TEST #2

(1) [4 Pts] $C^1[0,1]$ is the space of continuous functions on $[0,1]$, whose derivative is continuous on $(0,1)$. Show that:

(a) $C^1[0,1]$ is a normed space, with norm $\|x\| = \max_{t \in [0,1]} |x(t)| + \max_{t \in [0,1]} |x'(t)|$;

(b) $C^1[0,1]$ is complete (and thus, it is a Banach space).

(2) [4 Pts] Using the Banach fixed point theorem, set up an iteration solving $f(x) = 0$, where $f \in C^1[a,b]$, $f(a) < 0$, $f(b) > 0$, and $0 < k_1 \leq f'(x) \leq k_2$, for all $x \in [a,b]$. Hint: use $Tx = x - \lambda f(x)$, for a suitable $\lambda$.

(3) [4 Pts] Let $H$ be a Hilbert space, and $x, y \in H$. Prove that $x \perp y$ if and only if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{C}$. 