Final Exam

Solve the following problems.

(1) [4 Pts] Let $T$ be a bounded linear operator on a Hilbert space $H$. Let $\mathcal{N}(T)$ and $\mathcal{R}(T)$ be the null space and range of $T$, respectively. Show that
\[ \text{cl}(\mathcal{R}(T)) = (\mathcal{N}(T^*))^\perp. \]
(Hint: Show first that $(\mathcal{R}(T))^\perp = \mathcal{N}(T^*)$. Next show that $(\mathcal{R}(T))^{\perp\perp} = \text{cl}(\mathcal{R}(T))$.)

(2) [5 Pts] (a) Prove that if $T$ is a bounded linear operator on a Hilbert space $H$ and $\|T\| < 1$, then $T - I$ is invertible ($I$ is the identity operator) and
\[ (I - T)^{-1} = \sum_{k=0}^{\infty} T^k \]
with convergence in the operator norm (notice that in the series $T^0 = I$).
(b) Use part (a) to deduce that if if $T$ is a bounded linear operator on a Hilbert space $H$ and $\|I - T\| < 1$, then $T$ is invertible.

(3) [6 Pts] A sequence $(x_n)$ in a Hilbert space $H$ is a frame of $H$ if there exist real constants $A, B > 0$ such that, for all $x \in H$:
\[ A \|x\|^2 \leq \sum_{n=0}^{\infty} |\langle x, x_n \rangle|^2 \leq B \|x\|^2. \]
$A$ and $B$ are called the frame constants.
(a) Prove that if $(x_n)$ is a frame of $H$ with frame constants $A = B = 1$, and $\|x_n\| = 1$ for all $n$, then $(x_n)$ is an orthonormal basis of $H$.
(b) Prove that if $(x_n)$ is a frame of $H$, then the operator $S$, defined by $Sx = \sum_{n=0}^{\infty} \langle x, x_n \rangle x_n$, where $x \in H$, is a self-adjoint bounded linear operator on $H$.
(c) Prove that the operator $S$ is invertible on $H$. (Hint: you have to show that $S$ is one to one, and that the range $\mathcal{R}(S) = H$).