Some Open Problems in Classical Invariant Theory.

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1 Invariants.

Consider a quadratic polynomial $p(x) = a\,x^2 + b\,x + c$. Recall the quadratic formula and observe that the discriminant $D = b^2 - 4ac$ governs whether there are 0,1 or 2 real roots of $p(x)$. A substitution (translation) $x \to x + t$, where t is a number does change the polynomial: $\bar{p}(x) = p(x+t) = a(x+t)^2 + b(x+t) + c = ax^2 + (2ta+b)x + (t^2a + tb + c)$. It does not, however, change the discriminant: $(2ta+b)^2 - 4a(t^2a+tb+c) = b^2 - 4ac$. For this reason one says that the discriminant is *invariant* under translations. Moreover, two quadratics with the same leading coefficient a have the same discriminant if and only if they can be translated to each other.

Questions. If one considers polynomials of higher degree or larger number of variables, how many invariants, similar to the discriminant, are there? What properties do they characterize? What happens if a larger set of substitutions is considered (translations and scalings for example)?

2 Symmetries.

Consider polynomial $p(x,y) = x^8 + 14x^4y^4 + y^8$. Any of the following four complex linear substitution does not change this polynomial.

$$(x,y) \to \begin{cases} (\frac{\sqrt{2}}{2}(1+i)x,\ \frac{\sqrt{2}}{2}(1+i)y) \\ (\frac{\sqrt{2}}{2}i(x+y),\ \frac{\sqrt{2}}{2}(x-y)) \\ (ix,\ y) \\ \frac{\sqrt{2}}{2}(1+i)x,\ \frac{\sqrt{2}}{2}(1+i)y) \end{cases}$$

One can say that $p(x,y)$ is symmetric with respect to these four linear transformations. It is also symmetric with respect to the group of transformations they generate, consisting of 192 elements.

The polynomial $p(x,y) = x^2 + y^2$ has even larger, in fact, infinite symmetry group, since it contains all rotations in the xy-plane, while $x^5 - 4xy^4 - 2y^5$ is preserved only by scalings by a fifth root of 1: $x \to wx, y \to wy$, where $w^5 = 1$.

Question. Given a multivariable polynomial, how can one efficiently find the size of its symmetry group? Compute it explicitly?

3 Hesse "Theorem".

The transformation $x \to x - y, y \to y$ transforms the polynomial $p(x,y) = x^2 + 2xy + y^2 = (x+y)^2$ in two variables to the polynomial $\bar{p}(x,y) = x^2$ is one variable only. Note that its Hessian $\det\left(\frac{\partial^2 p}{\partial x \partial y}\right) = 0$. In fact, the Hessian of a homogeneous polynomial in two variables is zero if and only if it can be transformed to a polynomial of a single variable by a linear change of variables. Hesse claimed that the similar result is true for any number of variables: a homogeneous polynomial $p(x_1, \ldots, x_m)$ can be transformed to a polynomial of fewer than m variables if and only if its Hessian $\det\left(\frac{\partial^2 p}{\partial x_i \partial x_j}\right) = 0$. This conjecture was assumed to be true for 25 years until Noether and Gordan showed that it is true for $m \leq 4$ only. The polynomial $x_1^2 x_3 + x_1 x_2 x_4 + x_2^2 x_5$, whose Hessian is zero, is a counter-example for this claim.

Question. How can one determine efficiently that a given polynomial essentially depends on a fewer number of variables than it seems to be? How to find the corresponding change of variables?

4 Remark

In the general case these problems are difficult and still unsolved. Some special cases and sub-problems, however, are suitable for an undergraduate project.