control problems, Section 7.
8. Fractional wave equations ($0 < \alpha < 2$), Section 8.
2 Wellposedness for a closed linear operator $A$

In this section we consider the case $A$ is m-dissipative in a Banach space $X$ and show that $A$ defined by (1.8)–(1.9) generates the $C_0$-semigroup on appropriate state spaces $Z$.

2.1 Weighted $Z = L_g(-\infty, 0; H)$ state space

First, we consider $A$ is a closed linear operator in $X$. Let $X = H$ be a Hilbert space. Let $A$ be the linear operator defined by

$$A\phi = \phi'(\theta)$$

on the weighted history space $Z = L^2_g(-\infty, 0; H)$ with norm

$$|\phi|_Z^2 = \int_{-\infty}^{0} g(\theta) |\phi'(\theta)|^2_H d\theta$$

with domain

$$\text{dom}(A) = \{ \phi \in Z : \phi' \in Z \text{ and } \int_{-\infty}^{0} g(\theta) \phi'(\theta) d\theta = A\phi(0), \ \phi(0) \in \text{dom}(A) \}.$$  

The evolution equation for $u$ is embedded into the non-local boundary condition at $\theta = 0^+$ in the domain of the closed operator $A$.

**Theorem 2.1.** Assume $A$ is $m$-dissipative. Then, $A$ is $m$-dissipative, i.e.,

$$R(\lambda I - A) = Z \quad \text{for } \lambda > 0.$$  

Thus, $A$ generates the $C_0$-semigroup on $Z$.

Proof: Define

$$g_\epsilon(\theta) = \frac{1}{\epsilon} \int_{\theta-\epsilon}^{\theta} g(\theta) d\theta$$

For $\phi \in \text{dom}(A)$

$$\langle A\phi, \phi \rangle_Z = \int_{-\infty}^{0} g(\theta)(\phi'(\theta), \phi'(\theta) - \phi(0))_H d\theta + \int_{-\infty}^{0} g(\theta) \phi'(\theta) d\theta, \phi(0) \rangle_H.$$
where
\[
\int_{-\infty}^{0} g(\theta)(\phi'(\theta), \phi(\theta) - \phi(0))_H = \lim_{\epsilon \to 0^+} \int_{-\infty}^{0} g_\epsilon(\theta)(\phi'(\theta), \phi(\theta) - \phi(0))_H \, d\theta
\]
(2.1)
\[
= -\frac{1}{2} \lim_{\epsilon \to 0^+} \int_{-\infty}^{0} g_\epsilon'(\theta)|\phi(\theta) - \phi(0)|^2 \, d\theta \leq 0,
\]
since \(g_\epsilon'(\theta) \geq 0\). Since
\[
(\int_{-\infty}^{0} g(\theta)\phi'(\theta) \, ds, \phi(0))_H = (A\phi(0), \phi(0)) \leq 0
\]
we have \((A\phi, \phi)_Z \leq 0\). If we define for \(\lambda > 0\)
\[
\psi(\theta) = \int_{\theta}^{0} e^{\lambda(\theta - \xi)} f(\xi) \, d\xi,
\]
(2.2)
then
\[
\lambda \psi - \psi' = f \in Z, \quad \psi(0) = 0,
\]
and \(|\psi|_Z \leq \frac{1}{\lambda} |f|_Z\). Let
\[
I = \int_{-R}^{-\delta} g(\theta)\psi'(\theta) \, d\theta = g(-\delta)\psi(-\delta) - g(-R)\psi(-R) - \int_{-R}^{-\delta} g'(\theta)\psi(\theta) \, d\theta.
\]
(2.3)
Since for all \(-\infty < R < \delta < 0\)
\[
|\psi(\theta)| \leq \frac{1}{\sqrt{\lambda g(\theta)}} |f|_Z \quad \text{and} \quad \int_{-R}^{-\delta} \frac{(g')^2}{g} \, d\theta \text{ is bounded},
\]
it follows that for some \(M > 0\)
\[
|\int_{-\infty}^{0} g(\theta)\psi'(\theta) \, d\theta| \leq M |\psi|_Z.
\]
Let
\[
\Delta(\lambda) = \lambda \int_{-\infty}^{0} e^{\lambda \theta} g(\theta) \, d\theta.
\]
(2.4)
Since \(A\) is m-dissipative, \(\lambda \phi - A\phi = f\) has the solution
\[
\phi = (\lambda I - A)^{-1} f = e^{\lambda \theta} \phi(0) + \psi(\theta) \in \text{dom}(A),
\]
where
\[
\phi(0) = (\Delta(\lambda) I - A)^{-1} \int_{-\infty}^{0} g'(\theta)\psi(\theta) \, d\theta.
\]
The theorem now follows from the Lumer-Phillips theorem \([\text{LK} 12]\). \(\square\)
2.2 State space  $Z = C((-\infty, 0]; H)$

Let $X = H$ be a Hilbert space and consider the state space  

$Z = C((-\infty, 0]; H)$.

Define a linear operator $A$ by  

$A \phi = \phi'$

with domain  

$\text{dom}(A) = \{ \phi' \in Z : \phi(0) \in \text{dom}(A), \int_{-\infty}^{0} g(\theta)\phi'(\theta) d\theta = A\phi(0) \}$

Then,  

$$
(\int_{-\infty}^{0} g_\epsilon(\theta)\phi'(\theta) d\theta, \phi(0)) = (\int_{-\infty}^{0} \frac{g(\theta) - g(\theta - \epsilon)}{\epsilon} (\phi(0) - \phi(\theta)) d\theta, \phi(0))
$$

$$
= \int_{-\infty}^{0} \frac{g(\theta) - g(\theta - \epsilon)}{\epsilon} ((\phi(0), \phi(0)) - (\phi(\theta), \phi(0))) d\theta
$$

$$
\geq \int_{-\infty}^{0} \frac{g(\theta) - g(\theta - \epsilon)}{\epsilon} (|\phi(0)|^2 - |\phi(\theta)||\phi(0)|) d\theta
$$

Note that  

$$
\lim_{\epsilon \to 0^+} (\int_{-\infty}^{0} g_\epsilon(\theta)\phi'(\theta) d\theta, \phi(0)) = (A\phi(0), \phi(0))
$$

Suppose $|\phi(0)| > |\phi(\theta)|$, then it follows from (2.5)–(2.6) that  

$$(A\phi(0), \phi(0)) > 0$$

which is a contradiction to the fact that $A$ is dissipative. That is, max $|\phi| = |\phi(\theta_0)|$ for some $\theta_0 < 0$ and  

$$
|\lambda \phi - A\phi|_Z \geq |\lambda \phi(\theta_0) - \phi'(\theta_0)|_Z \geq \lambda |\phi(\theta_0)| = \lambda |\phi|_Z
$$

since $(\phi'(\theta_0), \phi(\theta_0))_H = 0$. Thus, $A$ is dissipative. Moreover, for $\lambda > 0$ from (2.3)  

$$
|I| \leq 2(g(-\delta) + g(-R))|\psi|_Z
$$

and thus $\phi = e^{\lambda \theta}\phi(0) + \psi \in \text{dom}(A)$ satisfies  

$$(\lambda I - A)\phi = f \text{ in } Z,$$

where  

$$
\phi(0) = (\Delta(\lambda) - A)^{-1} \int_{-\infty}^{0} g(\theta) f(\theta) - \lambda \int_{\theta}^{0} e^{\lambda(\theta - \xi)} f(\xi) d\xi d\theta
$$

$$
\phi = (\lambda I - A^\phi)^{-1} f = e^{\lambda \theta} \phi(0) + \int_{\theta}^{0} e^{\lambda(\theta - \xi)} f(\xi) d\xi. \square
$$

Hence, we have

**Theorem 2.2.** Assume $A$ is m-dissipative. Then, $A$ is dissipative and $R(\lambda I - A) = Z$ for $\lambda > 0$. Thus, $A$ generates the $C_0$-semigroup $T(t)$ on $Z = C((-\infty, 0]; H)$. 


2.3 Banach space-valued solution

In this case section we discuss the case when $X$ is a Banach space and $A$ is $m$-dissipative linear operator in $X$. Let $X^*$ is the dual space of $X$, $\langle \cdot, \cdot \rangle_{X \times X^*}$ denote the dual product and $F$ be the duality mapping

$$ F(x) = \{ x^* \in X^* : |\langle x, x^* \rangle| = |x|_X |x^*|_{X^*} \}. $$

Then, $A$ is dissipative if and only if for all $x \in \text{dom}(A)$ there exists a $x^* \in F(x)$, the such that

$$ \langle Ax, x^* \rangle \leq 0 $$

and we assume

$$ \text{range}(\lambda I - A) = X \quad \text{for all } \lambda > 0. $$

Then we have the generation theory:

**Theorem 2.3.** Assume $A$ is $m$-dissipative in a Banach space $X$. Then, $A$ is dissipative and $R(\lambda I - A) = Z$ for $\lambda > 0$. Thus, $A$ generates the $C_0$-semigroup $T(t)$ on $Z = C((-\infty, 0]; X)$. Also, it follows from (2.7) that

$$(\lambda I - A)^{-1}x(0) = (\Delta(\lambda) I - A)^{-1} \Delta(\lambda) x. \quad (2.8)$$

Proof: First we show that $A$ is dissipative. For $\phi \in \text{dom}(A)$ suppose $|\phi(0)| > |\phi(\theta)|$ for all $\theta < 0$. For all $x^* \in F(\phi(0))$

$$ \langle \int_{-\infty}^{0} g_\epsilon(\theta)(\phi')d\theta, x^* \rangle $$

$$ = \langle \int_{-\infty}^{0} g(\theta) - g(\theta - \epsilon) \frac{\epsilon}{\epsilon} (\phi(\theta) - \phi(0)), x^* \rangle d\theta \leq 0 $$

since

$$ \langle \phi(\theta) - \phi(0), x^* \rangle \leq (|\phi(\theta)| - |\phi(0)|) |\phi(0)| < 0, \quad \theta < 0. $$

Thus,

$$ \langle \int_{-\infty}^{0} g(\theta)\phi' d\theta, x^* \rangle < 0. \quad (2.9) $$

But, since there exists a $x^* \in F(\phi(0))$ such that

$$ \langle Ax, x^* \rangle \geq 0 $$

which contradicts to (2.9). Thus, there exists $\theta_0$ such that $|\phi(\theta_0)| = |\phi|_Z$. Since $\langle \phi(\theta), x^* \rangle \leq |\phi(\theta)|$ for $x^* \in F(\phi(\theta_0))$, $\theta \rightarrow \langle \phi(\theta), x^* \rangle$ attains the maximum at $\theta_0$ and thus $\langle \phi'(\theta_0), x^* \rangle = 0$. Hence,

$$ |\lambda \phi - \phi'|_Z \geq (\lambda \phi(\theta_0) - \phi'(\theta_0), x^*) = \lambda |\phi(\theta_0)| = \lambda |\phi|_Z. \quad (2.10) $$

The range condition is exactly the same as the one in the proof of Theorem (2.2) and the theorem follows from the Lumer-Phillips theorem. □
2.4 Mild solution to (\textcolor{red}{\textbf{frag}} 1)

Consider the case with the initial value \( \phi(\theta) = x \in \mathbb{Z} \). It follows from (\textcolor{red}{\textbf{resl}} 2.7) that for \( \mu > 0 \)

\[
(I - \mu A)^{-1} x = \phi_\mu, \quad \phi_\mu(\theta) = x + e^{\theta \phi_\mu(0) - x}
\]

with

\[
\phi_\mu(0) = (\Delta(1/\mu) I - A)^{-1} \Delta(1/\mu) x
\]

since

\[
\lambda \int_0^\theta e^{(\theta - \xi) \mu} d\xi = 1
\]

Note that

\[
\phi_\mu(0) - x = (I - \frac{1}{\Delta(1/\mu)} A)^{-1} \frac{1}{\Delta(1/\mu)} Ax
\]

Thus,

\[
|\phi_\mu(0) - x| \leq \frac{1}{\Delta(1/\mu)} |Ax| X \to 0 \quad \text{as } \mu \to 0
\]

for all \( x \in \text{dom}(A) \). Since \( \phi_\mu \in \text{dom}(A) \), \( z_\mu = u_\mu(t + \cdot) = T(t) \phi_\mu \) is a strong solution to

\[
\frac{d}{dt} z_\mu(t) = Az_\mu(t), \quad z_\mu(0) = \phi_\mu.
\]

That is, \( u_\mu \in C^1((-\infty, T]; X) \) satisfies

\[
\int_0^t g(t - s) u'_\mu(s) \, ds = Au_\mu(t) - \int_{-\infty}^0 g(-t + \theta) \phi'_\mu(\theta) \, d\theta.
\]

where

\[
|\int_{-\infty}^0 g(-t + \theta) \phi'_\mu(\theta) \, d\theta| \leq g(t)|\phi_\mu(0) - x| \to 0 \quad \text{as } \mu \to 0^+.
\]

Since \( D^t_\mu \) is closed on \( C(0, T; X) \), \( u(t) = \lim_{\mu \to 0^+} u_\mu(t) \) satisfies (\textcolor{red}{\textbf{de}} 1.6);

\[
D^t_\mu u = Au(t)
\]

Let \( g^* = \mathcal{L}^{-1}(\frac{1}{\Delta(\lambda)}) \), i.e.,

\[
\int_s^\tau g^*(\tau - t) g(t - s) \, dt = 1. \quad (2.11) \]

Thus,

\[
\int_0^\tau g^*(\tau - s) \int_0^t g(t - s) u'(s) \, ds = \int_0^\tau u'(s) \int_s^\tau g^*(\tau - t) g(t - s) \, dt \, ds = \int_0^\tau u' \, ds = u(\tau) - u(0). \quad (2.12)
\]

Since \( AT(t) = T(t) A \), \( Au_\mu(t) \to Au(t) \) in \( C(0, T; X) \) and the limit \( u(t) = \lim_{\mu \to 0^+} u_\mu(t) \) satisfies

\[
u(t) = x + A \int_0^t g^*(t - s) u(s) \, ds. \quad (2.13) \]

Since \( \text{dom}(A) \) is dense in \( X \) and \( u(t) \in C(0, T; X) \) defines the mild solution to (\textcolor{red}{\textbf{frag}} 1) with \( f = 0 \).
In the case of \( \text{fra} 1.1 \), \( g^*(t) = t^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha) \Gamma(1-\alpha)} \), i.e.,
\[
\int_0^t (t-s)^{\alpha-1} u(s) ds = 1 \quad (2.14)
\]
and we have if \( x \in \text{dom}(A) \), then \( u(t) \in C(0,T; \text{dom}(A)) \) satisfies
\[
u(t) = x + A \int_0^t (t-s)^{\alpha-1} u(s) ds. \quad (2.15)
\]
for \( \text{fra} \) with \( f = 0 \).

**Theorem 2.4.** Equation \( \text{int} 2.13 \) holds for all \( x \in X \), i.e.,
\[
\int_0^t g^*(t-s) u(s) ds \in C(0,T; \text{dom}(A)).
\]

In the case of \( \text{fra} 1.1 \) if \( x \in \text{dom}(A) \), then \( t \to u(t) = S(t)x \in X \) is absolutely continuous and \( \text{int} 2.15 \) holds for all \( x \in X \)

Proof: The first assertion follows since \( \text{dom}(A) \) is dense in \( X \) and \( A \) is a closed operator in \( X \). The second one follows since
\[
u'(t) = \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} Au(s) ds = \int_0^t (t-s)^{\alpha-1} Au'(s) ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} Ax. \square
\]

### 3 Nonlinear Monotone Equations in Banach spaces

In this section we consider a nonlinear fractional inclusion of the form
\[
\int_0^t g(t-s) u'(s) ds \in Au(t).
\]

Let a graph \( A \subset X \times X \) be dissipative, i.e., for any \([x_1, y_1] \in A\) there exists \( x^* \in F(x_1 - x_2) \) such that \( \text{Re}(y_1 - y_2, x^*) \leq 0 \), where \( F : X \rightarrow X^* \) the duality mapping. Or, equivalently
\[
|x - \lambda y| \geq |x| \quad \text{for all} \ \lambda > 0 \ \text{and} \ \langle x, y \rangle \in A.
\]

Define \( A \) in \( Z = C([0, \infty); X) \) by
\[
A\phi = \phi'
\]
with domain
\[
\text{dom}(A) = \{ \phi' \in Z : \phi(0) \in \text{dom}(A), \int_{-\infty}^0 g(\theta) \phi'(\theta) d\theta \in A\phi(0) \}.
\]

**Theorem 3.1.** Assume \( A \) is dissipative and \( \text{Range}(\lambda I - A) \) for all sufficiently small \( \lambda > 0 \). Then, \( A \) is dissipative and \( \text{Range}(\lambda I - A) = Z \) for all sufficiently small \( \lambda > 0 \). Thus \( A \) generates the nonlinear semigroup of contraction on \( Z \).
Proof: For $\phi_1, \phi_2 \in \text{dom}(A)$ suppose $|\phi_1(0) - \phi_2(0)| > |\phi_1(\theta) - \phi_2(\theta)|$ for all $\theta < 0$.
For all $x^* \in F(\phi_1(0) - \phi_2(0))$

$$\langle \int_{-\infty}^0 g_\epsilon(\theta)(\phi_1' - \phi_2') d\theta, x^* \rangle$$

$$= \langle \int_{-\infty}^0 \frac{g(\theta) - g(\theta - \epsilon)}{\epsilon} (\phi_1(\theta) - \phi_2(\theta) - (\phi_1(0) - \phi_2(0)), x^*) d\theta \leq 0$$

since

$$\langle \phi_1(\theta) - \phi_2(\theta) - (\phi_1(0) - \phi_2(0), x^*) \rangle$$

$$\leq (|\phi_1(\theta) - \phi_2(\theta)| - |\phi_1(0) - \phi_2(0)|) |\phi_1(0) - \phi_2(0)| < 0, \quad \theta < 0.$$ Thus,

$$\langle \int_{-\infty}^0 g_\epsilon(\theta)(\phi_1' - \phi_2') d\theta, x^* \rangle < 0. \quad (3.1)$$

But, since for $y_1 \in A\phi_1(0)$, $y_2 \in A\phi_2(0)$ there exists a $x^* \in F(\phi_1(0) - \phi_2(0))$ such that

$$\langle y_1 - y_2, x^* \rangle \geq 0$$

which contradicts to $\langle y_1 - y_2, x^* \rangle < 0$. Thus, there exists $\theta_0$ such that $|\phi_1(\theta_0) - \phi_2(\theta_0)| = |\phi_1 - \phi_2|_{\mathbb{Z}}$ and thus $\langle \phi_1(\theta_0), x^* \rangle = 0$ for all $x^* \in F(\phi_1(\theta_0) - \phi_2(\theta_0))$. Thus,

$$|\lambda (\phi_1 - \phi_2) - (\phi'_1 - \phi'_2)|_{\mathbb{Z}} \geq \langle \lambda (\phi_1(\theta_0) - \phi_2(\theta_0)) - (\phi'_1(\theta_0) - \phi'_2(\theta_0)), x^* \rangle$$

$$= \lambda |\phi_1(\theta_0) - \phi_2(\theta_0)| = \lambda |\phi_1 - \phi_2|_{\mathbb{Z}}. \quad (3.2)$$

For the range condition

$$\lambda \phi - \phi' = f, \quad \int_{-\infty}^0 g(\theta) \phi'(\theta) d\theta \in A\phi(0)$$

we have $\phi = e^{\lambda \theta} \phi(0) + \psi$ and

$$\Delta(\lambda) \phi(0) - \int_{-\infty}^0 g(\theta) \psi'(\theta) d\theta \in A\phi(0)$$

where $\psi$ is defined by $(2.2)$. Since $A$ is m-dissipative,

$$\phi(0) = (\Delta(\lambda) I - A)^{-1} \int_{-\infty}^0 g(\theta) \psi'(\theta) d\theta$$

exists and $\text{range}(\lambda I - A) = \mathbb{Z}$. Thus, the theorem follows from the Carandall and Ligget theorem $(2)$. \quad \Box

### 3.1 Cone preserving

Let $C$ be a closed cone in $X$ and $A$ is cone preserving, i.e.,

$$(I - s A)^{-1} C \subset C$$

for all $s > 0$. 

Theorem 3.2. $\mathcal{A}$ is cone preserving and $T(t)\mathcal{C} \subset \mathcal{C}$.

Proof: From
\[
\phi = (I - \mu A)^{-1} f = e^{\frac{\lambda}{\mu} \theta} \phi(0) + \psi(\theta)
\]
with
\[
\phi(0) = (\Delta \left( \frac{1}{\mu} \right) - A)^{-1} \int_{-\infty}^{0} g'(\theta) \psi(\theta) \, d\theta
\]
\[
\psi(\theta) = \frac{1}{\mu} \int_{0}^{\theta} e^{\frac{\lambda}{\mu}(\theta - \xi)} f(\xi) \, d\xi
\]
Thus, since $g' \geq 0$ if $f \in \mathcal{C}$, then $\phi \in \mathcal{C}$. \( \square \)

4 Nonlinear Fractional Evolution Equations

In this section we consider the case of a class of nonlinear evolution operators $A = A(t)$ for $t \geq t_i$. Let $X$ be a Banach space.

\[
D^q_t u = \int_{0}^{t} g(t - s) u'(s) \, ds \in A(t)u(t), \quad u(0) = x. \tag{4.1}
\]

We assume a family of dissipative operators $A(t) \subset X \times X$, $t \in [0, T]$ satisfy $\text{dom} (A(t)) = X$. Define the operator $A(t)$ in $Z = C((\infty, 0]; X)$ by

\[
A(t)\phi = \phi'
\]
with domain

\[
\text{dom} (A(t)) = \{ \int_{0}^{\theta} g(\theta) \phi(\theta) \, d\theta \in A(t)\phi(0) \}.
\]

For $\lambda > 0$ define the resolvent

\[
J_\lambda(t_i)z = (I - \lambda A(t_i))^{-1} z, \quad z \in Z
\]
and $\phi_i = J_\lambda(A(t_i))z$. Then, $\psi = \phi_1 - \phi_2 \in Z$ satisfies

\[
\Delta \left( \frac{1}{\lambda} \right) \psi(0) = y_1 - y_2
\]
for $y_i \in A(t_i)\phi_i(0)$. We assume there exist a continuous function $f : [0, T] \to X$ and a constant $L > 0$ such that

\[
|\psi(0)| \leq \lambda^a L |f(t_1) - f(t_2)|. \tag{4.2}
\]

Since $\lambda \psi - \psi' = 0$, it follows from (3.2) that

\[
|A_\lambda(t_1)z - A_\lambda(t_2)z| \leq \lambda^{a-1} L |f(t_1) - f(t_2)|, \tag{4.3}
\]
where the Yoshida approximation $A_\lambda(t_1)$ is defined by

\[
A_\lambda(t_i)z = \frac{1}{\lambda}(J_\lambda(t_i)z - z).
\]
For $\lambda > 0$ let $\{z^\lambda_k\}$ be the sequence generated by

$$z^\lambda_k = J_\lambda(t^\lambda_k)z^\lambda_{k-1}, \quad z^\lambda_0 = \phi \in \mathbb{Z}.$$ 

That is, the product formula $z^\lambda_k = \prod_{i=1}^{m} J_\lambda(t_i)z$ defines an approximation sequence and satisfies

$$|\prod_{i=1}^{m} J_\lambda(t_i)z_1 - \prod_{i=1}^{m} J_\lambda(t_i)z_2| \leq |z_1 - z_2|. \quad (4.4)$$

From (8.4)

$$|A_\lambda(t^\lambda_k)z^\lambda_k| = |A_\lambda(t^\lambda_k)J_\lambda(t^\lambda_k)z^\lambda_{k-1}| \leq |A_\lambda(t^\lambda_k)z^\lambda_{k-1}|$$

$$\leq |A_\lambda(t^\lambda_{k-1})z^\lambda_{k-1}| + \lambda^{n-1}L|f(t^\lambda_k) - f(t^\lambda_{k-1})|.$$ 

If we assume $f$ is of bounded variation on $[0, T]$, then, for

$$a_k = |A_\lambda(t_k)z^\lambda_k|, \quad b_k = L|f(t^\lambda_k) - f(t^\lambda_{k-1})|$$

we have

$$a_k = a_{k-1} + \lambda^{n-1}b_k$$

and thus

$$|A_\lambda(t^\lambda_k)| \leq M_0 \lambda^{n-1} \quad (4.5)$$

for all $k$ and $\lambda$.

Let $\lambda = 2^{-n}$, $\mu = 2^{-m}$ and $N = 2^{m-n}$ with $t^\lambda_k = k \lambda$. For $1 \leq j \leq N$ define $z^\mu_{iN+j}$ by

$$z^\mu_{iN+j} = J_\mu(t^\lambda_{(i+1)N})z^\mu_{iN+j-1}.$$ 

It follows from (4.2) that

$$|z^\mu_{iN+j} - z^\mu_{iN+j}| \leq |z^\mu_{iN+j-1} - z^\mu_{iN+j-1}| + \mu^\alpha |f(t^\mu_{(i+1)N}) - f(t^\mu_{iN+j})|L.$$ 

If we assume $f$ is Hölder continuous with order $1 - \alpha + \gamma$, $\gamma > 0$, then

$$\sum_{j=1}^{N} \mu^\alpha |f(t^\lambda_{(i+1)N}) - f(t^\lambda_{iN+j})| \leq \lambda^{2^{m-n} - \gamma m}.$$ 

Define the piecewise constant functions by

$$z_\mu(t) = z^\mu_{iN+j}, \quad \tilde{z}_\mu(t) = z^\mu_{iN+j} \text{ for } t \in [t_{iN+j}, t_{iN+j+1}).$$

Then,

$$|z_\mu(t) - \tilde{z}_\mu(t)| \leq 2^{(1-\gamma)n-m} L \lambda. \quad (4.6)$$

It follows from Theorem 5.3 (Crandall-Liggett theorem) that

$$|z_\lambda(t) - \tilde{z}_\mu(t)| \leq C \lambda |A_\lambda(t^\lambda_{iN})z^\lambda_{iN}|. \quad (4.7)$$

It thus follows from (8.4) that

$$|z_\lambda(t) - \tilde{z}_\mu(t)| \leq \tilde{C} \lambda^\alpha. \quad (4.8)$$

Hence, from (4.6) (8.8) we have

$$|z_\lambda(t) - z_\mu(t)| \leq 2^{(1-\gamma)n-m} L \lambda^{\alpha} + \tilde{C} \lambda^\alpha \leq C \lambda^\alpha.$$
if $m \leq \frac{2 - \alpha}{1 - \gamma} n$. Let us define the sequence \( \{m_k\} \) by

\[
m_k = \left[ \frac{2 - \alpha}{1 - \gamma} m_{k-1} \right], \quad m_0 = n.
\]

Then, by induction in \( k \)

\[
|z_{\mu_k}(t) - z_{\mu_{k-1}}(t)| \leq C 2^{-\alpha m_{k-1}},
\]

where \( \mu_k = 2^{-m_k} \). Since \( m_k - m_{k-1} \geq \frac{1 - \alpha + \gamma}{1 - \gamma} m_0 > 0 \),

\[
|z_{\mu_k} - z_{\mu_0}| \leq M \lambda^\alpha
\]

for some \( M > 0 \) and thus \( \{z_\lambda\} \), \( \lambda = 2^{-n} \) is a Cauchy sequence. Consequently, \( z(t) = \lim_{\lambda \to 0} z_\lambda \) exits in \( C(0, T; Z) \) which defines the solution to (4.1) with \( z(0) = \phi \in \text{dom}(A(0)) \) and

\[
|z_\lambda(t) - z(t)| \leq M \lambda^\alpha.
\]

It follows from (4.1) that

\[
|z_1(t) - z_2(t)| \leq |\phi_1 - \phi_2| \quad \text{for } \phi_1, \phi_2 \in \text{dom}(A(0))
\]

and \( \text{dom}(A(0)) \) is dense in \( Z \), (4.1) has the unique mild solution for all \( x \in X \). In summary we have

**Theorem 4.1.** Assume (A2) and \( f \) is of bounded variation and Hölder continuous of order \( 1 - \alpha + \gamma \), \( \gamma > 0 \) on \( [0, T] \). Then,

\[
z(t) = \lim_{\lambda \to 0} \prod_{i=1}^{[t/\lambda]} J_\lambda(t_i^\lambda) z(0)
\]

exists for all \( z(0) \in X \). Moreover,

\[
|z_\lambda(t) - z(t)| \leq C \lambda^\alpha
\]

for \( z(0) \in \text{dom}(A(0)) \).

The followings are specific cases for which Theorem 4.1 applies. Assume \( A(t) \) is of the form

\[
A(t)u = Au + g(t, u),
\]

where \( A \) is a monotone graph and \( u \to g(t, u) \) is a montone operator. If

\[
|g(t_1, u) - g(t_2, u)|_X \leq L |f(t_1) - f(t_2)| (1 + |u|_X)
\]

then, condition (A.2) holds.

Assume for \( c > 0 \)

\[
|A(0)A(t)x - A(t)A(0)x| \leq c |x|.
\]

\[
|A(t_1)x - A(t_2)x| \leq |f(t_1) - f(t_2)||A(0)x|.
\]

(4.9)
For $F(t)x = A(0)A(t)x - A(t)A(0)x$ we have

$$D^t_0 A(0)x = A(t)A(0)x(t) + F(t) x(t).$$

Thus, if $A(0)x(0) \in X$, one can show that there exists $M > 0$ such that

$$|A(0)x(t)| \leq M, \quad t \in [0, T]. \quad (4.10)$$

Since $\psi(0) = \phi_1(0) - \phi_2(0) \in X$ satisfies

$$\Delta(\lambda)\psi(0) = A(t_2)\psi(0) + (A(t_1) - A(t_2))\phi_1(0),$$

it follows from $(5.15)$–$(5.1)$ that condition $(5.16)$ holds.

5 Operator Theoretic Representation

In this section we develop the solution representation for the linear equation;

$$\int_0^t g(t-s)u'(s) ds = Au(t) + f(t), \quad u(0) = x. \quad (5.1)$$

Assume that $A$ is a closed densely defined linear operator $A$ in the Banach space $X$ and there exist $M \geq 1$ and $\omega_0 \in \mathbb{R}$ such that

$$|\lambda I - A|^{-1} \leq \frac{M}{\text{Re} \lambda - \omega_0} \quad \text{for all } \text{Re} \lambda \omega_0. \quad (5.2)$$

For example, $(5.2)$ holds if $A$ generates a $C_0$ semigroup of $G(M, \omega_0)$ type on $X$. Define the Yoshida approximation $A_\mu \in \mathcal{L}(X)$ of $A$ by

$$A_\mu = A(I - \mu A)^{-1} \quad \text{for } \mu \omega_0 < 1. \quad (5.3)$$

Then, $A_\mu \in \mathcal{L}(X)$ and consider the equation

$$\int_0^t g(t-s)u_\mu'(s) ds = A_\mu u_\mu(t) + f(t), \quad u_\mu(0) = x.$$

Taking the Laplace transform of the equation, we obtain

$$\Delta(\lambda)\hat{u}_\mu - \lambda^{-1}\Delta(\lambda)x = A_\mu \hat{u}_\mu + \hat{f},$$

and thus

$$\hat{u}_\mu = (\Delta(\lambda) I - A_\mu)^{-1} \lambda^{-1}\Delta(\lambda)x + (\Delta(\lambda) I - A_\mu)^{-1}\hat{f}.$$

Let $f = 0$. Since $\text{Re} \sigma(A_\mu) \leq \frac{\omega_0}{1 - \mu \omega_0} < \gamma$, we have

$$u_\mu(t) = S_\mu(t)x = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} (\Delta(\lambda) - A_\mu)^{-1} \lambda^{-1}\Delta(\lambda)x \, d\lambda.$$

Note that

$$\Delta(\lambda)(\Delta(\lambda) I - A)^{-1} = I + (\Delta(\lambda) I - A)^{-1}A \quad (5.4)$$

16
and in general
\[
\Delta(\lambda)(\Delta(\lambda) I - A)^{-1} = \sum_{k=0}^{n-1} \Delta(\lambda)^{-k} A^k + (\Delta(\lambda) I - A)^{-1} \Delta(\lambda)^{-n} A^n. \tag{5.5}
\]
Since
\[
\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{\lambda} \, d\lambda = 1
\]
and
\[
\int_{\gamma-i\infty}^{\gamma+i\infty} |\lambda^{-1} \Delta(\lambda)^{-1}| \, d\lambda < \infty,
\]
we have
\[
|S_\mu(t)x| \leq M |Ax|,
\]
uniformly in \(\mu > 0\). Since from (5.4)
\[
\Delta(\lambda)^{-1}(\Delta(\lambda) I - A)^{-1} = \frac{1}{\lambda} + \lambda^{-1} A(\Delta(\lambda) I - A)^{-1},
\]
if \(\mathcal{L}(\Delta(\lambda)^{-1}) = g^*\) we have
\[
u u_\mu(t) = x + \int_0^t g^*(t-s) A\mu S_\mu(s)x \, ds.
\]
Moreover, since
\[
(\Delta(\lambda) I - A_\mu)^{-1}x - (\Delta(\lambda) I - A)^{-1}x = \frac{\mu}{1 + \mu\Delta(\lambda)}(\nu I - A)^{-1}(\Delta(\lambda) I - A)^{-1}A^2x,
\]
where \(\nu = \frac{\Delta(\lambda)}{1 + \mu\Delta(\lambda)}\), \(\{u_\mu(t)\}\) is Cauchy in \(C(0, T; X)\) provided that \(x \in \text{dom}(A^3)\).

Letting \(\mu \to 0^+\), we obtain

**Theorem 5.1.** For \(x \in \text{dom}(A^2)\)
\[
u u(t) = S(t)x = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}(\Delta(\lambda) I - A)^{-1} \lambda^{-1} \Delta(\lambda)x \, d\lambda
\]
\[
u u(t) = x + \int_0^t g^*(t-s) A\mu S_\mu(s)x \, ds.
\]

**Corollary 5.1.** For \(x \in \text{dom}(A^{n+2})\),
\[
u u(t) = S(t)x = x + \sum_{k=1}^{n-1} \mathcal{L}^{-1}(\lambda^{-1} \Delta(\lambda)^{-k}) A^k x + \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}(\Delta(\lambda) I - A)^{-1} \lambda^{-1} \Delta(\lambda)^{-n} A^nx \, d\lambda,
\]
x where \(\mathcal{L}^{-1}(\cdot)\) is the inverse Laplace transform. In the case of the Caputo \(g = g_{1-\alpha} = \frac{[\theta]_{1-\alpha}}{\Gamma(1-\alpha)}\)
\[
u u(t) = S(t)x = \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k\alpha + 1)} A^k x + \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}(\lambda^\alpha I - A)^{-1} \lambda^{-1-n\alpha} A^nx \, d\lambda,
\]

17
and
\[ u(t) = x + \int_0^t (t-s)^{\alpha-1} AS(t) x \, ds. \]

Proof: The first assertion follows from (5.5). For the Caputo case \( \Delta(\lambda) = \lambda^\alpha \) and \( g^*(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}. \) □

### 5.1 Riemann-Liouville equation

In this section we consider the Riemann-Liouville equation
\[
\frac{d}{dt} \int_0^t g(t-s) u(s) \, ds = Au(t) + f(t) \tag{5.8} \]
with
\[
(\int_0^t g(t-s) u(s) \, ds)(0^+) = y.
\]
Taking the Laplace transform, we have
\[
\Delta(\lambda) \hat{u} = y + A \hat{u} + \hat{f}
\]
and thus
\[
\hat{u} = (\Delta(\lambda) I - A)^{-1} (y + \hat{f}) \tag{5.9} \]
Note that
\[
\Delta(\lambda)(\Delta(\lambda) I - A)^{-1} = I + (\Delta(\lambda) I - A)^{-1} A
\]
Let the linear operator \( P(t) \)
\[
P(t)y = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (\Delta(\lambda) I - A)^{-1} y \, d\lambda,
\]
for \( y \in \text{dom}(A). \) Hence, using the same arguments as above, from (5.9) we have;

**Theorem 5.2.** The solution to (5.8) is given by
\[
u(t) = P(t)y + \int_0^t P(t-s)f(s) \, ds
\]
for \( y \in \text{dom}(A) \) and \( f \in L^2(0,T;\text{dom}(A)), \) and for the case of fractional derivative \( g = g_{1-\alpha} = \frac{|\theta|^{1-\alpha}}{\Gamma(1-\alpha)}, \)
\[
P(t)y = t^{\alpha-1} \sum_{k=0}^{n-1} \frac{t^{\kappa\alpha}}{\Gamma(k\alpha + \alpha)} A^k y + \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} (\lambda^\alpha I - A)^{-1} \lambda^{-n\alpha} A^n y \, d\lambda.
\]
for \( y \in \text{dom}(A^{n+2}). \)
Proof: The second assertion follows from
\[
\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \lambda^{-\alpha} d\lambda = \frac{t^{\alpha-1}}{\Gamma(\alpha)}
\]
and the fact that
\[
(\lambda^\alpha I - A)^{-1} x = \lambda^{-\alpha} x + (\lambda^\alpha I - A)^{-1} \lambda^{-\alpha}
\]
and by induction
\[
(\lambda^\alpha I - A)^{-1} x = \lambda^{-\alpha} \sum_{k=0}^{n-1} \left( \frac{A}{\lambda^\alpha} \right)^k + (\lambda^\alpha I - A)^{-1} \left( \frac{A}{\lambda^\alpha} \right)^n x. \square
\]
Similarly, for (Cap 5.1) the solution is given by
\[
u(t) = S(t)x + \int_0^t P(t-s) f(s) ds \tag{5.10}
\]
for \(x \in \text{dom}(A^2)\) and \(f \in C([0,T];X)\)

**Corollary 5.2.**
\[
\frac{d}{dt} S(t) = P(t)A, \quad S(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} P(s) ds.
\]
Proof: The first assertion follows from the fact that
\[
\lambda^\alpha (\lambda^\alpha I - A)^{-1} = I + (\lambda^\alpha I - A)^{-1} A.
\]
The second one follows simply follows from
\[
S(t)x = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \lambda^{-\alpha-1} (\lambda^\alpha I - A)^{-1} x d\lambda
\]
and
\[
\mathcal{L}^{-1}(\lambda^{-\alpha}) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \square
\]

**5.2 Sectorial Calculus and Asymptotic Estimates**

Let a closed and densely defined linear operator \(A\) satisfy
\[
|\Delta(\lambda)| \leq \frac{M}{\Re \lambda} \quad \text{for all } \Re \lambda > 0 \tag{5.11}
\]
Assume there exits \(\alpha \in (0,1)\) such that for \(0 < c_1 \leq c_2 < \infty\)
\[
c_1 |\lambda^\alpha| \leq |\Delta(\lambda)| \leq c_2 |\lambda^\alpha|, \quad \Re \lambda > \omega_0. \tag{5.12}
\]
We recall \(\Delta(\lambda) = \lambda^\alpha\) for
\[
g = g_{1-\alpha} = \frac{|t|^{-\alpha}}{\Gamma(1-\alpha)}.
\]
Then, for $0 < \alpha < 1$ there exists $\theta_0 > 0$ such that

$$|(\Delta(\lambda) I - A)^{-1}| \leq \frac{M_\alpha}{|\lambda|^{\alpha}} \text{ on } \Sigma_\theta = \{ z \in C : \arg(z) < \frac{\pi}{2} + \theta_0 \} \cap \{ z \neq 0 \}$$

since

$$Re \lambda^\alpha \geq |\lambda|^\alpha \cos(\alpha \theta) \quad \text{for } \lambda = |\lambda| e^{i \theta}.$$ 

Let $\Gamma_{\gamma,\delta}$ be the integration path defined by

$$\Gamma^\pm = \{ z \in C : |z| \geq \delta, \arg(z) = \pm(\frac{\pi}{2} + \theta_0) \}, \quad \Gamma_0 = \{ z \in C : |z| = \delta, |\arg(z)| \leq \frac{\pi}{2} + \theta_0 \}$$

for some $\delta > 0$ and $0 < \theta \leq \theta_0$. Then, the solution map $S(t) : x \in X \to u(t) \in X$ is given by

$$S(t)x = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (\Delta(\lambda) I - A)^{-1} \Delta(\lambda) \lambda^{-1} x \, d\lambda,$$  \hspace{1cm} (5.13) \hspace{1cm} \text{(Rep1)}$$

using the Cauchy integral representation (5.6) and the analytic continuation. On $\Gamma^\pm$

$$|e^{\lambda t} (\Delta(\lambda) I - A)^{-1} \Delta(\lambda) \lambda^{-1}| \leq \frac{M}{r} e^{-r \sin \theta t}$$

for $\lambda = r(cos(\theta) + i \sin(\theta)), \ r \geq \delta$. On $\Gamma_0$

$$|e^{i \lambda t} (\Delta(\lambda) I - A)^{-1} \Delta(\lambda) \lambda^{-1}| \leq \frac{M}{\delta} e^{\delta \sin \phi t}$$

for $\lambda = \delta(cos(\phi) + i \sin(\phi)), \ |\phi| \leq \frac{\pi}{2} + \theta$. Hence (5.13) holds for all $x \in X$. Let

$$P(t) \in L(X), \ t > 0$$

be

$$P(t)x = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (\Delta(\lambda) I - A)^{-1} x \, d\lambda.$$

Since

$$\Delta(\lambda) (\Delta(\lambda) I - A)^{-1} = I + A(\Delta(\lambda) I - A)^{-1},$$

if $x \in dom(A)$, then $u(t) = S(t)x \in C(0, T; X)$ for $x \in dom(A)$ satisfies

$$\frac{d}{dt} S(t)x = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (\Delta(\lambda) I - A)^{-1} \Delta(\lambda) x \, d\lambda = P(t)Ax.$$ 

and if $g^* = L^{-1}(\Delta(A)^{-1})$, then

$$g^*(t)P(t)x \in C(0, T; X).$$

For $x \in X$ and $f \in C(0, T; X)$ we have \textbf{(5.10)}:

$$u(t) = S(t)x + \int_0^t P(t-s)f(s) \, ds$$

for the solution to \textbf{(5.1)}. 

\hspace{1cm}  

20
**Theorem 5.3.** Assume $0 \in \rho(A)$. Then, for some $C > 0$

$$\left| \frac{d}{dt} S(t)x \right| \leq \frac{C}{t} |x|,$$

$$|P(t)x| \leq \frac{C}{t^{1-\alpha}} |x|,$$

$$AS(t)x = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t}((\Delta(\lambda) I - A)^{-1}\Delta(\lambda) - \Delta(\lambda)) \frac{1}{\lambda} x \, d\lambda.$$

**Proof:** Since $0 \in \rho(A)$, one can let $\delta = 0$ for the integral path $\Gamma$ and

$$|\Delta(\lambda)(\Delta(\lambda) I - A)^{-1}| \leq M \text{ on } \Gamma.$$

Thus,

$$\left| \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t}((\Delta(\lambda) I - A)^{-1}\Delta(\lambda)) x \, d\lambda \right| \leq \frac{M}{\pi} \int_{0}^{\infty} e^{-r \sin \theta t} \, dr \leq \frac{M}{\pi \sin \theta t},$$

$$|P(t)| = \left| \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t}(\Delta(\lambda) I - A)^{-1} d\lambda \right| \leq \frac{M}{\pi} \int_{0}^{\infty} r^{-\alpha} e^{-r \sin \theta t} \, dr \leq \frac{M}{\pi} \Gamma(1-\alpha)(\sin \theta t)^{\alpha-1}. \quad \blacksquare$$

### 5.3 Caputo equation and Inverse inequality

Consider the case when

$$g(t-s) = g_{1-\alpha}(t-s) = \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}.$$

**Theorem 5.4.** For $t > 0 \ R(S(t)) \subset R(A^{-1})$. If $R(A^{-1})$ is pre-compact and $S(t)$ is injective, then $R(S(t)) = R(A^{-1})$.

**Proof:** Since

$$A(\lambda^\alpha I - A)^{-1} = \lambda^\alpha (\lambda^\alpha I - A)^{-1} - I,$$

we have

$$AS(t)x = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t}\lambda^{2\alpha-1}(\lambda^\alpha I - A)^{-1} x \, d\lambda - \frac{1}{\Gamma(1-\alpha)} t^{-\alpha} x = Kx - g_{1-\alpha}(t)x$$

where

$$Kx = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t}\lambda^{2\alpha-1}(\lambda^\alpha I - A)^{-1} x \, d\lambda$$

Then, $R(S(t)) \subset R(A^{-1})$. Since

$$AK = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t}\lambda^{3\alpha-1}(\lambda^\alpha I - A)^{-1} x \, d\lambda - \frac{t^{-2\alpha}}{\Gamma(1-2\alpha)} \in \mathcal{L}(X),$$

if $R(A^{-1})$ is pre-compact, $K$ is a compact operator. Moreover, if $S(t)$ is injective, then it follows from the Fredholm alternative theorem that $R(S(t)) = R(A^{-1})$. \quad \blacksquare
Corollary 5.3. For $t > 0$

$$S(t) = -\sum_{k=1}^{n-1} \frac{t^{-k\alpha}}{\Gamma(1-k\alpha)} A^{-k} + \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{-1} (\lambda^\alpha I - A)^{-1} \lambda^{-n} d\lambda. \quad (5.15)$$

For $t > 0$ sufficiently large $AS(t) = K - g_{1-\alpha}(t)I \in \mathcal{L}(X)$ is bounded invertible, i.e., there exists a constant $c$ such that

$$|x| \leq c |AS(t)x|.$$  

Moreover, for $x \in X$

$$|S(t)x| \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)} |A^{-1}x|.$$  

Proof: Equation (5.15) follows from

$$A^n(\lambda^\alpha I - A)^{-1} = \lambda^{n\alpha}(\lambda^\alpha I - A)^{-1} - \sum_{k=0}^{n-1} \lambda^{k\alpha} A^k.$$  

Thus, $g_{1-\alpha}(t) - |K| > 0$ for sufficiently large $t > 0$. and $|(g_{1-\alpha}(t) I - K)^{-1}| \leq (g_{1-\alpha}(t) - |K|)^{-1}$. \qed

Similarly, we have

Corollary 5.4. For $t > 0$

$$|AS(t)| \leq \frac{M}{\Gamma(1-\alpha)} t^{-\alpha}$$

$$|P(t)| \leq \frac{M}{\Gamma(\alpha)} t^{-(1-\alpha)}.$$  

$$|AP(t)| \leq M t^{-1}$$

Moreover, assume $A$ is a sectorial operator, i.e., there exist $M > 0$, $\theta_0 > 0$ such that

$$|(z I - A)^{-1}| \leq \frac{M}{|z|} \quad \text{on} \quad \Sigma_{\theta_0} = \{z \in \mathbb{C} : \arg(z) \leq \frac{\pi}{2} + \theta_0\} \cap \{z \neq 0\}$$

Then,

$$A^\beta x = \int_{\Gamma} \lambda^\beta (z I - A)^{-1} x$$

and for $0 \leq \beta \leq 1$

$$|A^\beta P(t)| \leq \frac{M}{t^{1-\alpha+\beta\alpha}}. \quad (5.16)$$

Proof: It follows from Eq (5.14). The last assertion uses $|A^\beta x| \leq M |Ax|^\beta |x|^{1-\beta}$ for $x \in \text{dom}(A)$. \qed
5.4 Basset equation

Consider the Basset equation

\[ u'(t) + kD^\alpha_t u = Au(t) + f(t), \quad u(0) = x. \]

In this case

\[ \Delta(\lambda) = \lambda + k\lambda^\alpha. \]

Assume \( A \) a sectorial operator so that (5.13) holds. Since

\[ A(\Delta(\lambda) - A)^{-1}\lambda^{-1}\Delta(\lambda) = (\Delta(\lambda) - A)^{-1}\lambda^{-1}\Delta(\lambda)^2 - (1 + k\lambda^{-1})I \]

we have

\[ AS(t)x = Kx - kg_1\alpha(t)x, \]

where

\[ Kx = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t}\lambda^{-1}\Delta(\lambda)^2(\Delta(\lambda)I - A)^{-1}x d\lambda. \]

Moreover, we have the estimate:

\[ |K| \leq M ((t\sin \theta)^{-1} + k\Gamma(\alpha)(t\sin \theta)^{-\alpha}). \]

Thus, if \( A^{-1} \) is compact, Theorem 5.4 holds.

6 Series Solution

Define the operator:

\[ J^\alpha_t \phi = \int_0^t (t-s)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \phi(s) ds \]

Since \( J^{1-\alpha}_t J^\alpha_t = J^1_t \), we have

\[ D^\alpha_t J^\alpha_t \phi = \phi \quad \text{and} \quad J^\alpha_t D^\alpha_t \phi = \phi - \phi(0). \]

Thus, (6.1) is equivalently written as

\[ u(t) = x + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Au(s) ds \quad (6.1) \]

Thus,

\[ S(t)x = x + A \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S(s)x ds. \quad (6.2) \]

Since \( Au \in C(0,T;H) \) for \( x \in \text{dom}(A) \) it follows from (6.1) that

\[ \lim_{t \to 0^+} \frac{u(t) - x}{t^\alpha} = \frac{1}{\Gamma(\alpha + 1)} Ax \]

one has the first order approximation

\[ u(t) \sim (I - \frac{t^\alpha}{\Gamma(\alpha + 1)} A)^{-1}x. \]
Moreover, we have
\[
\lim_{t \to 0^+} \frac{u(t) - x - \frac{1}{\Gamma(\alpha + 1)} A x}{t^{2\alpha}} = \frac{1}{\Gamma(2\alpha + 1)} A^2 x
\]
we look for an second order approximation of the form
\[
u(t) \sim (I + b_1 t^\alpha A)^{-1}(I + c_1 t^\alpha A)x
\]
with
\[
c_1 - b_1 = \frac{1}{\Gamma(\alpha + 1)}, \quad b_1^2 - c_1 b_1 = \frac{1}{\Gamma(2\alpha + 1)}.
\]
Thus, we obtain
\[
b_1 = -\frac{\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)}, \quad c_1 = \frac{1}{\Gamma(\alpha + 1)} - \frac{\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)}.
\]
In general we can find the Pade\((n,n)\) approximation of the form
\[
u(t) \sim \left(\sum_{k=0}^n b_k t^{k\alpha} A^k\right)^{-1}x + \left(\sum_{k=1}^n a_k t^{k\alpha} A^k\right)x.
\]
That is, since
\[
E_{\alpha,1}(\lambda t) = \sum_{j=0}^\infty c_j \lambda^j t^{j\alpha}, \quad c_j = \frac{1}{\Gamma(j\alpha + 1)}.
\]
we have
\[
\left(\sum_{k=0}^n b_k t^{k\alpha}\right)\left(\sum_{j=0}^n c_j t^{j\alpha}\right) = \sum_{\ell=0}^n a_\ell t^{\ell\alpha}
\]
in term by term, i.e.,
\[
a_\ell = \sum_{k=0}^n b_k c_{\ell-k}, \quad 0 \leq \ell \leq n + n.
\]
Thus, \(\{b_k\}\) satisfies
\[
\begin{pmatrix}
c_n & c_{n-1} & \ldots & c_0 \\
c_{n+1} & c_{n-1} & \ldots & c_1 \\
\vdots & \vdots & \ddots & \vdots \\
c_{n+n} & c_{n-1} & \ldots & c_n
\end{pmatrix}
\begin{pmatrix}
b_0 \\
b_1 \\
\vdots \\
b_n
\end{pmatrix} =
\begin{pmatrix}
a_n \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]

Next, we consider the series expansion in terms of the resolvent \((\lambda^\alpha I - A)^{-1}\). The Post-Widder inversion theory is given as:

**Theorem 6.1.** Let \(u(t)\) be a \(X\)-valued continuous function on \(t \geq 0\) such that \(u(t) = O(e^{\gamma t})\) as \(t \to \infty\) for some \(\gamma\) and \(\hat{u}\) be the Laplace transform of \(u(t)\). Then,
\[
u(t) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \left(\frac{\partial^n}{\partial \lambda^n} \hat{u}\right)\left(\frac{n}{t}\right),
\]
uniformly on any compact sets of \(t > 0\).
Theorem 6.2. If $A$ generates a $C_0$ semigroup on $X$, then

$$\frac{\partial^n}{\partial \lambda^n} (\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}) = (-1)^{(n+1)} \lambda^{-(n+1)} \sum_{k=1}^{n+1} b_{k,n+1}^\alpha (\lambda^\alpha (\lambda^\alpha I - A)^{-1})^k$$

where $b_{k,n}^\alpha$ are given by the recurrence

$$b_{k,n}^\alpha = (n - k\alpha) b_{k,n-1}^\alpha + \alpha(k-1) b_{k-1,n-1}^\alpha, \quad 1 \leq k \leq n, \quad n \geq 2$$

with $b_{1,1}^\alpha = 1$ and $b_{k,n}^\alpha = 0$, $k > n$, and

$$S(t)x = \lim_{n \to \infty} \frac{1}{n!} \sum_{k=1}^{n+1} b_{k,n+1}^\alpha (I - (\frac{t}{n})^\alpha A)^{-k} x$$

where the convergence is uniform on bounded intervals of $t \geq 0$.

Proof: By induction in $n$ we have

$$\frac{\partial^n}{\partial \lambda^n} (\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}) = (-1)^n \lambda^{-(n+1)} \sum_{k=1}^{n+1} b_{k,n+1}^\alpha (\lambda^\alpha (\lambda^\alpha I - A)^{-1})^k$$

Since $b_{k,n}^\alpha > 0$ for $\alpha \in (0,1)$, we have

$$|\frac{\partial^n}{\partial \lambda^n} (\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1})| \leq \sum_{k=1}^{n+1} b_{k,n+1}^\alpha \lambda^{k\alpha-n-1} |(\lambda^\alpha I - A)^{-k}|$$

$$\leq M \sum_{k=1}^{n+1} b_{k,n+1}^\alpha \lambda^{k\alpha-n-1} (\lambda^\alpha - \omega)^k = M(-1)^n \frac{\partial^n}{\partial \lambda^n} (\frac{\lambda^{\alpha-1}}{\lambda^\alpha - \omega})$$

Since

$$\frac{\lambda^{\alpha-1}}{\lambda^\alpha - \omega} = \int_0^\infty e^{-\lambda t} E_{\alpha,1}(t) dt,$$

$$(-1)^n \frac{\partial^n}{\partial \lambda^n} (\frac{\lambda^{\alpha-1}}{\lambda^\alpha - \omega}) = \int_0^\infty t^n e^{-\lambda t} E_{\alpha,1}(t) dt \leq C \frac{n!}{(\lambda - \omega^\frac{1}{\alpha})^{n+1}}$$

since $E_{\alpha,1}(\omega^\alpha) \leq C e^{\omega^\alpha t}$. Thus,

$$|\frac{\partial^n}{\partial \lambda^n} (\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1})| \leq CM \frac{n!}{(\lambda - \omega^\frac{1}{\alpha})^{n+1}}$$

Since $u(t) = S(t)x$ is continuous function on $t \geq 0$ and $u(t) = O(e^{\gamma t}$ as $t \to \infty$, the theorem follows from the Post-Widder inversion theory. $\square$

Similarly, we have

Corollary 6.1.

$$P(t) = \frac{d}{dt} S(t)A^{-1} = \lim_{n \to \infty} \frac{t^{\alpha-1}}{n!} \sum_{k=1}^{n+1} \frac{k}{n^k} b_{k,n+1}^\alpha (I - (\frac{t}{n})^\alpha A)^{-(k+1)} x.$$
where the convergence is uniform on bounded intervals of \( t > 0 \). The solution to \((\text{P}?)\) is given by

\[
    u(t) = S(t)x + \int_0^t P(t-s)f(s) \, ds.
\]

**Corollary 6.2.** For the Caputo equation with \( A \in \mathcal{L}(X) \)

\[
    S(t) = E_{\alpha,1}(At^\alpha) = \sum_{n=0}^{\infty} \frac{A^n t^{n\alpha}}{\Gamma(n\alpha + 1)}
\]

and

\[
    P(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha) = t^{\alpha-1} \sum_{n=0}^{\infty} \frac{A^n t^{n\alpha}}{\Gamma(n\alpha + \alpha)}.
\]

### 7 Nonhomogeneous equation

Consider the nonhomogeneous equation

\[
    \int_0^t g(t-s)u'(s) \, ds = Au(t) + f(t), \quad u(0) = 0 \tag{7.1}
\]

Suppose \( u \in L^2_{\text{loc}}(0, \infty, X) \) be a solution. Then, it is unique and satisfies the Laplace transform of \((\text{nhom}\, \text{7.1})\)

\[
    (\Delta(\lambda) I - A)\hat{u} = \hat{f} \tag{7.2}
\]

It follows from Theorem 2.3 that \( A \) generates the \( C_0 \) semigroup \( T(t), t \geq 0 \) on \( Z = C(({-\infty}; X) \).

**Theorem 7.1.** For \( \omega \geq 0 \) we define the operator

\[
    \Psi_\omega = e^{\omega s}(A - \Delta(\omega) I)^{-1} \in \mathcal{L}(X, Z)
\]

and define

\[
    u(t) = \left((A - \omega I) \int_0^t T(t-s)\Psi_\omega f(s) \, ds\right)(0) \tag{7.3}
\]

Then, \((\text{nhom}\, \text{7.3})\) defines the solution to \((\text{nhom}\, \text{7.1})\) if \( f \in C^2(0,T; X) \).

Proof: Note that

\[
    z(t) = (A - \omega I) \int_0^t T(t-s)\Psi_\omega f(s) \, ds
\]

\[
    = T(t)\Psi_\omega f(0) - \Psi_\omega f(t) + \int_0^t T(t-s)\Psi_\omega f'(s) \, ds - \omega \int_0^t T(t-s)\Psi_\omega f(s) \, ds
\]

and thus, \( z \in C(0,T; Z) \) and \( z(t) = u(t + \cdot) \). Taking the Laplace transform of \((\text{nhom}\, \text{7.4})\), we obtain

\[
    \hat{z}(\lambda) = (A - \omega I)(\lambda I - A)^{-1}\Psi_\omega \hat{f}(\lambda)
\]

\[
    = (\lambda - \omega)(\lambda I - A)^{-1}\Psi_\omega \hat{f}(\lambda) - \Psi_\omega \hat{f}(\lambda).
\]
Since from (5.6) and
\[ \Delta(\lambda) = \lambda \int_{-\infty}^{0} e^{\lambda \theta} g(\theta) \, d\theta, \]
\[ ((\lambda I - A)^{-1} \Psi_{\omega})(0) = -(\Delta(\lambda) I - A)^{-1} (\Delta(\omega) I - A)^{-1} \int_{-\infty}^{0} g(\theta)(e^{\omega \theta} - \lambda \int_{\theta}^{0} e^{\lambda(\theta - \xi)} e^{\omega \xi} \, d\xi) \]
\[ = (\Delta(\lambda) I - A)^{-1} (\Delta(\omega) I - A)^{-1} \frac{\Delta_{\omega} - \Delta_{\lambda}}{\lambda - \omega}, \]
Thus we have
\[ \hat{u}(\lambda) = (\Delta(\lambda) I - A)^{-1} \hat{f}(\lambda) \]
and the claim holds. □

When \( 0 \in \rho(A) \), we let \( \omega = 0 \) and obtain
\[ u(t) = S(t) A^{-1} f(0) - A^{-1} f(t) + \int_{0}^{t} S(t - s) A^{-1} f'(s) \, ds \]
\[ = \int_{0}^{t} S(t - s) A^{-1} f(s) \, ds = \int_{0}^{t} P(t - s) f(s) \, ds, \]
for \( f \in W^{11}(0, T; X) \), since
\[ \dot{S} = \frac{d}{dt} S(t) = P(t) A. \]
Otherwise, for \( \omega \geq 0 \) let \( y(t) = e^{-\omega t} u(t) \). It can be easily seen that \( y \) satisfies
\[ \int_{-\infty}^{t} g(t - s) e^{-\omega(t-s)} y'(s) \, ds = -\omega \int_{-\infty}^{t} g(t - s) e^{-\omega(t-s)} y(s) \, ds + Ay(t) + e^{-\omega t} f(t) \]
with \( y(s) = 0, \ s \leq 0. \) Since
\[ \int_{0}^{t} g(t - s) e^{-\omega(t-s)} y'(s) \, ds = \int_{-\infty}^{0} (g'(t - s) - \omega g(t - s)) e^{-\omega(t-s)} (y(s) - y(t)) \, dt, \]
we have
\[ (\Delta(\omega) I - A)y(t) = \lim_{\epsilon \to 0^+} \int_{-\infty}^{t} g_{\epsilon}'(t - s) e^{-\omega(t-s)} (y(t) - y(s)) \, ds = e^{-\omega t} f(t) \]
Suppose \( |y(t)| = \max_{s \leq T} |y(s)| \) we have
\[ \langle (\Delta(\omega) I - A)y(t), x^* \rangle \leq e^{-\omega t} \langle f(t), x^* \rangle \]
for \( x^* \in F(y(t)) \) since \( g_{\epsilon}' \leq 0 \) on \( \mathbb{R}^+ \) and
\[ \langle y(t) - y(s), x^* \rangle \geq (|y(t)| - |y(s)|) |y(t)| \geq 0. \]
Since \( A \) is dissipative we have
\[ |y(t)| \leq |\Delta(\omega)^{-1}| e^{-\omega t} |f(t)|. \]
Theorem 7.2. For \( f \in C(0, T; X) \) there exists a unique solution \( u \) in \( C(0, T; X) \) to (7.1) and it has the representation (7.3):

\[
    u(t) = \int_0^t P(t - s)f(s)\,ds.
\]

and for some \( M > 0 \)

\[
    |u(t)| \leq M |f|_{C(0, T; X)}.
\]

Proof: Since \( C^2(0, T; X) \) is dense in \( C(0, T; X) \) and \( \mathcal{A} \) is closed, the theorem follows from (7.6).

7.1 \( f \in L^2(0, T; H) \)

Let \( X = H \) be a Hilbert space. Since for \( R > T \) and \( 0 \leq t \leq T \) and \( C^1(0, T; H) \),

\[
    \int_{t-R}^t g(t - s)(u'(s), u(s) - u(t))\,ds = \int_{-R}^0 g(-\theta)(u'(t + \theta), u(t + \theta) - u(t))\,d\theta
\]

\[
    = \frac{1}{2} \left( \int_{-R}^0 g'(-\theta)|u(t + \theta) - u(t)|^2\,d\theta - g(R)|u(t) - u(0)|^2 \right)
\]

and

\[
    \int_{t-R}^t g(t - s)(u'(s), u(s))\,ds = \int_{-R}^0 g(-\theta)(u'(t + \theta), u(t + \theta))\,d\theta
\]

\[
    = \frac{d}{dt} \frac{1}{2} \int_{-R}^0 g(-\theta)|u(t + \theta)|^2\,d\theta,
\]

it follows from

\[
    \int_0^T \left( \int_0^t g(t-s)u'(s)\,ds, u(t) \right) dt \int_0^T \int_0^t g(t-s)(u'(s), u(t)-u(s))\,ds\,dt + \int_0^T \int_0^t g(t-s)u'(s), u(s))\,ds\,dt
\]

that

\[
    \int_{-R}^0 g(\theta)|u(T + \theta)|^2\,d\theta - \int_{-R}^0 g(\theta)|x|^2\,d\theta
\]

\[
    + \int_0^T \left( \int_{-R}^0 g'(\theta)|u(t + \theta) - u(t)|^2\,d\theta + g(-R)|x - u(t)|^2 \right) dt
\]

\[
    = 2 \int_0^T \int_0^t g(t-s)u'(s)\,ds, u(t))\,dt.
\]

Letting \( R \to T^+ \), we obtain

\[
    \int_{-T}^0 g(\theta)|u(T + \theta)|^2\,d\theta - \int_{-T}^0 g(\theta)|x|^2\,d\theta
\]

\[
    + \int_0^T \left( \int_{-T}^0 g'(\theta)|u(t + \theta) - u(t)|^2\,d\theta + g(T)|x - u(t)|^2 \right) dt = 2 \int_0^T \int_0^t g(t-s)u'(s)\,ds, u(t))\,dt.
\]

Thus, we have
Theorem 7.3. The energy identity (7.5) holds for all \( u \in L^2(0, T; H) \) satisfying \( \int_0^t g(t-s)u'(s) ds \in L^2(0, T; H) \) and for all \( u(0) = 0 \).

\[
\int_0^T \left( \int_0^t g(t-s)u'(s) ds, u(t) \right) dt \geq g(T) \int_0^T |u(t)|^2 dt.
\]

Assume that there exists \( \omega > 0 \) such that

\[
(A\phi, \phi) = -\omega |\phi|^2, \quad \text{for all } \phi \in \text{dom}(A).
\]

Suppose \( f(0) = 0 \) and \( f \in C^2(0, T; H) \) it follows from (7.3) that \( u \in C^1(0, T; H) \cap C(0, T; \text{dom}(A)) \). Then, from (7.3)

\[
\int_{-T}^0 g(\theta)|u(T + \theta)|^2 d\theta - \int_{-T}^0 g(\theta)|x|^2 d\theta
\]

\[
+ \int_0^T (- \int_{-T}^0 g'(\theta)|u(t + \theta) - u(t)|^2 d\theta + g(T)|x - u(t)|^2 + \omega |u(t)|^2) dt \leq \frac{1}{\omega} \int_0^T |f(t)|^2 dt.
\]

Otherwise, using exactly the same arguments as for the estimate for \( C(0, T; H) \), we have

\[
\int_{-\infty}^0 g(\theta)e^{\omega\theta}|y(T + \theta)|^2 d\theta - \frac{\Delta(\omega)}{\omega} |x|^2
\]

\[
- \int_0^T \int_{-\infty}^0 g'(t-s)e^{-\omega(t-s)}|y(s) - y(t)|^2 ds dt + \omega \int_0^T \int_{-\infty}^0 g(t-s)e^{-\omega(t-s)}|y(s)|^2 ds dt
\]

\[
+ \int_0^T (\Delta(\omega) I - 2A)y(t), y(t)) dt = 2 \int_0^T (e^{\omega t} f(t), y(t)) dt
\]

Hence there exists \( M_T \) such that

\[
\int_0^T |u(t)|^2 dt \leq M_T \int_0^T |f(t)|^2
\]

(7.10)

Consequently, we have

Theorem 7.4. For \( f \in L^2(0, T; H) \) there exists a unique solution \( u \) in \( L^2(0, T; H) \) to (7.1) and it has the representation (7.3) and the energy identity (7.8).

Proof: Since \( C^2(0, T; H) \) is dense in \( L^2(0, T; H) \) and \( A \) is closed, the theorem follows from (7.10).

Corollary 7.1. Assume there exist a closed subspace \( V \) of \( H \) and \( \delta > 0 \) such that

\[
(A\phi, \phi) \leq -\delta |\phi|^2_V.
\]

Then, for \( f \in L^2(0, T; V) \) there exists a unique solution \( u \) in \( L^2(0, T; V) \) to (7.1) and it has the representation (7.3) and the energy identity

\[
\int_0^t |u(s)|^2_V ds \leq \frac{1}{\delta} \int_0^t |f(s)|^2_V ds.
\]
8 Dual System and Optimal Control Problems

In this section we consider the dual system to (8.6) and its application to a class of optimal control problems.

Let $X = H$ be a Hilbert space. For $u, p \in C^1(0, T; H)$ we have
\[
\int_0^T \left( \int_0^t g(t-s)u'(s) \, ds, p(t) \right) \, dt = \int_0^T \left( \int_s^T g(t-s)v(t) \, dt, u'(s) \right) \, ds
\]
\[
= -(u(0), \int_0^T g(t)v(t) \, dt) - \int_0^T \left( u(s), \frac{d}{ds} \int_s^T g(t-s)p(t) \, dt \right) \, ds
\]
\[
= (\int_0^T g(T-s)u(s) \, ds, p(T)) - (u(0), \int_0^T g(t)p(t) \, dt) - \int_0^T \left( u(s), \int_s^T g(t-s)p'(t) \, dt \right) \, ds.
\]
(8.1) dual

Theorem 8.1. Let $D_t^g : L^2(0, T; H) \to L^2(0, T; H)$ be
\[
(D_t^g u)(t) = \int_0^t g(t-s)u'(s) \, ds
\]
with domain
\[
dom (D_t^g) = \{ u \in L^2(0, T; H) : D_t^g u \in L^2(0, T; H), \, u(0^+) = 0 \}
\]
Then, $D_t^g$ is a densely defined, closed operator on $L^2(0, T; H)$ and the adjoint $(D_t^g)^*$ is given by
\[
((D_t^g)^* p)(s) = \frac{d}{ds} \int_s^T g(t-s)p(t) \, dt
\]
with domain
\[
dom ((D_t^g)^*) = \{ p \in L^2(0, T; H) : \int_0^T g(t-s)p(t) \, dt \in H^1(0, T; X), \ \ (\int_s^T g(t-s)p(t) \, dt)(T^-) = 0 \}.
\]

Proof: Since $g^* \in L^1(0, T)$ is defined by (8.11) and from (8.12)
\[
u(t) = \int_0^t g^*(t-s)(D_t^g u) \, ds,
\]
$D_t^g$ is closed. For $y = (D_t^g)^* u$,
\[
\int_0^T (y(t), u(t)) \, dt = \int_0^T \left( \int_s^T y(t) \, dt, u'(s) \right) \, ds
\]
and
\[
\int_0^T (D_t^g u, p(t)) \, dt = \int_0^T \left( \int_s^T g(t-s)p(t) \, dt, u'(s) \right) \, ds.
\]
Since $u \in dom(D_t^g)$ are arbitrary,
\[
\int_s^T g(t-s)p(t) \, dt = \int_s^T y(t) \, dt, \ \text{for a.e.} \ s \in (0, T).
\]

30
Hence, we obtain
\[
\lim_{s \uparrow T} \int_{s}^{T} g(t-s)p(t) \, dt = 0
\]
and
\[
y(s) = (D_t^g)^*_p = \frac{d}{ds} \int_{s}^{T} g(t-s)p(t) \, dt.
\]

**Definition 7.1 (Weak solution)** A function \( u \in L^2(0,T;H) \) is a weak solution to (fde 1.6) if
\[
\int_{0}^{T} \left(- \int_{s}^{T} g(t-s)p'(t) \, dt + A^* p(s), u(s)\right) + (p(s), f(s)) \, ds - (u(0), \int_{0}^{T} g(t)p(t) \, dt) = 0
\]
for all \( p \in C^1(0,T;H) \cap C(0,T;\text{dom}(A^*)) \) satisfying \( p(T) = 0 \).

**Theorem 8.2.** Weak solution to (fde 1.6) is unique

Proof: It follows from Theorem 7.4 that the dual system
\[
\int_{s}^{T} g(t-s)p'(t) \, dt = A^* p(s) + f(s), \quad p(T) = 0
\]
has a solution \( p \in C^1(0,T;H) \cap C(0,T;\text{dom}(A^*)) \) for all \( f \in C^1(0,T;H) \) with \( f(T) = 0 \). Since \( \{ f \in C^1(0,T;H) \cap C(0,T;\text{dom}(A^*)) \} \) is dense in \( L^2(0,T;H) \) the uniqueness follows from (8.2). □

Consider the control problem
\[
\min \int_{0}^{T} \left(\ell(x(t)) + h(u(t))\right) \, dt,
\]
subject to (fde 1.6);
\[
\int_{0}^{t} g(t-s)x'(s) \, ds = Au(t) + Bu(t), \quad x(0) = x^0,
\]
where \( x(t) \in X \), a Hilbert space and \( A \) is a maximal monotone operator in \( X \) Let \( U \) be a Hilbert space and \( \hat{U} \) is a closed convex set in \( U \) and
\[
u \in \mathcal{C} = \{ u \in L^2(0,T;U) : u(t) \in \hat{U}, \text{a.e.}\}
\]
denote the control function and \( B \in \mathcal{L}(U,X) \). The functional \( \ell \) and \( h \) are convex on \( X \) and \( U \), respectively.

**Theorem 8.3.** Problem (8.4)–(8.5) has an optimal control \( u^* \in \mathcal{C} \).

Proof: Since given \( u \in \mathcal{C} \) (8.5) has a unique solution \( x(\cdot;u) \in C(0,T;X) \). Thus, the optimal control problem is equivalent to minimizing
\[
J(u) = \int_{0}^{T} \left(\ell(x(t;u)) + h(u(t))\right) \, dt
\]
over \( u \in \mathcal{C} \). Suppose \( u_n \in \mathcal{C} \) is a minimizing sequence of \( J \) over \( \mathcal{C} \), i.e. \( J(u_n) \downarrow \) and \( \lim_{n \to \infty} J(u_n) = \eta = \inf_{u \in \mathcal{C}} J(u) \). Then there exists a weak convergent subsequence of \((u_n, \eta)\) to \((u^*, x^*)\) in \( L^2(0, T; U \times X) \). Since \( \mathcal{C} \) is weakly closed, \( u^* \in \mathcal{C} \). From (8.2) it follows that \( x^* \) is a weak solution of (8.5) corresponding to \( u^* \). Since convex functionals are weakly sequentially lower semi-continuous, \( J(u^*) \leq \eta \), i.e., \( u^* \in \mathcal{C} \) is optimal. \( \square \)

Let \( \partial \ell(x^*) \) be the sub-differential of \( \ell \) at \( x^* \in X \), i.e.
\[
\partial \ell(x^*) = \{ \lambda \in \mathbb{R}^n : \ell(x) - \ell(x^*) \geq \langle \lambda, x - x^* \rangle \} \text{ for all } x \in X.
\]

Define the Lagrange functional
\[
L(x,u,p) = \int_0^T (\ell(x(s)) + h(u(s))) \, ds + \int_0^T (p(s), A(s) + Bu(s) - (D^2_\ell x)(s)) \, ds.
\]

**Theorem 8.4.** Let \( u^* \in \mathcal{C} \) be an optimal to problem (8.4)-(8.5) and assume \( \ell \) is \( C^1 \). Then,
\[
h(u) + (u, B^* p(t)) \geq h(u^*(t)) + (u^*(t), Bp(t)) \text{ for all } u \in \hat{U}, \text{ a.e. } t \in (0, T)
\]
where the adjoint state satisfies
\[
(D^2_\ell)^* p = A^* p + \ell'(x^*), \quad p(T) = 0. \tag{8.6}
\]

**Proof:** Since \( \ell'(x^*) \in C(0, T; X) \) there exists a unique solution to (8.6). Let \( u = u^* + t(v - u^*) \in \mathcal{C} \) with \( v \in \mathcal{C} \) and \( t \in (0, 1) \) and \( x \) be the corresponding solution of (8.5) to \( u \). Note that
\[
(\ell'(x^*), x - x^*) = ((D^2_\ell)^* p - A^* p, x - x^*) = (p, D^2_\ell(x - x^*) - A(x - x^*)) = (p, B(u - u^*)).
\]

Then,
\[
0 \leq J(u) - J(u^*) = \int_0^T (E(x(s), x^*(s)) + (\ell'(x^*(s)), x(s) - x^*(s))) \, ds \tag{8.7}
\]
\[
+ \int_0^T (h(u(s)) - h(u^*(s)) + (p(t), B(u(s) - u^*(s)))) \, ds
\]
where
\[
E(x, x^*) = \ell(x) - \ell(x^*) - \ell'(x^*)(x - x^*).
\]

Since
\[
\frac{1}{T} \int_0^T E(x(s), x^*(s)) \, ds \to 0 \text{ as } t \to 0^+
\]

Since \( h \) is convex,
\[
\int_0^T (h(u(s)) - h(u^*(s)) + (p(s), B(u(s) - u^*(s)))) \, ds \leq t \int_0^T (h(v(s)) - h(u^*(s)) + (p(s), B(v(s) - u^*(s)))) \, ds,
\]

Now, since
\[
\frac{1}{T} \int_0^T E(x(s), x^*(s)) \, ds \to 0 \text{ as } t \to 0^+
\]

letting \( t \to 0^+ \) in (8.7), we obtain
\[
\int_0^T (h(v(s)) - h(u^*(s)) + (p(s), B(v(s) - u^*(s))) \, ds \geq 0
\]
for all \( v \in \mathcal{C} \), which implies the necessary optimality. \( \square \)
9 Case: $1 < \alpha < 2$

In this section we consider the case when $1 < \alpha < 2;)

\begin{align*}
D_t^\alpha u &= \int_0^t g_2(t - s)u''(s) \, ds = Au(t) + f(t), \quad u(0) = u_0, \; u'(0) = v_0,
\end{align*}
or in general

\begin{align*}
\int_0^t g(t - s)u''(s) \, ds = Au(t) + f(t), \quad u(0) = u_0, \; u'(0) = v_0. \tag{9.1}
\end{align*}

Equivalently, we have

\begin{align*}
\frac{d}{dt} u &= v \quad \text{and} \quad \int_0^t g(t - s)v'(s) \, ds = Au(t) + f(t).
\end{align*}

with

\begin{align*}
u(0) = u_0, \quad v(0) = v_0.
\end{align*}

Assume $-A$ is self-adjoint and positive on a Hilbert space $H$ and define

\begin{align*}
V = dom((-A)^{\frac{1}{2}}) \text{ with } |\phi|_V^2 = \langle -A\phi, \phi \rangle.
\end{align*}

Define a linear operator $A$ on $X = V \times L^2_g(-\infty, 0; H)$ by

\begin{align*}
A(u, z) &= (z(0), z')
\end{align*}

with

\begin{align*}
\text{dom}(A) &= \{(u, z) \in X : A(u, z) \in X \text{ and } \int_{-\infty}^0 g(\theta) z'(\theta) \, d\theta = Au\}.
\end{align*}

**Theorem 9.1.** The linear operator $A$ is $m$-dissipative and $A$ generates a $C_0$ semigroup $T(t)$ on $X$. $u(t) = (T(t)(u_0, v_0))_1 \in C(0, T; V)$ defines a mild solution to (9.1).

Proof: For $(u, z) \in \text{dom}(A)$

\begin{align*}
(A(u, z), (u, z)) &= (-Au, z(0)) + \int_{-\infty}^0 g(\theta)(z'(\theta), z(\theta)) \, d\theta
\end{align*}

\begin{align*}
&= \int_{-\infty}^0 g(\theta)(z'(\theta), z(\theta) - z(0)) \, d\theta.
\end{align*}

From (2.1)

\begin{align*}
(A(u, z), (u, z)) &= -\lim_{\epsilon \to 0^+} \int_{-\infty}^0 \frac{g(\theta) - g(\theta - \epsilon)}{\epsilon} |z(\theta) - z(0)|^2 \, d\theta,
\end{align*}

and thus

\begin{align*}
(A(u, z), (u, z)) &= -\int_{-\infty}^0 g'(\theta)|z(\theta) - z(0)|^2 \, d\theta \leq 0.
\end{align*}

For the resolvent

\begin{align*}
\lambda (u, z) - A(u, z) = (f^1, f^2)
\end{align*}

33
is equivalent to
\[ \lambda z - z' = f^2, \quad \lambda u - z(0) = f^1. \]

From the first equation
\[ z(\theta) = e^{\lambda \theta} z(0) + \int_{\theta}^{0} e^{\lambda(\theta - \xi)} f^2(\xi) d\xi \quad (9.2) \]
and
\[ \lambda(\int_{-\infty}^{0} e^{\lambda \theta} g(\theta) d\theta) z(0) + \int_{-\infty}^{0} g(\theta)(\lambda \int_{\theta}^{0} e^{\lambda(\xi - \theta)} f^2(\xi) d\xi - f^2(\theta)) d\theta = Au. \]

From the second equation
\[ u = (\lambda \Delta(\lambda) I - A)^{-1}(\Delta(\lambda) f^1 + \int_{-\infty}^{0} g(\theta)(f^2(\theta) - \lambda \int_{\theta}^{0} e^{\lambda(\theta - \xi)} f^2(\xi) d\xi) d\theta) \quad (9.3) \]
and
\[ z(0) = \lambda u - f^1. \quad (9.4) \]

Thus, \( R(\lambda I - A) = X. \) □

It follows from \( (9.3) \) that \( u(t) = S_1(t)u + S_2(t)v \) with
\[ S_1(t)u = \frac{1}{2\pi i} \int_{\gamma + i\infty}^{\gamma - i\infty} e^{\lambda t}(\lambda \Delta(\lambda) I - A)^{-1}\Delta(\lambda) u d\lambda \]
\[ S_2(t)v = \frac{1}{2\pi i} \int_{\gamma + i\infty}^{\gamma - i\infty} e^{\lambda t}(\lambda \Delta(\lambda) I - A)^{-1}\lambda^{-1}\Delta(\lambda) v d\lambda. \quad (9.5) \]

Assume \( A \) is a sectorial operator, i.e., there exist \( M > 0, \theta_0 > 0 \) such that
\[ |(\lambda I - A)^{-1}| \leq \frac{M}{|\lambda|} \quad \text{on} \quad \Sigma_{\theta_0} = \{ \lambda \in C : \arg(\lambda) \leq \pi / 2 + \theta_0 \} \cap \{ \lambda \neq 0 \} \]

Assume that if \( \lambda \in \Sigma_{\theta_0}, \) then \( \lambda \Delta(\lambda) \in \Sigma_{\tilde{\theta}} \) for \( \tilde{\theta} > 0. \) Let \( \Gamma_{\theta,\delta} \) be the integration path defined by
\[ \Gamma_{\theta,\delta} = \{ z \in C : |\lambda| \geq \delta, arg(\lambda) = \pm\left(\frac{\pi}{2} + \lambda\right)\}, \quad \Gamma_0 = \{ z \in C : |z| = \delta, |arg(z)| \leq \frac{\pi}{2} + \theta \} \]
for some \( \delta > 0 \) and \( 0 < \theta \leq \tilde{\theta}. \) Then, the solution map \( S_1(t) \) and \( S_2(t) \) is given by
\[ S_1(t)x = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t}(\lambda \Delta(\lambda) I - A)^{-1}\Delta(\lambda)x d\lambda, \]
\[ S_2(t)x = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t}(\lambda \Delta(\lambda) I - A)^{-1}\lambda^{-1}\Delta(\lambda)x d\lambda, \]
using the Cauchy integral representation \( (9.5) \) and the analytic continuation. Let \( P(t) \in L(X), \) \( t > 0 \) be
\[ P(t)x = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t}(\lambda \Delta(\lambda) I - A)^{-1}x d\lambda. \]

34
Then
\[ \frac{d}{dt} S_1(t) = P(t) A \]
and the solution \( u(t) \) to (1.2) is given by
\[ u(t) = S_1(t) u_0 + S_2(t) v_0 + \int_0^t P(t - s) f(s) \, ds \]
for \( u_0, v_0 \in X \) and \( f \in C(0, T; X) \).

## 10 Finite Difference Method

In this section we develop the finite difference scheme for (1.6) and analyze the stability and the convergence of the scheme for a general class of (1.6) with maximal monotone operators \( A \).

Let \( h > 0 \) be a stepsize and define
\[ g_j = \int_{-(j+1)h}^{-jh} g(s) \, ds, \quad j \leq 0 \]
The sequence \( \{u_j^k, j \leq 0\} \) approximating \( u(kh, -jh) \in X \) is generated by
\[ g_0 \frac{u_0^k - u_{-1}^k}{h} + \sum_{j=-1}^{-(k-1)} g_j \frac{u_j^k - u_{j-1}^k}{h} = A u_0^k + f^k \quad (10.1) \]
That is,
\[ u_{j-1}^k = u_j^{k-1}, \quad j \leq 0, \quad u_j^k = x \] for \( j \leq -k \).

Thus, (10.1) is equivalent to
\[ g_0 \frac{u_k^k - u_{k-1}^k}{h} + \sum_{j=-1}^{-(k-1)} g_j \frac{u_{k+j}^k - u_{k+j-1}^k}{h} = A u_k^k + f^k \quad (10.3) \]
which is an approximation for (1.6) directly. We have the following stability results for (10.1);

**Theorem 10.1.** If \( A \) is dissipative, for \( f = 0 \) we have \( |u_0^k| \leq |x| \). In general we assume that for all \( u \in dom(A) \), there \( u^* \in F(u) \) such that
\[ \langle Au, u^* \rangle \leq -\delta |u|^2, \quad (10.4) \]
then
\[ \max_{0 \leq k \leq N} |u^k| \leq \frac{1}{\delta} \max_{0 \leq k \leq N} |f^k|. \]
Proof: For $f = 0$ since

$$
\sum_{j=0}^{-(k-1)} g_j \frac{u_j - u_{j-1}}{h} = - \sum_{j=0}^{-k-2} g_j - g_j \frac{(u_{j-1} - u_0)}{h},
$$

it follows from (10.1) that for $u^* \in F(u_0^k)$

$$
\langle Au_0^k, u^* \rangle = - \sum_{j=0}^{-(k-2)} \frac{g_j - g_j}{h} ((u_j^{k-1}, u^*) - |u_j^k|^2).
$$

Suppose $|u_0| > |u_j|$, $j < 0$ then

$$
\langle Au_0, u^* \rangle > \sum_{j=0}^{-(k-2)} \frac{g_j - g_j}{h} (|u_0|^2 - |u_j^k||u_0|) > 0
$$

which contradicts to the fact that $A$ is dissipative. Thus, $\max_{j \leq 0} |u_j^k| \leq \max_{j \leq 0} |u_j^{k-1}|$.

In general, suppose $|u^k| \geq |u_j|$ for $0 \leq j \leq N$. From (10.5),

$$
-\langle Au_0^k, u^* \rangle \leq |u_0^k||f^k|
$$

for $u^* \in F(u_0^k)$ and thus from (10.3)

$$
|u_0^k| \leq \frac{1}{\delta} |f^k|.
$$

Hence we obtain

$$
\max_{0 \leq k \leq N} |u^k| \leq \frac{1}{\delta} \max_{0 \leq k \leq N} |f^k|.
$$

Assume $X = H$ is a Hilbert space.

**Theorem 10.2** Assume for $\delta > 0$

$$(A\phi, \phi) \leq -\delta|\phi|^2, \quad \phi \in \text{dom}(A).$$

For all $k \geq 1$

$$
\sum_{j=0}^{-(k-1)} g_j |u_j^k|^2 + \sum_{\ell=1}^{-(k-2)} g_j - \sum_{j=0}^{k} |u_\ell^\ell|^2 h + \delta \sum_{\ell=0}^{k} |u_\ell^\ell|^2 h \leq \sum_{j=0}^{-(k-1)} g_j |x|^2 + \frac{1}{\delta} \sum_{\ell=1}^{k} |f^\ell|^2 h.
$$

Proof: From (10.3)

$$
(u_j^k - u_j^{k-1}) \frac{1}{2} (u_j^k + u_j^{k-1}) = \frac{1}{2} |u_j^k|^2 - |u_j^{k-1}|^2 = \frac{1}{2} (u_j^k - u_j^{k-1}) \frac{1}{2} (u_j^k + u_j^{k-1}),
$$

where

$$
\sum_{j=0}^{-(k-1)} g_j \frac{u_j^k - u_j^{k-1}}{h}, \frac{1}{2} (u_j^k + u_j^{k-1}) - u_0^k = \frac{1}{2} \sum_{j=0}^{-(k-2)} g_j - g_j \frac{|u_j^k + u_j^{k-1}|^2}{h}.
$$

36
Since from (10.1)
\[ \sum_{j \leq 0} g_j \left( \frac{u^k_j - u_{j-1}^k}{h} \right, u_0^k) = (Au_0^k + f^k, u_0^k), \]
summing this over \( k \) we obtain
\[ -\sum_{j=0}^{-(k-1)} g_j |u_j^k|^2 + \sum_{\ell=1}^{k} \sum_{j=0}^{-(\ell-2)} \frac{g_j - g_{j-1}}{h} |u_j^\ell - u_{j-1}^\ell|^2 h = \sum_{j=0}^{-(k-1)} g_j |x_j|^2 + 2 \sum_{\ell=1}^{k} (Au_0^\ell + f^\ell, u_0^\ell) h. \]
Hence, from the assumption we obtain the desired estimate. \( \square \)

Now, we have the convergence results;

**Theorem 10.3.** We assume
\[ \langle Ax_1 - Ax_2, x^* \rangle \leq -\delta |x_1 - x_2|^2. \] (10.7) \( \text{dif2} \)
for some \( x^* \in F(x_1 - x_2) \). Define the linear interpolation
\[ U_h(t) = u(kh) + \frac{t - kh}{h} (u(kh) - u((k-1)h)) \text{ if } t \in (k-1)h, kh]. \]
Then,
\[ |u_h - U_h|_{L^2(0,T;H)} \to 0 \text{ as } h \to 0^+. \]
Proof: If we let \( U_j^k = U((k-j)h) = u((k-j)h) \), then we have
\[ \sum_{j=0}^{-(k-1)} g_j |u_j^k|^2 + \sum_{\ell=1}^{k} \sum_{j=0}^{-(\ell-2)} \frac{g_j - g_{j-1}}{h} |u_j^\ell - u_{j-1}^\ell|^2 h = \sum_{j=0}^{-(k-1)} g_j |x_j|^2 + 2 \sum_{\ell=1}^{k} (Au_0^\ell + f^\ell, u_0^\ell) h. \] (10.8) \( \text{form} \)
where
\[ E^k = \int_{-kh}^{0} g(\theta)(u'(kh + \theta) - U'(kh\theta)) d\theta \to 0 \]
Let \( X = H \) be a Hilbert space. The, we have
\[ \sum_{\ell=1}^{N} |E^\ell|^2 h \to 0 \text{ as } h \to 0^+ \]
If we define
\[ u_h(t) = u^k + \frac{t - kh}{h} (u^k - u^{k-1}) \text{ if } t \in (k-1)h, kh] \]
then it follows from (10.6) and (10.7) that
\[ |u_h - U_h|_{L^2(0,T;H)} \to \frac{1}{\delta} |E_h|_{L^2(0,T;H)} \to 0 \]
as \( h \to 0^+ \). \( \square \)

In general, let \( X \) be a Banach space and then we have;

**Corollary 10.1.**
\[ |u_h - U_h|_{C(0,T;X)} \to 0, \text{ as } h \to 0^+, \]
assuming (10.7) and \( f \in C(0,T;X) \).

Proof: Using the same arguments as above, from (10.7) and (10.8) we have
\[ |u_h - U_h|_{C(0,T;X)} \leq \frac{1}{\delta} |E_h|_{C(0,T;X)} \to 0. \] \( \square \)
10.1 Cone invariance and Maximum Principle

Let $H = L^2(\Omega)$ and $\mathcal{C}$ be a a closed cone in $H$. $A$ is cone preserving, i.e.,
\[(I - sA)^{-1}\mathcal{C} \subset \mathcal{C}\]
for all sufficiently small $s > 0$. Since
\[u^k = (I - \frac{h}{g_0}A)^{-1} \left( \sum_{j=0}^{-(k-2)} \frac{g_j - g_{j-1}}{g_0} u^{k+j} + \frac{hg-(k-1)}{g_0} u^0 + f^k \right)\]
By induction in $k$, we have $u^k \in \mathcal{C}$ if $u^0 \in \mathcal{C}$ and $f^k \in \mathcal{C}$.

11 Semi-linear equations

In this section we consider the semilinear equation
\[\frac{D_\alpha t}{x} = Ax(t) + F(x(t)), \quad x(0) = x_0\]
or equivalently
\[x(t) = S(t)x_0 + \int_0^t P(t-s)F(x(s)) \, ds, \quad t \geq 0\]
with the locally Lipschitz function $F$ in $X$. Assume
\[|S(t)| \leq \psi(t) = C \min(1, t^{-\alpha}), \quad |P(t)| \leq C \min(t^{-1+\alpha}, t^{-1}).\]
and that $F(0) = 0$ and
\[|F(x) - F(y)| \leq \rho(M) |x - y| \quad \text{for} \quad |x|, |y| \leq M.\]
For given $f \in C(0, T; X)$ consider the map from $C(0, \infty, X)$ to $C(0, \infty; X)$ by
\[(\Psi x)(t) = f(t) + \int_0^t P(t-s)F(x(s)) \, ds\]
First, we establish the local existence of solutions.

**Theorem (Local Existence)** Assume $|f(t)| \leq M_0(1 + t^\alpha)$. Then, there exists a $\tau > 0$ such that $(\Psi x)$ has a fixed point $x \in C(0, \tau; X)$.

**Proof:** For $|x(t)| \leq M$ on $[0, \tau]$ for some $M \geq 2M_0$ we have
\[\left| \int_0^t P(t-s)F(x(s)) \, ds \right| \leq \int_0^t C(t-s)^{-1+\alpha} M \rho(M) \, ds \leq C M \rho(M) \frac{t^\alpha}{\alpha}\]
Let $\tau = \tau(M_0)$ is chosen so that
\[M_0(1 + \tau^\alpha) + C M \rho(M) \frac{\tau^\alpha}{\alpha} \leq M,\]
given $M_0 > 0$. It follows from (11.3)

$$|(\Psi x_1)(t) - (\Psi x_2)(t)| \leq C\rho(M)\frac{\tau}{\alpha} |x_1 - x_2|_{C(0,\tau;X)}$$

for $|x_1|, |x_2| \leq M$. Thus, $\Psi$ has a unique fixed point $x \in C(0,\tau;X)$ satisfying $|x(t)| \leq M$, $t \in [0,\tau]$, provided that (11.4) holds, since $C\rho(M)\frac{\tau^\alpha}{\alpha} < 1$. □

Hence, given $t \geq 0$ for $h \geq 0$

$$x(t + h) = S(t + h)x_0 + \int_0^t P(t + h - s)F(x(s)) + \int_t^{t+h} P(t + h - s)F(x(s)) \, ds$$

$$= f(t + h) + \int_0^h P(h - \sigma)F(x(t + \sigma)) \, d\sigma$$

has a unique solution $x(t + h)$, $0 \leq h \leq \tau$ as a fixed point to (11.5) with

$$f(t + h) = S(t + h)x_0 + \int_0^t P(t + h - s)F(x(s)) \, ds; \ h \geq 0$$

provided that $f(t + h) \leq M_0(1 + h^\alpha)$ and some $M_0$.

Next, we establish a priori bound of $x(t)$. Assume $x \in M \min(1,t^{-\alpha})$ for some $M > 0$ and that

$$\rho(s)s = s^\gamma.$$ 

We will use

$$\int_0^t (t - s)^{-1+\delta} s^{-\delta} \, ds = \frac{1}{\Gamma(1-\delta)\Gamma(\delta)}$$

for $0 < \delta < 1$. Let

$$I(t) = |\int_0^t P(t-s)F(x(s)) \, ds|$$

For $t \leq 1$

$$I(t) \leq M^\gamma \int_0^t C(t-s)^{-1+\alpha} \, ds = \frac{CM^\gamma}{\alpha}. $$

For $1 \leq t \leq 2$

$$I(t) \leq \int_1^t C(t-s)^{-1+\alpha} M^\gamma s^{-\gamma} \, ds + \int_0^1 C M^\gamma (t-s)^{-1+\alpha} \, ds$$

$$\leq CM^\gamma [(t-1)^\alpha - t^\alpha + \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} t^{-\alpha(\gamma-1)}].$$

For $t \geq 2$

$$I(t) \leq \int_{t-1}^t C(t-s)^{-1+\alpha} M^\gamma s^{-\gamma} \, ds + \int_1^{t-1} C(t-s)^{-1} M^\gamma s^{-\gamma} \, ds + \int_0^1 C(t-s)^{-1} M^\gamma \, ds$$

$$\leq CM^\gamma [(t-1)^{-\gamma} t^\alpha + \frac{1}{\Gamma(\delta)\Gamma(1-\delta)} M^\gamma + (t-1)^{-1} t^{-\alpha}].$$

with $\delta = \alpha(\gamma - 1)$. It thus follows that there exists $\beta$ independent of $t \geq 0$ such that

$$I(t) \leq \beta CM^\gamma \min(1,t^{-\alpha})$$
It can be proved that if $M_0 > 0$ is sufficiently small there exists $M > 0$ such that

$$M_0 + \beta CM^\gamma \leq M.$$  \hfill (11.5)

**Theorem (Asymptotic Stability)** Assume $\rho(s)s = s^\gamma$ and condition (cond) holds. Then, (11.1) has a unique solution in $C(0, \infty; X)$ satisfying $|x(t)| \leq M \min(1, t^{-\alpha}).$

Proof: Using exactly the same arguments as above, without loss of the generality with same $\beta > 0$ we have

$$|\Psi x_1 - \Psi x_2| \leq \beta CM^{\gamma - 1}|x_1 - x_2|.$$  

The claim follows since $\beta CM^{\gamma - 1} < 1$. \hfill $\square$

### 11.1 Sectorial Case

Assume there exists $\alpha_1, \alpha_2 \geq 0$ such that

$$|A^{-\alpha_1}(F(x) - F(y))| \leq \rho(M) |A^{\alpha_2}(x - y)| \text{ for } |A^{\alpha_2}x|, |A^{\alpha_2}y| \leq M.$$  \hfill (11.6)  

and

$$|A^{-\alpha_1}F(0)|_X \leq c.$$  \hfill (11.7)

with $0 \leq \alpha_1 + \alpha_2 < 1$. Let $X_{\alpha_2} = dom(A^{\alpha_2})$. For $u \in C([0, \tau], X_{\alpha_2})$ define the map

$$(\Psi(u))(t) = S(t)x_0 + \int_0^t P(t-s)F(u(s)) \, ds.$$  

**Theorem 11.1. (Local Existence)** For $x_0 \in X_{\alpha_2}$ there exists a $\tau > 0$ such (11.1) has a unique solution $u \in C(0, \tau; X_{\alpha_2})$.

Proof: For $u(0, \tau, X_{\alpha_2})$ satisfying $|u| \leq M$ on $[0, \tau]$ it follows from Corollary 5.4 that

$$|A^{\alpha_2}\Psi(u)| = |S(t)||A^{\alpha_2}x_0| + \int_0^t A^{\alpha_1 + \alpha_2}P(t-s)A^{-\alpha_1}F(u(s)) \, ds|.$$

$$\leq |A^{\alpha_2}x| + \int_0^t C(t-s)^{-\beta}(\rho(M)M + c) \, ds.$$

$$\leq |A^{\alpha_2}x| + \left( \frac{C}{1 - \beta} \right) \tau^{1-\beta}(\rho(M)M + c)$$

where $\beta = \alpha \left( 1 - (\alpha_1 + \alpha_2) \right)$. Let $\tau = \tau(M_0)$ such that

$$M_0 + \frac{C}{1 - \beta} \tau^{1-\beta}(\rho(M)M + c) \leq M$$

given $M_0 = |A^{\alpha_2}x_0|$. It follows from (cond1) that

$$|A^{\alpha_2}(\Psi(u_1) - \Psi(u_2))| \leq \left( \frac{C}{1 - \beta} \right) \tau^{1-\beta}\rho(M)|A^{\alpha_2}(u_1 - u_2)|$$

Thus, $\Psi$ has a unique fixed point in $x \in C(0, \tau, X_{\alpha_2})$ satisfying $|A^{\alpha_2}x(t)| \leq M$ on $[0, \tau]$, which defines a solution to (11.1) \hfill $\square$
12 Examples

In this section we discuss the application of our theory for concrete examples.

12.1 Fractional Parabolic equations

Let \( \Omega \) is a bounded open set in \( \mathbb{R}^d \). Consider the fractional parabolic equation

\[
D_\alpha^t u = \nabla \cdot (a(x)\nabla u) + b(x) \cdot \nabla u + f(x, u),
\]

(12.1)

where \( u \in C([0, \tau] \times \Omega) \) and \( f : \mathbb{R}^d \times R \rightarrow R \) is locally Lipschitz function. Define the second order elliptic operator

\[
Au = \nabla (a(x)\nabla u) + b(x) \cdot \nabla u
\]

with \( a \in \mathbb{R}^{d \times d} \in C^1(\Omega) \) is symmetric and positive definite and \( b \in C(\Omega) \). The linear operator \( A \) with \( \text{dom}(A) = \{u \in C^2(\Omega) \cap H^1_0(\Omega)\} \) is dissipative in \( X = C(\Omega) \). In fact, if \( u(\zeta_0) \geq |u| \) for \( \zeta_0 \in \Omega \), then \( (\nabla u)(\zeta_0) = 0 \) and

\[
(H_i) = \text{tr} a(\zeta_0) H(\zeta_0) + (\nabla \cdot a + b) \cdot (\nabla u)(\zeta_0) \leq 0
\]

where \( H_{i,j} = u_{x_i, x_j} \) is the Hessian of \( u \). Similarly, if \( u(\zeta_0) \leq |u| \), then \( (Au)(\zeta_0) \geq 0 \).

Define the nonlinear operator by

\[
(F(u))(x) = f(x, u(x))
\]

We assume \( f(0) = 0 \) and

\[
|f(x) - f(y)| \leq \rho(M) |x - y| \quad \text{for } |x|, |y| \leq M.
\]

(12.2)

Then, (cond.1.5) is satisfied and it follows from Theorem that (parab.1.1) has a local in time solution \( u \in C([0, \tau] \times \Omega) \). Moreover, if \( f(x, u)u \leq 0 \), then the solution \( u \) is global in time and \( |u(t)|_X \leq |u_0|_X \).

12.2 Fractional Scalar conservation law

In this section we consider the scalar conservation law

\[
D_\alpha^t u + (f(u))_x + f_0(x, u) = 0, \quad t > 0 \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^d
\]

(12.3)

where \( f : \mathbb{R} \rightarrow \mathbb{R}^d \) is \( C^1 \). Let \( X = L^1(\mathbb{R}^d) \) and define

\[
Au = -(f(u))_x,
\]

where we assume \( f_0 = 0 \) for the sake of simplicity of our presentation. Define

\[
A\phi = \phi'(\theta)
\]

with domain

\[
\text{dom}(A) = \{ \int_{-\infty}^0 g(\theta)\phi'(\theta) d\theta = A\phi(0) \}.
\]
Let 
$$C = \{ \phi \in \mathbb{Z} : \phi \geq 0 \}.$$ 

Since \( Ac = 0 \) for all constant \( c \), it follows that 
$$\phi - c \in C \implies (I - \lambda A)^{-1} \phi - c \in C.$$ 

Similarly, 
$$c - \phi \in C \implies c - (I - \lambda A)^{-1} \phi \in C.$$ 

Thus, without loss of generality, one can assume \( f \) is bounded. Let \( \rho \in C^2(R) \) be a monotonically increasing function satisfying \( \rho(0) = 0 \) and \( \rho(x) = sgn(x), \ |x| \geq 1 \). Note that 
$$-(f(u_1)_x - f(u_2)_x, \rho(u_1 - u_2)) = (f(u_1) - f(u_2), \rho'(u_1 - u_2)(u_1 - u_2)_x),$$ 

$$= (\eta, \rho'(u_1 - u_2)(u_1 - u_2)_x(u_1 - u_2)),$$
where 
$$\eta = \int_0^1 f_u(u_2 + \tau(u_1 - u_2)) d\tau.$$ 

If we define \( \Psi(x) = \int_0^x \sigma \rho'(\sigma) d\sigma \), then 
$$\eta(u_1 - u_2), \rho'(u_1 - u_2)(u_1 - u_2)_x = -(\Psi(u_1 - u_2), \eta_x)$$ 

where \( u_\tau = u_2 + \tau(u_1 - u_2) \) and 
$$\eta_x = \int_0^1 (f_{uu}(s, x, u_\tau) + f_{uu}(s, x, u_\tau)(u_\tau)_x) d\tau.$$ 

Define \( \rho_\epsilon(x) = \rho(\frac{x}{\epsilon}) \) for \( \epsilon > 0 \). Then 
$$|(\eta(u_1 - u_2), \rho'_\epsilon(u_1 - u_2)(u_1 - u_2)_x)| = \epsilon (\Psi(\frac{u_1 - u_2}{\epsilon}), \eta_x) \leq const \epsilon |\eta_x|_1 \to 0$$ 
as \( \epsilon \to 0 \). Note that for \( u \in L^1(R^d) \) 
$$(u, \rho_\epsilon(u)) \to |u| \quad \text{and} \quad (\psi, \rho_\epsilon(u)) \to (\psi, sgn_0(u)) \quad \text{for} \quad \psi \in L^1(R^d)$$ 
as \( \epsilon \to 0^+ \). Thus, 
$$\langle Au_1 - Au_2, sgn_0(u_1 - u_2) \rangle \leq 0$$ 
and \( A \) is monotone. It is show in [2] that 
$$range(\lambda I - A) = X,$$
i.e., for any \( g \in X \) there exists an entropy solution satisfying 
$$(\text{sign}(u - k)(\lambda u - g), \psi) \leq (\text{sign}(u - k)(f(u) - f(k)), \psi_x)$$
for all \( \psi \in C_0^1(R^d) \) and \( k \in R \). Hence \( A \) has a maximal monotone extension in \( L^1(R^d) \).
12.3 Fractional Hamilton-Jacobi equation

Let $u$ is a solution to a scalar conservation in $R^1$, then $v = \int^t u \, dx$ satisfies the fractional Hamilton-Jacobi equation

$$D^\alpha_t v + f(v_x) = 0.$$ 

Let $X = C_0(R^d)$ and

$$Av = -f(v_x), \quad \text{dom}(A) = \{ f(v_x) \in X \}$$

Then, for $v_1, v_2 \in C_1(R^d)$

$$\langle A(v_1 - v_2), \delta_{x_0} \rangle = -(f((v_1)_x(x_0)) - f((v_2)_x(x_0))) = 0$$

where $x_0 \in R^n$ such that $|v|_X = |v(x_0)|$. It also can be proved $[2]$ that

$$\text{range}(\lambda I - A) = X \quad \text{for } \lambda > 0.$$ 

That is, there exists a unique viscosity solution to $\lambda v - f(v_x) = g$; for all $\phi \in C^1(\Omega)$ if $v - \phi$ attains a local maximum at $x_0 \in R^d$, then

$$\lambda v(x_0) - g(x_0) + f(\phi_x(x_0)) \leq 0$$

and if $v - \phi$ attains a local minimum at $x_0 \in R^d$, then

$$\lambda v(x_0) - g(x_0) + f(\phi_x(x_0)) \geq 0.$$

12.4 Fractional Semilinear wave equation

In this section we consider the fractional semilinear wave equation of the form;

$$D_t^\alpha u'(s) \, ds = A_0 u(t) + F(u(t)), \quad u(0) = u_0, \quad u'(0) = v_0.$$ 

(12.4) \hspace{1cm} a1-2

Let $-A_0$ be a positive self-adjoint operator on a Hilbert space $H$. $V = \text{dom}(-(A_0)^{\frac{1}{2}})$

12.5 Fractional Navier Stokes

In this section we discuss a fractional incompressible Navier Stokes equation

$$D_t^\alpha u + u \cdot \nabla u + \text{grad} p = \nu \Delta u$$

(12.5) \hspace{1cm} NS

$$\text{div} u = 0$$

where $u$ is the velocity field defined on domain $\Omega$ with Lipschitz boundary $\Gamma$ and $p$ is the pressure. Let $V$ be the divergence free closed subspace of $H_0^1(\Omega)^d$:

$$V = \{ u \in H_0^1(\Omega)^d : \text{div} u = 0 \}$$

and $X$ be the closure of $V$ with respect to $L^2(\Omega)^d$ norm:

$$X = \{ u \in H_0^1(\Omega)^d : \text{div} u = 0, \ n \cdot u = 0 \text{ at } \Gamma \}$$

43
Let $P$ be the orthogonal projection of $L^2(\Omega)^d$ onto $X$ and define the Stokes operator $-A$ by

$$-Au = P\Delta u \text{ with } \text{dom}(A) = H^2(\Omega)^d \cap V$$

and the convection

$$F(u) = -P(u \cdot \nabla u) \text{ for } u \in V$$

Then, $(\text{NS})_{12.5}$ is equivalently written as

$$D_\alpha t u = Au + F(u).$$

The Stokes operator $-A$ is positive self-adjoint operator $X$ with $\text{dom}((-A)^{1/2}) = V$. Moreover,

$$|F(u) - F(v)|_{V_{-1/2}} \leq c |u - v|_V (|u|_V + |v|_V),$$

where $V_{1/2} = \text{dom}((-A)^{1/4})$ and thus $(\text{cond})_{11.6}$ is satisfied with $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$. It follows from Theorem that $(\text{NS})_{12.5}$ has a local solution $u \in C(0, \tau; V)$ in time, satisfying

$$u(t) = S(t)x + \int_s^t P(t - s)F(u(s)) \, ds.$$

### 13 Space varying model

In this section we consider space varying cases;

**Case 1** (Space varying fraction $0 \leq \alpha(x) < 1$)

$$\int_0^t \frac{u'(x, s)}{(t - s)^{-\alpha(x)}} \, ds \in Au(x, t) + f(t). \quad (13.1) \text{ fdes1}$$

**Case 2** (Space varying weight $0 \leq \tilde{g}(x, s)$)

$$\int_0^t \frac{(t - s)^{-\alpha}}{\Gamma(1 - \alpha)} u'(x, s) \, ds + \int_0^t \tilde{g}(x, t - s)u'(x, s) \, ds \in Au(x, t) + f(t). \quad (13.2) \text{ fdes2}$$

Assume $X$ is a Banach space and $A \subset X \times X$ is dissipative. Let $Z = C(-\infty, 0]; X)$ and $Az = z'(\theta)$ in $Z$. For Case 1 define $g(x, \theta) = |\theta|^{-\alpha(x)}$ for $x \in \Omega$ and $\theta \in (0, -\infty)$. For Case 2 assume $s \to g(x, s)$ is decreasing for every $x \in \Omega$ and set $g(x, \theta) = g_a(|\theta|) + \tilde{g}(x, |\theta|)$ Note that $\theta \to g(x, \theta)$ is monotonically increasing for every $x \in \Omega$. Define

$$\text{dom}(A) = \{z' \in Z, \ z(0) \in \text{dom}(A) \text{ and } \int_{-\infty}^0 g(x, \theta)z'(x, \theta) \in Az(0)\}$$

Using exactly the same arguments as for Theorem 3.1, $A$ is maximal monotone in $Z$ and generate a nonlinear semigroup on $Z$. Thus Both $(\text{fdes1})$ and $(\text{fdes2})$ has a unique mild solution $u \in C(0, T; X)$. 

44
14 Fractional equations in Space

In this section we consider the nonlocal diffusion equation of the form

\[ u_t = Au = \int_{\mathbb{R}^d} J(z)(u(x + z) - u(x)) \, dz. \]

Or, equivalently

\[ Au = \int_{(\mathbb{R}^d)^{+}} J(z)(u(x + z) - 2u(x) + u(x - z)) \, dz \]

for the symmetric kernel \( J \) in \( \mathbb{R} \). It will be shown that

\[ (Au, \phi)_{L^2} = \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^{+}} J(z)(u(x + z) - u(x))(\phi(x + z) - \phi(x)) \, dz \, dx \]

and thus \( A \) has a maximum extension.

Also, the nonlocal Fourier law is given by

\[ Au = \nabla \cdot (\int_{\mathbb{R}^d} J(z)\nabla u(x + z) \, dz). \]

Thus,

\[ (Au, \phi)_{L^2} = \int_{\mathbb{R}^d \times \mathbb{R}^d} J(z)\nabla u(x + z) \cdot \nabla \phi(x) \, dz \, dx \]

Under the kernel \( J \) is completely monotone, one can prove that \( A \) is a maximal monotone extension.

14.1 Jump diffusion Model for American option

In this section we discuss the American option for the jump diffusion model

\[ u_t + (x - \sigma^2/2)u_x + \sigma^2 x^2 u_{xx} + Bu + \lambda = 0, \quad u(T, x) = \psi, \]

\((\lambda, u - \psi) = 0, \quad \lambda \leq 0, \quad u \geq \psi\)

where the generator \( B \) for the jump process is given by

\[ Bu = \int_{-\infty}^{\infty} k(s)(u(x + s) - u(x) + (e^s - 1)u_x) \, ds. \]

The CMGY model for the jump kernel \( k \) is given by

\[ k(s) = \begin{cases} 
Ce^{-M|s|}1_{Y} \quad & s \geq 0 \\
Ce^{-G|s|}1_{-Y} \quad & s \leq 0 
\end{cases} \]

Since

\[ \int_{-\infty}^{\infty} k(s)(u(x + s) - u(x)) \, ds = \int_{0}^{\infty} k^+(s)(u(x + s) - u(x)) \, ds + \int_{0}^{\infty} k^-(s)(u(x - s) - u(x)) \, ds \]

\[ = \int_{0}^{\infty} \frac{k^+(s) + k^-(s)}{2}(u(x + s) - 2u(x) + u(x - s)) \, ds + \int_{0}^{\infty} \frac{k^+(s) - k^-(s)}{2}(u(x + s) - u(x - s)) \, ds. \]
Thus,
\[
\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} k(s)(s)(u(x + s) - u(x)) \, ds \right) \phi \, dx \\
= \int_{-\infty}^{\infty} \int_{0}^{\infty} k_s(s)(u(x + s) - u(x))(\phi(x + s) - \phi(s)) \, ds \, dx \\
+ \int_{-\infty}^{\infty} \left( \int_{0}^{\infty} k_u(s)(u(x + s) - u(s)) \phi(x) \, ds \right) \phi \, dx
\]
where
\[
k_s(s) = \frac{k^+(s) + k^-(s)}{2}, \quad k_u(s) = \frac{k^+(s) - k^-(s)}{2}
\]
and hence
\[
(Bu, \phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_s(s)(u(x + s) - u(x))(\phi(x + s) - \phi(s)) \, ds \, dx \\
+ \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} k_u(s)(u(x + s) - u(s)) \phi(x) \, ds \right) \phi \, dx + \omega \int_{-\infty}^{\infty} u_x \phi \, dx.
\]
where
\[
\omega = \int_{-\infty}^{\infty} (e^s - 1)k(s) \, ds.
\]
If we equip \( V = H^1(R) \) by
\[
|u|^2_V = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_s(s)|u(x + s) - u(x)|^2 \, ds \, dx + \frac{\sigma^2}{2} \int_{-\infty}^{\infty} |u_x|^2 \, dx
\]
then \( A + B \in \mathcal{L}(V, V^*) \) and \( A + B \) generates the analytic semigroup on \( X = L^2(R) \).

### 14.2 Numerical approximation of nonlocal operator

In this section we describe our higher order integration method for the convolution;
\[
\int_{0}^{\infty} \frac{k^+(s) + k^-(s)}{2}(u(x+s) - 2u(x) + u(x-s)) \, ds + \int_{0}^{\infty} \frac{k^+(s) - k^-(s)}{2}(u(x+s) - u(x-s)) \, ds.
\]
For the symmetric part,
\[
\int_{-\infty}^{\infty} s^2 k_s(s) \frac{u(x + s) - 2u(x) + u(x - s)}{s^2} \, ds,
\]
where we have
\[
\frac{u(x + s) - 2u(x) + u(x - s)}{s^2} \sim u_{xx}(x) + \frac{s^2}{12} u_{xxxx}(x) + O(s^4)
\]
We apply the fourth order approximation of \( u_{xx} \) by
\[
u_{xx}(x) \sim \frac{u(x + h) - 2u(x) + u(x - h)}{h^2} - \frac{1}{12} u(x + 2h) - 4u(x) + 6u(x) - 4u(x - h) + u(x - 2h)
\]
46
and we apply the second order approximation of \( u_{xxxx}(x) \) by
\[
u_{xxxx}(x) \sim \frac{u(x + 2h) - 4u(x) + 6u(x - h) + u(x - 2h)}{h^4}.
\]

Thus, one can approximate
\[
\int_{-\frac{h}{2}}^{\frac{h}{2}} s^2 k_s(s) \frac{u(x + s) - 2u(x) + u(x - s)}{s^2} ds
\]
by
\[
\rho_0 \left( \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} - \frac{1}{12} \frac{u_{k+2} - 4u_{k+1} + 6u_k - 4u_{k-1} + u_{k-2}}{h^2} \right) + \frac{\rho_1}{12} \frac{u_{k+2} - 4u_{k+1} + 6u_k - 4u_{k-1} + u_{k-2}}{h^2},
\]
where
\[
\rho_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} s^2 k_s(s) ds \quad \text{and} \quad \rho_1 = \frac{1}{h^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} s^4 k_s(s) ds.
\]

The remaining part of the convolution
\[
\int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} u(x_{k+j} - s) k_s(s) ds
\]
can be approximated by three point quadrature rule based on
\[
u(x_{k+j} + s) \sim u(x_{k+j}) + u'(x_{k+j}) s + \frac{s^2}{2} u''(x_{k+j})
\]
with
\[
u'(x_{k+j}) \sim \frac{u_{k+j+1} - u_{k+j-1}}{2h} \quad \text{and} \quad u''(x_{k+j}) \sim \frac{u_{k+j+1} - 2u_{k+j} + u_{k+j-1}}{h^2}.
\]

That is,
\[
\int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} u(x_{k+j} + s) k_s(s) ds
\]
\[
\sim \rho_0^k u_{k+j} + \rho_1^k \frac{u_{k+j+1} - u_{k+j-1}}{2} + \rho_2^k \frac{u_{j+k+1} - 2u_{k+j} + u_{j+k-1}}{2}
\]
where
\[
\rho_0^k = \int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} k_s(s) ds
\]
\[
\rho_1^k = \frac{1}{h} \int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} (s - x_k) k_s(s) ds
\]
\[
\rho_2^k = \frac{1}{h^2} \int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} (s - x_k)^2 k_s(s) ds.
\]
For the skew-symmetric integral
\[
\int_{-\frac{b}{2}}^{\frac{b}{2}} k_u(s)(u(x + s) - u(x - s)) \, ds \sim \rho_2 u_x(x) + \frac{\rho_3}{6} h^2 u_{xxx}(x)
\]
where
\[
\rho_2 = \int_{-\frac{b}{2}}^{\frac{b}{2}} 2sk_u(s) \, ds, \quad \rho_3 = \frac{1}{h^2} \int_{-\frac{b}{2}}^{\frac{b}{2}} 2s^3k_u(s) \, ds.
\]
We may use the fourth order difference approximation
\[
u_x(x) \sim \frac{u(x + h) - u(x - h)}{2h} - \frac{u(x + 2h) - 2u(x + h) + 2u(x - h) - u(x - 2h)}{6h}
\]
and the second order difference approximation
\[
u_{xxx}(x) \sim \frac{u(x + 2h) - 2u(x + h) + 2u(x - h) - u(x - 2h)}{h^3}
\]
and obtain
\[
\int_{-\frac{b}{2}}^{\frac{b}{2}} k_u(s)(u(x + s) - u(x - s)) \, ds
\]
\[
\sim \rho_2 \left( \frac{u_{k+1} - u_{k-1}}{2h} - \frac{u_{k+2} - 2u_{k+1} + 2u_{k-1} - u_{k-1}}{6h} \right) + \frac{\rho_3}{6} \frac{u_{k+2} - 2u_{k+1} + 2u_{k-1} - u_{k-1}}{h}.
\]

15 Eigenvalue Problems for Fractional Operators

In this section we consider the eigenvalue problem for the fractional differential operator. Given the potential function \( q \in L^\infty(0,1) \) consider the eigenvalue problem
\[
Au = -\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} u''(s) \, ds + q(t)u(t) = \lambda u(t)
\]
with
\[\text{dom}(A) = \{ u \in H^1(0,1) : u' \in \text{dom}(D_t^\alpha) \text{ with } u(0) = u(1) = 0 \}.\]
Since
\[
\int_0^1 \left( \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} u''(s) \, ds \right) \phi(t) \, dt = \int_0^1 \left( \int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \phi(t) \, dt \right) u''(s) \, ds
\]
\[
= u'(1) \int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \phi(t) \, dt (1) - u'(0) \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \phi(t) \, dt - \int_0^1 u'(s) \frac{d}{ds} \int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \phi(t) \, dt 
\]
\[
= u'(1) \int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \phi(t) \, dt (1) - u'(0) \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \phi(t) \, dt + \int_0^1 u(s) \frac{d^2}{ds^2} \int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \phi(t) \, dt \, ds,
\]
the adjoint operator of \( A \) is given by
\[
A^* \phi = -\frac{d}{ds} \int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \phi'(t) \, dt + q(s)\phi(s)
\]
with
\[
\text{dom} (A^*) = \{ \phi \in L^2(0, 1) : \int_{s}^{1} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \phi(t) \, dt \in H^2(0, 1) \cap H^1_0(0, 1) \}
\]
\[
= \{ \phi \in H^1(0, 1) : \phi(1) = 0; \int_{0}^{1} t^{-\alpha} \phi(t) \, dt = 0, \int_{s}^{1} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \phi'(t) \, dt \in H^1(0, 1) \}
\]

Thus, \( \text{dom}(A) \neq \text{dom}(A^*) \) and \( A \) is not self-adjoint. Note that
\[
(\phi, A^* \phi) = \int_{0}^{1} \phi'(s) D_1 s \phi'(s) \, ds.
\]

With zero extension of \( \phi \) to we have
\[
\int_{0}^{1} \phi'(s) D_1 s \phi'(s) \, ds = \int (i \omega)^{1+\alpha} |\hat{\phi}(\omega)|^2
\]

where \( \hat{\phi} \) is the Fourier transform of \( \phi \).

We develop a numerical method that approximates \( A \) and \( A^* \) simultaneously. It is based on the Legendre-tau method \[.\] We use the Legendre approximation
\[
\tilde{u}^N(t) = \sum_{k=0}^{N} u_k L_k(2t-1)
\]

where \( L_k(\cdot) \) is the \( k \)-th Legendre polynomial on \([-1, 1]\). The boundary condition \( u^N(0) = u^N(1) = 0 \) implies
\[
u_N = - \sum_{k: \text{even}} u_k, \quad u_{N-1} = \sum_{k: \text{odd}} u_k \tag{15.1} \text{Leg1}
\]

if \( N \) is even and
\[
u_N = - \sum_{k: \text{odd}} u_k, \quad u_{N-1} = \sum_{k: \text{even}} u_k \tag{15.2} \text{Leg2}
\]

if \( N \) is odd. Thus, \( \{u_k\}_{k=0}^{N-2} \) defines the approximations
\[
u^N = \sum_{k=0}^{N-2} u_k L_k(2t-1)
\]

and
\[
\tilde{u}^N = u^N + u_{N-1} L_{N-1}(2t-1) + u_N L_N(2t-1).
\]

where \( u_{N-1} \) and \( u_N \) are defined by \( (15.1)-(15.2) \). The Legendre-tau approximation
\( A^N : X^{n-2} \rightarrow X^{n-2} \) is defined by
\[
A^N u^N = P^{n-2} A \tilde{u}^N,
\]

where \( P^{n-2} \) is the orthogonal projection of \( L^2(0, 1) \) onto
\[
X^{n-2} = \{ u \in L^2(0, 1) ; u = \sum_{k=0}^{N-2} u_k L_k(2t-1) \}.
\]
That is,

$$(A^Nu^N)_j = \int_0^1 L_j(2t-1) \left( - \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} (\tilde{u}^N)'(s) \, ds + q(t)u^N(t) \right) \, dt.$$

Since $(\tilde{u}^N)'' \in X^{N-2}$, for $\phi^N \in X^{N-2}$,

$$(A^Nu^N,\phi^N) = \int_0^1 (\tilde{u}^N)''(P^{N-2} \int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \phi^N(t) \, dt + \beta_{N-1}L_{N-1}(2s-1) + \beta_N L_N(2s-1))$$

where we let

$$\beta_N = - \sum_{k\text{ even}} \beta_k, \quad \beta_{N-1} = - \sum_{k\text{ odd}} \beta_k$$

and

$$P^{N-2} \left( \int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \phi^N(t) \, dt \right) = \sum_{k=0}^{N-2} \beta_k L_k(2s-1).$$

It thus follows that $(A^*)^N \in \mathcal{L}(X^{N-2},X^{N-2})$ is given by

$$(A^*)^N \phi^N = \frac{d^2}{ds^2} \left( P^{N-2} \int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \phi^N(t) \, dt + \beta_{N-1}L_{N-1}(2s-1) + \beta_N L_N(2s-1) \right) + P^{N-2}(q(s)\phi^N).$$

## 16 Appendix: Derivation

Let us consider the continuous time random walk (CTRW) process $x(t)$ on $R$ which is characterized by the joint probability density function (pdf) $\varphi(\xi,\tau)$ of jumps $\xi = x(t_j) - x(t_{j-1})$ and waiting times $\tau_j = t_j - t_{j-1}$. We assume jumps and waiting times are independent, i.e., $\varphi = \lambda(\xi)\psi(\tau)$. The jump pdf $\lambda(\xi)$ represents the pdf of jump size $\xi$ and the waiting time pdf $\psi(\tau)$ represents the pdf of waiting time $\tau$. Thus,

$$\int_0^t \psi(\tau) \, d\tau$$

gives the probability that at least one jump is taken in $(0,t)$.

$$\Psi(t) = 1 - \int_0^t \psi(\tau) \, d\tau$$

defines the probability of no jump occurs during $(0,t]$. Let us denote by $p(x,t)$ the pdf of reaching to position $x$ after time $t$, i.e. $p(x,0) = \delta(x)$. The master equation of CTRW is given by

$$p(x,t) = \delta(x)\Psi(t) + \int_0^t \psi(t-t') \int_{-\infty}^\infty \lambda(x-x')p(x',t') \, dx' \, dt'. \quad (16.1) \text{CTRW}$$

Taking the Laplace transform in $t$ and the Fourier transform in $x$ of (16.1), we obtain the Montroll-Weiss equation

$$\hat{p}(k,s) = \frac{\hat{\Psi}(s)}{1 - \hat{\lambda}(k)\hat{\psi}(s)} = \frac{1 - \psi(s)}{s} \frac{1}{1 - \lambda(k)\hat{\psi}(s)}. \quad (16.2) \text{MW}$$
In order to derive an evolution equation of Fokker-Planck-Kolmogorov type we rewrite (16.2) as
\[ \hat{\Phi}(s)(s^2 \hat{p}(k,s) - 1) = (\hat{\lambda}(k) - 1) \hat{p}(k,s), \] (16.3)
where
\[ \hat{\Phi}(s) = \frac{1 - \hat{\psi}(s)}{s \hat{\psi}(s)} = \frac{\hat{\Psi}(s)}{1 - s \hat{\Psi}(s)}. \]
Taking the inverse transforms of (16.3), we obtain the evolution equation for \( p \);
\[ \int_0^t \Phi(t - s) \frac{\partial}{\partial t} p(x,s) \, ds = -p(x,t) + \int_{-\infty}^{\infty} \lambda(x - x') p(x',t) \, dx', \] (16.4)
where
\[ \Phi(t) = \mathcal{L}^{-1}\left\{ \frac{\hat{\Psi}(s)}{1 - s \hat{\Psi}(s)} \right\} \]
and
\[ \Psi(t) = \int_0^t \Phi(t - s) \psi(s) \, ds. \]
If \( \hat{\Phi}(s) = 1 \) and thus \( \psi(t) = \Psi(t) = e^{-t} \), it reduces to the Kolmogorov-Feller equation
\[ \frac{\partial}{\partial t} p(x,t) = -p(x,t) + \int_{-\infty}^{\infty} \lambda(x - x') p(x',t) \, dx'. \]
If we assume \( \lambda \) is symmetric,
\[ -p(x,t) + \int_{-\infty}^{\infty} \lambda(x - x') p(x',t) \, dx' = \int_{-\infty}^{\infty} \lambda(s)(p(x+s) - 2p(x) + p(x-s)) \, ds \]
and thus
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda(s)(p(x+s) - 2p(x) + p(x-s)) \, ds \psi(x) \, dx \]
\[ = -\int_{-\infty}^{\infty} \lambda(s) \int_{-\infty}^{\infty} (p(x+s) - p(x)) (\psi(x+s) - \psi(x)) \, dx. \]
Hence, the right hand side defines a self-adjoint nonnegative definite operator \( A \) on \( L^2(R) \).

17 Fractional diffusion equation via Homogenization

In this section we discuss an example of fractional diffusion equation which is derived by the homogenization method. is presented. This method uses a small parameter which measures the characteristic length of the period (i.e. of the heterogeneities) compared to a macroscopic length. In Section 3, the classical
Consider a diffusive mass transport of chemical species through a rigid porous saturated composite, consisting of two porous materials;

\[ \frac{\partial c}{\partial t} = \nabla \cdot (D(\frac{x}{\epsilon})\nabla c + b(\frac{x}{\epsilon})c) \]  

(17.1) \text{model}

where we assume the periodic diffusive media \( D(\frac{x}{\epsilon}) \) and the periodic advection \( b(\frac{x}{\epsilon}) \) with period \( \epsilon \) and they are given by

\[
D(y) = \epsilon^2 D_2 \chi_{\Omega_2}(y) + D_1 \chi_{\Omega_1}(y), \quad b(y) = \epsilon^2 b_2 \chi_{\Omega_2}(y) + b_1 \chi_{\Omega_1}(y),
\]

Here subdomains \( \Omega_1 \) and \( \Omega_2 \) are for each composite and are disjoint and \( \overline{\Omega_1 \cup \Omega_2} = [0, 1]^d = \Omega \). \( \Gamma = \Omega_1 \cap \Omega_2 \).

Here \( \epsilon > 0 \) is a small parameter which measures the characteristic length of the period of the heterogeneities compared to a macroscopic length. The heterogeneity reflects that \( \Omega_1 \) is a diffusive (fluid) medium and \( \Omega_2 \) is a less diffusive (solid) medium with ratio \( \epsilon^2 \). It follows from [Auriault & Lewandowka] that the homogenized equation as \( \epsilon \to + \) is given by

\[
\frac{|\Omega_1|}{|\Omega|} \frac{\partial c}{\partial t} + \int_0^t K(t - s) \frac{\partial c}{\partial s} ds = \nabla \cdot (D_{\text{eff}} \nabla c + b_{\text{eff}} c),
\]

(17.2) \text{hom0}

where

\[
\mathcal{L}(K) = (1 - \hat{k}) = \frac{1}{|\Omega|} \int_{\Omega_2} (1 - \hat{k}) dy
\]

and the \( Y \)-periodic \( \hat{k} \) satisfies

\[
\nabla_y \cdot (D_2 \nabla_y \hat{k}) = p(\hat{k} - 1), \quad y \in \Omega_2 \quad \hat{k} = 0 \text{ at } \Gamma.
\]

and \( D_{\text{eff}} \) and \( b_{\text{eff}} \) are defined by (17.11).

In what follows we give a sketch of the derivation of (17.2). Let \( y = \frac{x}{\epsilon} \) and assume the expansion;

\[
c_1 = c_0^0(x, y, t) + \epsilon c_1^0(x, y, t) + \epsilon^2 c_1^1(x, y, t) + \cdots
\]

\[
c_2 = c_0^0(x, y, t) + \epsilon c_1^0(x, y, t) + \epsilon^2 c_1^0(x, y, t) + \cdots.
\]

Substituting this into (17.1) and using the calculus;

\[
\nabla c(x, y, t) = \nabla_x c(x, y, t) + \frac{1}{\epsilon} \nabla_y c(x, y, t)
\]

and taking the Laplace transform of the resulting equation in time, it results in the following order terms.

\( \epsilon^{-2} \) order:

\[
\nabla_y \cdot (D_1 \nabla_y c_1^0) = 0, \quad y \in \Omega_1 \quad \text{and} \quad n \cdot (D_1 \nabla_y c_1^0) = 0, \quad y \in \Gamma. \quad \text{(17.3) order-2}
\]

\( \epsilon^{-1} \) order:

\[
\nabla_y \cdot (D_1 \nabla_x c_1^0 + b_1 c_1^0) + \nabla_x \cdot (D_1 \nabla_y c_1^0) + \nabla_y \cdot (D_1 \nabla_x c_1^1) = 0, \quad y \in \Omega_1
\]

\[
n \cdot (D_1 \nabla_y c_1^1 + D_1 \nabla_x c_1^0 + b_1 c_1^0) = 0, \quad y \in \Gamma. \quad \text{(17.4) order-1}
\]
\[\nabla_x \cdot (D_1 \nabla_x c_1^0 + b_1 c_1^0) + \nabla_x \cdot (D_1 \nabla_y c_1^1) + \nabla_y \cdot (D_1 \nabla_x c_1^1 + b_1 c_1^1) + \nabla_y \cdot (D_1 \nabla_y c_2^1) = p c_1^0, \quad y \in \Omega_1\]

\[\n abla_y \cdot (D_2 \nabla_y c_2^0) = p c_2^0, \quad y \in \Omega_2.\]

From (17.3) \(c_1^1 = \hat{c}(x, p)\) in which \((x, p)\) acts as the parameter. Thus, from (17.4) we have

\[\nabla_y \cdot (D_1 \nabla_x c_1^0 + b_1 c_1^0) + \nabla_y \cdot (D_1 \nabla_y c_1^1) = 0, \quad y \in \Omega_1\]

and for \(\psi \in H^1_{\text{per}}(Y)\) we have

\[\int_{\Omega_1} (D_1 \nabla_y c_1^1, \nabla_y \psi) = -\sum_{ij} \frac{\partial}{\partial x_j} \int_{\Gamma} (D_1)_{ij} \frac{\partial \psi}{\partial y_i} - c_1^0 \int_{\Gamma} b_1, \nabla_y \psi) dy \]  

(17.7) \text{id1}

Let \(Y - \text{periodic functions} w^k, v \in H^1(\Omega_1)\) satisfy

\[\int_{\Omega_1} (D_1 \nabla_y w^k, \nabla_y \psi) = -\int_{\Omega_1} (D_1 e^k, \nabla_y \psi) dy \]

\[\int_{\Omega_1} (D_1 \nabla_y v, \nabla_y \psi) = -\int_{\Omega_1} (b_1, \nabla_y \psi) dy \]

for all \(\psi \in H^1_{\text{per}}(Y)\). It follows from (17.7) that

\[c_1^1 = \sum_k w^k(y) \frac{\partial}{\partial x_k} c_1^0 + v(y) \hat{c}(x, p).\]

(17.9) \text{homo1}

From (17.6)

\[\nabla_y \cdot (D_2 \nabla_y (\hat{c}_2^0 - \hat{c}) = p(\hat{c}_2^0 - \hat{c} + \hat{c}), \quad \hat{c}_2^0 - \hat{c} = 0 \text{ at } \Gamma,\]

and thus

\[\hat{c}_2^0 = (1 - \hat{k}) \hat{c},\]

(17.10) \text{homo2}

where \(Y - \text{periodic function} \hat{k}\) satisfies

\[\nabla_y \cdot (D_2 \nabla_y \hat{k}) = p(\hat{k} - 1), \quad y \in \Omega_2, \quad \hat{k} = 0 \text{ on } \Gamma.\]

Integrating (17.5) with respect to \(y\) on \(\Omega_1\), we obtain

\[\int_{\Omega_1} \nabla_x \cdot (D_1 \nabla_x c_1^0 + b_1 c_1^0) dy + \int_{\Omega_1} \nabla_x \cdot (D_1 \nabla_y c_1^1) dy + \int_{\Gamma} n \cdot (D_2 c_2^0) ds = p|\Omega_1| \hat{c}\]

From (17.9)

\[\frac{1}{|\Omega|} \left( \int_{\Omega_1} \nabla_x \cdot (D_1 \nabla_x c_1^0 + b_1 c_1^0) dy + \int_{\Omega_1} \nabla_x \cdot (D_1 \nabla_y c_1^1) dy \right) = \nabla_x (D_{\text{eff}} \nabla_x \hat{c} + b_{\text{eff}} \hat{c})\]

53
where
\[(D_{\text{eff}})_{kj} = \frac{1}{|\Omega|} \int_{\Omega_1} D_1(I_{kj} + \frac{\partial w^k}{\partial y_j}) \, dy\] (17.11) homo3

\[b_{\text{eff}} = \frac{1}{|\Omega|} \int_{\Omega_1} (b_1 + D_1 \nabla v) \, dy.\]

From (17.6) and (17.10)
\[\frac{1}{|\Omega|} \int_{\Gamma} n \cdot (D_2 \hat{c}_0^0) \, ds = -\frac{1}{|\Omega|} \int_{\Omega_1} p(1 - \hat{k}) \, dy \hat{c}\]

Hence we obtain
\[\nabla_x (D_{\text{eff}} \nabla_x \hat{c} + b_{\text{eff}} \hat{c}) + \frac{1}{|\Omega|} \int_{\Omega_1} (1 - \hat{k}) \, dy \hat{c} = \frac{|\Omega_1|}{|\Omega|} p \hat{c}.\]

which implies (17.2).

If \(\hat{u} = 1 - \hat{k}\) we have
\[D_2 \Delta \hat{u} = p \hat{u}\]
\[\hat{u} = 1 \quad \text{at } \Gamma\]

For \(\Omega_2 = \{|r| \leq r_0\}\) in \(R^3\) the radial solution \(\hat{u} = \hat{u}(r)\) satisfies
\[(ru)^{'''} = pr \hat{u}(r)\]

Thus,
\[\hat{u}(r) = \frac{\sinh(\sqrt{pr})}{r \sinh(\sqrt{pr_0})}\]

and
\[\langle \hat{u} \rangle = \frac{4\pi r_0^2}{\sqrt{p}} \coth(\sqrt{pr_0}) - \frac{4\pi r_0}{p}.\]

In general, from (17.12) by the divergence theory
\[p\langle \hat{u} \rangle = \int_{\Gamma} \frac{\partial}{\partial \nu} \hat{u} \, ds.\]

Since at \(x^* \in \Gamma\)
\[\frac{\partial^2}{\partial \nu^2} \hat{u} + \Delta_r \hat{u} = p \hat{u}\]
and \(\hat{u} = 1\) on \(\Gamma\), we have
\[\frac{\partial^2}{\partial \nu^2} \hat{u} - \kappa \frac{\partial}{\partial \nu} \hat{u} = p \hat{u},\]
where \(\kappa\) is the curvature of \(\Gamma\) at \(x^*\). It can be shown that
\[\frac{\partial}{\partial \nu} \hat{u} \sim \sqrt{p}\]
for \(p >> 1\), since
\[\hat{u} \sim e^{-\sqrt{p}(x-x^*) \nu} \text{ at } x^* \in \Gamma.\]

54
Thus, for $p >> 1$

$$\langle \hat{u} \rangle \sim \frac{\vert \Gamma \vert}{\vert \Omega \vert} \frac{1}{\sqrt{p}}$$

and

$$K(t) \sim \frac{\vert \Gamma \vert}{\vert \Omega \vert} t^{-\frac{1}{2}} \frac{1}{\Gamma(\frac{1}{2})}.$$ 

It thus follows from (17.2) that

$$\frac{\vert \Omega \vert}{\vert \Omega \vert} \frac{\partial}{\partial t} c + \frac{\vert \Gamma \vert}{\vert \Omega \vert} D_{t^\frac{1}{2}} c = \nabla \cdot (D_{\text{eff}} \nabla c + b_{\text{eff}} c).$$

References


