Gentle Invitation to
"Real Quantifier Elimination"

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Real Quantifier Elimination?

Input: $\forall x \quad x^2 + 1 > 0$

Output: $True$

Input: $\forall x \quad x^2 + 3x + 1 > 0$

Output: $False$

Input: $\forall x \quad x^2 + bx + 1 > 0$

Output: $-2 < b < 2$

Real Quantifier Elimination?

Input: $\forall x \exists y \quad x^2 + xy + b > 0$

$$\wedge$$

$$x + ay^2 + b \leq 0$$

Output: $a < 0 \wedge b > 0$

Real Quantifier Elimination?

Input: $\exists Y \quad F(X,Y)=0 \quad \wedge \quad G(X,Y)>0$

$$X=\{a,b\}$$

$$Y=\{c_1,s_1,c_2,s_2\}$$

$$F=\{c_1^2+s_1^2-1,\ c_2^2+s_2^2-1\}$$

$$G=\{g\}$$

$$\begin{aligned}
g = {} & 4a^6b^2c_1^4c_2^2 - 8a^5b^3s_1s_2c_1^3c_2 - 8a^5b^3s_1s_2c_1^2c_2^2 + 4a^4b^4c_1^4c_2^2 + \\
& 16a^4b^4c_1^3c_2^3 + 4a^4b^4c_1^2c_2^4 - 8a^3b^5s_1s_2c_1^2c_2^2 - 8a^3b^5s_1s_2c_1c_2^3 + \\
& 4a^2b^6c_1^2c_2^4 - 4a^7bs_1s_2c_1^3 + 4a^6b^2c_1^4c_2 - 4a^6b^2c_1^3c_2^2 + 8a^5b^3s_1s_2c_1^3 + \\
& 12a^5b^3s_1s_2c_1^2c_2 + 16a^5b^3s_1s_2c_1c_2^2 - 8a^4b^4c_1^4c_2 - 24a^4b^4c_1^3c_2^2 - \\
& 24a^4b^4c_1^2c_2^3 - 8a^4b^4c_1c_2^4 + 16a^3b^5s_1s_2c_1^2c_2 + 12a^3b^5s_1s_2c_1c_2^2 + \\
& 8a^3b^5s_1s_2c_2^3 - 4a^2b^6c_1^2c_2^3 + 4a^2b^6c_1c_2^4 - 4ab^7s_1s_2c_2^3 + a^8c_1^4 + \\
& 12a^7bs_1s_2c_1^2 - 8a^6b^2c_1^4 - 12a^6b^2c_1^3c_2 - 12a^6b^2c_1^2c_2^2 - 4a^5b^3s_1s_2c_1^2 - \\
& 8a^5b^3s_1s_2c_2^2 + 4a^4b^4c_1^4 + 22a^4b^4c_1^2c_2^2 + 4a^4b^4c_2^4 - 4a^4b^2c_1^4c_2^2 - \\
& 8a^3b^5s_1s_2c_1^2 - 4a^3b^5s_1s_2c_2^2 + 8a^3b^3s_1s_2c_1^2c_2^2 - 12a^2b^6c_1^2c_2^2 - \\
& 12a^2b^6c_1c_2^3 - 8a^2b^6c_2^4 - 4a^2b^4c_1^2c_2^4 + 12ab^7s_1s_2c_2^2 + b^8c_2^4 - 4a^8c_1^3 - \\
& 12a^7bs_1s_2c_1 + 16a^6b^2c_1^3 + 12a^6b^2c_1^2c_2 + 20a^6b^2c_1c_2^2 - 16a^5b^3s_1s_2c_1 - \\
& 4a^5b^3s_1s_2c_2 + 4a^5bs_1s_2c_1^3 + 8a^4b^4c_1^3 + 12a^4b^4c_1^2c_2 + 12a^4b^4c_1c_2^2 + \\
& 8a^4b^4c_2^3 + 4a^4b^2c_1^4c_2 + 4a^4b^2c_1^3c_2^2 - 4a^3b^5s_1s_2c_1 - 16a^3b^5s_1s_2c_2 - \\
& 12a^3b^3s_1s_2c_1^2c_2 - 12a^3b^3s_1s_2c_1c_2^2 + 20a^2b^6c_1^2c_2 + 12a^2b^6c_1c_2^2 + \\
& 16a^2b^6c_2^3 + 4a^2b^4c_1^2c_2^3 + 4a^2b^4c_1c_2^4 - 12ab^7s_1s_2c_2 + 4ab^5s_1s_2c_2^3 - \\
& 4b^8c_2^3 + 6a^8c_1^2 + 4a^7bs_1s_2 - 4a^6b^2c_1c_2 - 8a^6b^2c_2^2 - 2a^6c_1^4 + \\
& 12a^5b^3s_1s_2 - 12a^5bs_1s_2c_1^2 - 14a^4b^4c_1^2 + 8a^4b^4c_1c_2 - 14a^4b^4c_2^2 - \\
& 4a^4b^2c_1^3c_2 + 10a^4b^2c_1^2c_2^2 + 12a^3b^5s_1s_2 + 4a^3b^3s_1s_2c_1^2 + 16a^3b^3s_1s_2c_1c_2 + \\
& 4a^3b^3s_1s_2c_2^2 - 8a^2b^6c_1^2 - 4a^2b^6c_1c_2 + 10a^2b^4c_1^2c_2^2 - 4a^2b^4c_1c_2^3 + \\
& 4ab^7s_1s_2 - 12ab^5s_1s_2c_2^2 + 6b^8c_2^2 - 2b^6c_2^4 - 4a^8c_1 - 16a^6b^2c_1 + \\
& 8a^6c_1^3 + 12a^5bs_1s_2c_1 - 12a^4b^4c_1 - 12a^4b^4c_2 - 8a^4b^2c_1^2c_2 - 16a^4b^2c_1c_2^2 - \\
& 4a^3b^3s_1s_2c_1 - 4a^3b^3s_1s_2c_2 - 16a^2b^6c_2 - 16a^2b^4c_1^2c_2 - 8a^2b^4c_1c_2^2 + \\
& 12ab^5s_1s_2c_2 - 4b^8c_2 + 8b^6c_2^3 + a^8 + 8a^6b^2 - 12a^6c_1^2 - 4a^5bs_1s_2 + 14a^4b^4 - \\
& 2a^4b^2c_1^2 + 12a^4b^2c_1c_2 + 6a^4b^2c_2^2 + a^4c_1^4 + 8a^2b^6 + 6a^2b^4c_1^2 + 12a^2b^4c_1c_2 - \\
& 2a^2b^4c_2^2 + 2a^2b^2c_1^2c_2^2 - 4ab^5s_1s_2 + b^8 - 12b^6c_2^2 + b^4c_2^4 + 8a^6c_1 + 4a^4b^2c_1 - \\
& 4a^4b^2c_2 - 4a^4c_1^3 - 4a^3bs_1s_2c_1 - 4a^2b^4c_1 + 4a^2b^4c_2 - 4ab^3s_1s_2c_2 + 8b^6c_2 - \\
& 4b^4c_2^3 - 2a^6 - 2a^4b^2 + 8a^4c_1^2 + 4a^3bs_1s_2 - 2a^2b^4 - 2a^2b^2c_1^2 + 4a^2b^2c_1c_2 - \\
& 2a^2b^2c_2^2 + 4ab^3s_1s_2 - 2b^6 + 8b^4c_2^2 - 8a^4c_1 - 4a^2b^2c_1 - 4a^2b^2c_2 - 8b^4c_2 + \\
& 3a^4 + 6a^2b^2 - 2a^2c_1^2 + 3b^4 - 2b^2c_2^2 + 4a^2c_1 + 4b^2c_2 - 2a^2 - 2b^2
\end{aligned}$$

Output: $a^6+3a^4b^2+3a^2b^4+b^6-3a^4+21a^2b^2-3b^4+3a^2+3b^2>1$

Real Quantifier Elimination?

Input: *Expression* made of

Algebraic expressions

$= \quad \neq \quad > \quad < \quad \geq \quad \leq$

$\wedge \quad \vee \quad \neg \quad \Rightarrow \quad \Leftarrow \quad \Leftrightarrow$

$\forall \quad \exists$

Output: *Expression*

equivalent

without $\forall \quad \exists$

Why work on it?

Foundation of Mathematics

- Hilbert's Program (1900)

Applications in Science and Engineering

- Stability analysis of PDE and Finite differences
- Robust control system design
- Reachability analysis
- Parametric optimization
- Hybrid system analysis
- Parameter estimation
- Robot motion planning
- Computer vision
- Dynamic geometric constraint solving
- Education software for real analysis
- ….

Why applicable?

"Ability" $\longrightarrow$ $\exists$

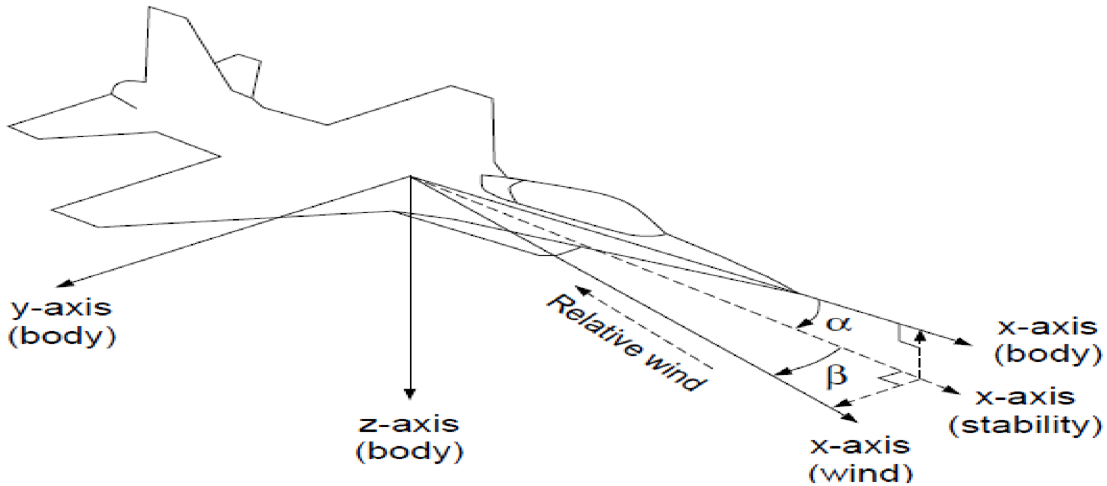
- Stablizability
- Reachability
- Satisfiability

"Robustness" $\longrightarrow$ $\forall$

- Robust control
- Tolerant system
- Stability

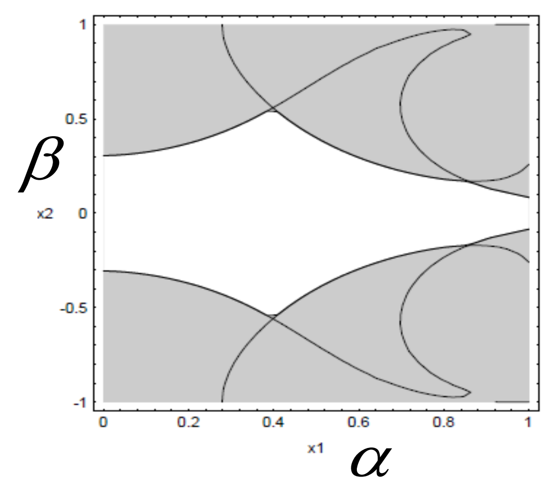
App: Nonlinear Control of Aircraft

M. Jirstrand *J. Symbolic Computation* (1997)



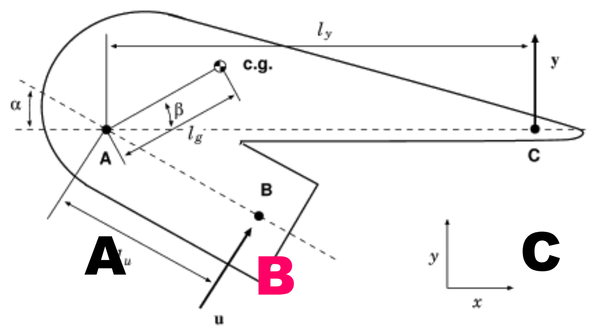
$$\exists u_1\, \exists u_2\, \exists u_3\, \left[C_L = 0 \;\wedge\; C_M = 0 \;\wedge\; C_N = 0 \;\wedge\; u_i^2 \leq 1,\; i = 1,2,3\right]$$

$$\begin{aligned}
C_L(x_1, x_2, u1, u3) = &- 38x_2 - 170x_1x_2 + 148x_1^2x_2 + 4x_2^3 \\
&+ u_1(-52 - 2x_1 + 114x_1^2 - 79x_1^3 + 7x_2^2 + 14x_1x_2^2) \\
&+ u_3(14 - 10x_1 + 37x_1^2 - 48x_1^3 + 8x_1^4 - 13x_2^2 - 13x_1x_2^2 \\
&+ 20x_1^2x_2^2 + 11x_2^4) \\
C_M(x_1, u_2) = &- 12 - 125u_2 + u_2^2 + 6u_2^3 + 95x_1 - 21u_2x_1 + 17u_2^2x_1 \\
&- 202x_1^2 + 81u_2x_1^2 + 139x_1^3 \\
C_N(x_1, x_2, u1, u3) = &139x_2 - 112x_1x_2 - 388x_1^2x_2 + 215x_1^3x_2 - 38x_2^3 + 185x_1x_2^3 \\
&+ u_1(-11 + 35x_1 - 22x_1^2 + 5x_2^2 + 10x_1^3 - 17x_1x_2^2) \\
&+ u_3(-44 + 3x_1 - 63x_1^2 + 34x_2^2 + 142x_1^3 + 63x_1x_2^2 - 54x_1^4 \\
&- 69x_1^2x_2^2 - 26x_2^4)
\end{aligned}$$

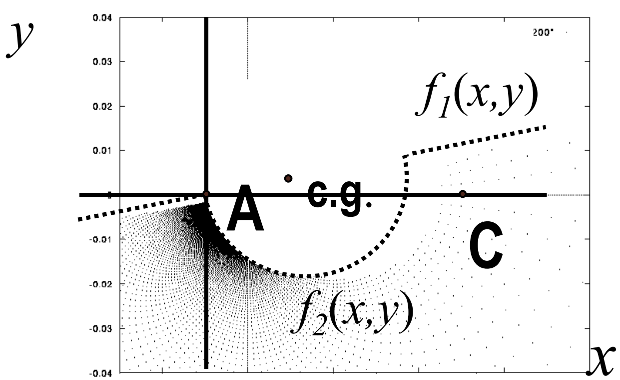


App: Design of HDD Swing-arm

H. Anai, S. Hara (2000) Fujitsu

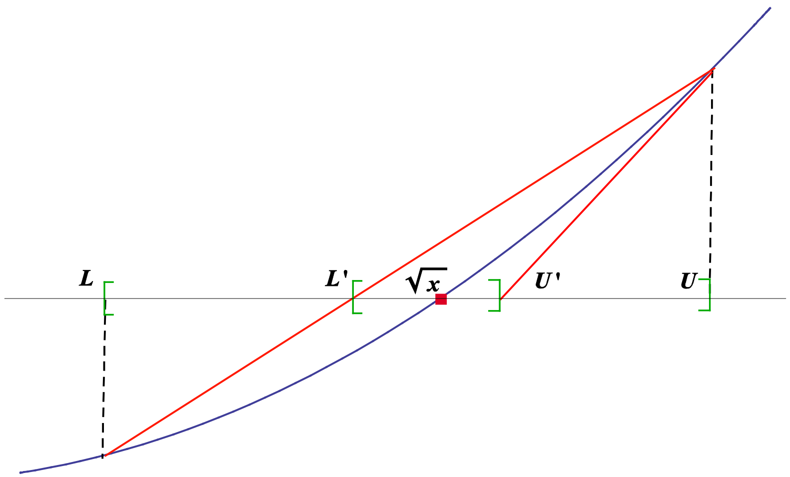


Where should the actuator **B** be located so that the arm satisfies "stability" and "positive realness (PR)" ?



App: Optimal Numerical Algorithm

M. Erascu, H. Hong, Reliable Computing (2013)



$$L' = L + \frac{x + p_0 L^2 + p_1 LU + p_2 U^2}{p_3 L + p_4 U} \qquad U' = U + \frac{x + q_0 U^2 + q_1 UL + q_2 L^2}{q_3 U + q_4 L}$$

$$Correctness(p,q) :\Longleftrightarrow \forall_{L,U,x}\ 0 < L \leq \sqrt{x} \leq U \implies 0 < L' \leq \sqrt{x} \leq U'$$

$$Termination(p,q) :\Longleftrightarrow \exists_{c\in(0,1)} \forall_{L,U,x}\ 0 < L \leq \sqrt{x} \leq U \implies 0 \leq U' - L' \leq c(U - L)$$

App: Stability of Numerical PDE

H. Hong and M. Safey-Eldin, *J. of Symbolic Computation* (2012)

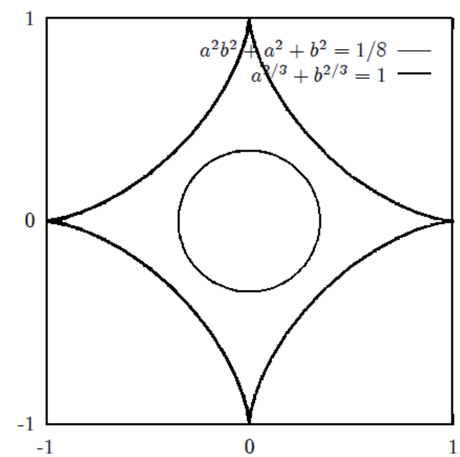
$$u(x, y, t + 2\Delta_t) \approx M u(x, y, t).$$

$$\begin{aligned}
G_{++} &= I + a(T_x - I) + b(T_y - I), \\
G_{--} &= I + a(I - T_x^{-1}) + b(I - T_y^{-} \\
G_{-+} &= I + a(I - T_x^{-1}) + b(T_y - I) \\
G_{+-} &= I + a(T_x - I) + b(I - T_y^{-1}) \\
M_1 &= \frac{1}{2}(I + G_{++}G_{--}), \\
M_2 &= \frac{1}{2}(I + G_{-+}G_{+-}), \\
M &= M_2 M_1.
\end{aligned}$$

$$\exists Y \quad F(X,Y) = 0 \quad \wedge \quad G(X,Y) > 0$$

$$X = \{a, b\}$$
$$Y = \{c_1, s_1, c_2, s_2\}$$
$$F = \{c_1^2 + s_1^2 - 1,\ c_2^2 + s_2^2 - 1\}$$
$$G = \{g\}$$

$$\begin{aligned}
g = {} & 4a^6b^2c_1{}^4c_2{}^2 - 8a^5b^3s_1s_2c_1{}^3c_2 - 8a^5b^3s_1s_2c_1{}^2c_2{}^2 + 4a^4b^4c_1{}^4c_2{}^2 + \\
& 16a^4b^4c_1{}^3c_2{}^3 + 4a^4b^4c_1{}^2c_2{}^4 - 8a^3b^5s_1s_2c_1{}^2c_2{}^2 - 8a^3b^5s_1s_2c_1c_2{}^3 + \\
& 4a^2b^6c_1{}^2c_2{}^4 - 4a^7bs_1s_2c_1{}^3 + 4a^6b^2c_1{}^4c_2 - 4 \\
& 12a^5b^3s_1s_2c_1{}^2c_2 + 16a^5b^3s_1s_2c_1c_2{}^2 - 8a \\
& 24a^4b^4c_1{}^2c_2{}^3 - 8a^4b^4c_1c_2{}^4 + 16a^3b^5s_1s_2 \\
& 8a^3b^5s_1s_2c_2{}^3 - 4a^2b^6c_1{}^2c_2{}^3 + 4a^2b^6c_1c_2 \\
& 12a^7bs_1s_2c_1{}^2 - 8a^6b^2c_1{}^4 - 12a^6b^2c_1{}^3c_2 - 12 \\
& 8a^5b^3s_1s_2c_2{}^2 + 4a^4b^4c_1{}^4 + 22a^4b^4c_1{}^2c_2{}^2 \\
& 8a^3b^5s_1s_2c_1{}^2 - 4a^3b^5s_1s_2c_2{}^2 + 8a^3b^3s_1 \\
& 12a^2b^6c_1c_2{}^3 - 8a^2b^6c_2{}^4 - 4a^2b^4c_1{}^2c_2{}^4 + 12a \\
& 12a^7bs_1s_2c_1 + 16a^6b^2c_1{}^3 + 12a^6b^2c_1{}^2c_2 + 2 \\
& 4a^5b^3s_1s_2c_2 + 4a^5bs_1s_2c_1{}^3 + 8a^4b^4c_1{}^3 + 1 \\
& 8a^4b^4c_2{}^3 + 4a^4b^2c_1{}^4c_2 + 4a^4b^2c_1{}^3c_2{}^2 - 4 \\
& 12a^3b^3s_1s_2c_1{}^2c_2 - 12a^3b^3s_1s_2c_1c_2{}^2 + 20 \\
& 16a^2b^6c_2{}^3 + 4a^2b^4c_1{}^2c_2{}^3 + 4a^2b^4c_1c_2{}^4 - 1 \\
& 4b^8c_2{}^3 + 6a^8c_1{}^2 + 4a^7bs_1s_2 - 4a^6b^2c_1 \\
& 12a^5b^3s_1s_2 - 12a^5bs_1s_2c_1{}^2 - 14a^4b^4c_1{}^2 \\
& 4a^4b^2c_1{}^3c_2 + 10a^4b^2c_1{}^2c_2{}^2 + 12a^3b^5s_1s_2 + 4a \\
& 4a^3b^3s_1s_2c_2{}^2 - 8a^2b^6c_1{}^2 - 4a^2b^6c_1c_2 + 1 \\
& 4ab^7s_1s_2 - 12ab^5s_1s_2c_2{}^2 + 6b^8c_2{}^2 - 2b^6 \\
& 8a^6c_1{}^3 + 12a^5bs_1s_2c_1 - 12a^4b^4c_1 - 12a^4b^4c_2 \\
& 4a^3b^3s_1s_2c_1 - 4a^3b^3s_1s_2c_2 - 16a^2b^6c_2 - \\
& 12ab^5s_1s_2c_2 - 4b^8c_2 + 8b^6c_2{}^3 + a^8 + 8a^6b^2 - 1 \\
& 2a^4b^2c_1{}^2 + 12a^4b^2c_1c_2 + 6a^4b^2c_2{}^2 + a^4c_1{}^4 + 8 \\
& 2a^2b^4c_2{}^2 + 2a^2b^2c_1{}^2c_2{}^2 - 4ab^5s_1s_2 + b^8 - 12b^6 \\
& 4a^4b^2c_2 - 4a^4c_1{}^3 - 4a^3bs_1s_2c_1 - 4a^2b^4c_1 + 4 \\
& 4b^4c_2{}^3 - 2a^6 - 2a^4b^2 + 8a^4c_1{}^2 + 4a^3bs_1s_2 - 2a^2b^4 - 2a^2b^2c_1{}^2 + 4a^2b^2c_1c_2 - \\
& 2a^2b^2c_2{}^2 + 4ab^3s_1s_2 - 2b^6 + 8b^4c_2{}^2 - 8a^4c_1 - 4a^2b^2c_1 - 4a^2b^2c_2 - 8b^4c_2 + \\
& 3a^4 + 6a^2b^2 - 2a^2c_1{}^2 + 3b^4 - 2b^2c_2{}^2 + 4a^2c_1 + 4b^2c_2 - 2a^2 - 2b^2
\end{aligned}$$

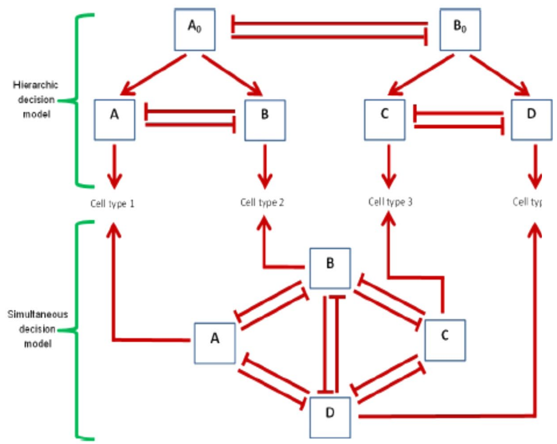


$$a^6 + 3a^4b^2 + 3a^2b^4 + b^6 - 3a^4 + 21a^2b^2 - 3b^4 + 3a^2 + 3b^2 > 1$$

App: Stability Analysis

Multi-stable model of Cell Differentiation

H, Hong, X. Tang, B. Xia (2013)



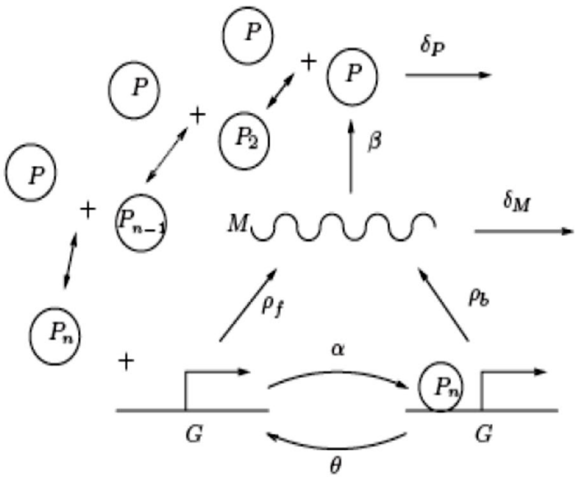
$$\frac{dx_k}{dt} = -x_k + \sigma \frac{1}{1 + \Sigma_{m=1}^{n} x_m^c - x_k^c}$$

$$\exists \mathbf{x} \left(\mathbf{f}(\sigma, \mathbf{x}) = 0 \ \wedge \ \det\left(J_{\mathbf{f}}\left(\sigma, \mathbf{x}\right)\right) \cdot \prod_{k=1}^{n} \Delta_k(\sigma, \mathbf{x}) = 0 \right)$$

App: Hopf Bifurcation

Gene regulatory network
Circadian clock of green alga

T. Sturm, et al (2013)



$$
\begin{aligned}
&\exists v_2 \exists v_1 \exists v_3 (0 < v_1 \wedge 0 < v_3 \wedge 0 < v_2 \wedge \vartheta \gamma_0 - v_1 - v_1 v_3{}^9 = 0 \wedge \\
&\lambda_1 v_1 + \gamma_0 \mu - v_2 = 0 \wedge 9 \alpha \gamma_0 - v_1 - v_1 v_3{}^9 + \delta v_2 - v_3 = 0 \wedge \\
&0 < \vartheta \delta + \vartheta v_3{}^9 \delta + 9 \lambda_1 \vartheta v_1 v_3{}^8 \delta \wedge \\
&162 \vartheta v_3{}^{17} \alpha v_1 + 162 \vartheta \alpha v_1 v_3{}^8 + 162 \alpha v_1 v_3{}^8 \delta + \vartheta + 2 \vartheta v_3{}^9 \delta + \vartheta^2 v_3{}^{18} \delta \\
&\quad + \vartheta v_3{}^9 \vartheta \delta + 81 \alpha v_1 v_3{}^8 \vartheta \delta + 81 \alpha v_1 v_3{}^{17} \vartheta \delta + \delta^2 + \vartheta \delta^2 + \vartheta^2 \delta + \vartheta^2 \\
&\quad + 2 \vartheta^2 v_3{}^9 + \vartheta^2 v_3{}^{18} + 6561 \alpha^2 v_1{}^2 v_3{}^{16} + 2 \vartheta^2 v_3{}^9 \delta + \delta + 81 \alpha v_1 v_3{}^8 \\
&\quad + \vartheta v_3{}^9 \delta^2 - 9 \lambda_1 \vartheta v_1 v_3{}^8 \delta = 0 \wedge \\
&0 < \vartheta \wedge 0 < \gamma_0 \wedge 0 < \mu \wedge 0 < \delta \wedge 0 < \alpha)
\end{aligned}
$$

Why work on it? (Recap)

- *Arise as a fundamental question in the logical foundation of mathematics*

- *Numerous problems from science and engineering can be reduced to QE problems.*

History/State of the Art

- ~1930: Tarski

  $$2^{2^{2^{\cdot^{\cdot^{n}}}}}$$

- ~1975: Collins

  $$2^{2^{n}}$$

- ~1990: Canny, Grigorev, Renegar, Roy, Basu,…

  2^{n} for existential inputs

- ----- : Arnon, McCallum, Hong, Brown, Strezbonski,…

  Efficient algorithms for moderate inputs

- ----- : Weispfenning, Sturm, Dolzman, Hong, Safey-Eldin, Anai, Xia, Chen, Kosta, . . .

  Efficient algorithms for special inputs

Software Packages

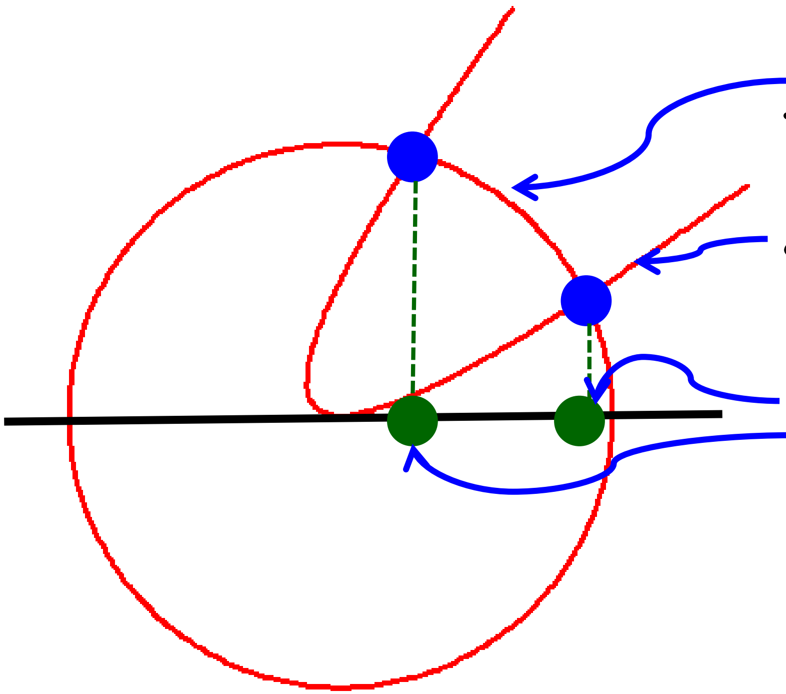
- General Inputs
  - QEPCAD (in C)
  - Resolve (in Mathematica)
  - Regular chain (in Maple)
  - Discoverer (in Maple)

- Special Inputs
  - Redlog (in REDUCE)
  - SyNRAC (in Maple)
  - …

How does it work?

- *Deep theories from*
  - (Real) Algebraic geometry
  - Commutative algebra
  - Real/complex analysis
- *Efficient operations from*
  - Computational algebra
  - Computational geometry
  - List processing

Taste of a mathematical theory…



$$f(x,y) = y^2 + x^2 - 4$$

$$g(x,y) = y^2 - (2x+1)y + x^2$$

$$h(x) = ?$$

$$res(f,g) = \begin{vmatrix} 1 & 0 & x^2-4 & \\ & 1 & 0 & x^2-4 \\ 1 & -2x-1 & x^2 & \\ & 1 & -2x-1 & x^2 \end{vmatrix}$$

$$= 4x^4 + 4x^3 - 15x^2 - 16x + 12$$

$$x = 0.538,\ 1.801$$

Challenges!

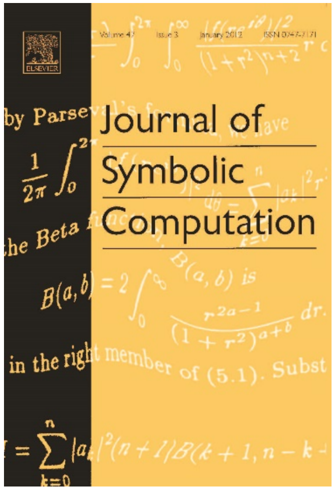
Improve the efficiency of QE

Tackle nontrivial engineering problems using QE

- *Exploit the inherent structure of the inputs*
- *Collaboration between*
  - Mathematicians (QE)
  - Engineers

Some References

- Computational Quantifier Elimination
  - *Computer Journal*
  - Edited by Hong
- Collins' 65th Birthday Conference
  - *RISC monograph*
  - Edited by Johnson and Caviness
- Application of Quantifier Elimination
  - *Journal of Symbolic Computation*
  - Edited by Hong
- Algorithms in Real Algebraic Geometry
  - *Springer*
  - Authored by Basu, Pollack, Roy
- Numerous individual articles
  - *Journal of Symbolic Computation*



Come and Join!